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The Class of Functionals which can be Represented by a Supremum

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We give a characterization of all lower semicontinuous functionals on L^∞_μ which can be represented in the form $\mu\text{-}\sup\{f(x,u)\colon x \in A\}$. We also show by a counterexample that the representation above may fail if the lower semicontinuity condition is dropped.

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1. Introduction

Functionals which can be written in *supremal* form

$$F(u,B) = \mu\text{-}\sup\big\{f\big(x,u(x)\big)\colon x \in B\big\} \tag{1}$$

received much attention in the last years (see References). In the applications they describe optimization problems whose criteria select solutions which minimize a given quantity in the worst possible situation. This is for instance the case of criteria like the maximum stress in elasticity, the maximum loss in economy, the maximum pressure in problems from fluidodynamics.

In order to apply the direct methods of the calculus of variations to this class of functionals, a first problem to be solved is the identification of qualitative conditions on the *supremand* f which imply the lower semicontinuity with respect to a convergence weak enough to provide the compactness in a large number of situations, say the weak* L^∞ convergence. This was already solved by Barron and Liu in [3] where they showed that a functional of the form (1) is weakly* L^∞ sequentially lower semicontinuous if and only if the function

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f is *level convex*, that is for every $t \in \mathbf{R}$ the level set $\{s \in \mathbf{R}^n : f(x,s) \leq t\}$ is convex for μ - a.e. x.

The problem we attack in the present paper is to characterize in a supremal form (1) all mappings $F(u,B)$ which fulfill certain intrinsic properties. A similar problem was considered by Buttazzo and Dal Maso in [5] for the case of *integral* functionals

$$F(u,B) = \int_B f\big(x,u(x)\big)\,d\mu. \tag{2}$$

We want to stress the fact that in the integral case (2) the assumption that the mappings $F(u,\cdot)$ are additive plays a crucial rôle. On the contrary, in the supremal case (1) additivity can no longer be reasonably assumed, and it must be replaced by the more natural assumption

$$F(u, A \cup B) = F(u,A) \vee F(u,B). \tag{3}$$

Our main result (Theorem 3.2) is that assumption (3), together with a lower semicontinuity hypothesis, implies for a mapping $F(u,B)$ the supremal representation formula (1) for a suitable supremand function f. A key tool in the proof is a result by Barron, Cardaliaguet and Jensen (see [1]) where a result analogous to the Radon-Nikodym theorem for measures is proved. The argument leading to the supremal representation can be concluded due to a new form of Moreau-Yosida transform which is suitable for this class of functionals.

2. Preliminary results

Let $(\Omega,\mathcal{F},\mu)$ be a measure space, with μ non-negative and σ-finite. We denote for brevity by L^∞_n the space $L^\infty_\mu(\Omega;\mathbf{R}^n)$, by $\mathcal{B}_n$ the Borel σ-field of $\mathbf{R}^n$, and by μ-sup the μ-essential supremum.

Definition 2.1. A function $f : \Omega \times \mathbf{R}^n \to]-\infty,+\infty]$ is said to be:

(a) a supremand if f is $\mathcal{F} \otimes \mathcal{B}_n$-measurable;

(b) a normal supremand if f is $\mathcal{F} \otimes \mathcal{B}_n$-measurable and $f(x,\cdot)$ is lower semicontinuous on $\mathbf{R}^n$ for μ - a.e. $x \in \Omega$;

(c) a level convex normal supremand if f is a normal supremand such that for μ - a.e. $x \in \Omega$ and for every $t \in \mathbf{R}$ the level set $\{s \in \mathbf{R}^n : f(x,s) \leq t\}$ is convex.

Remark 2.2. We prefer to use here the terminology *level convex* instead of *quasiconvex*, as was used in [1], [2], [3]. We want indeed to avoid every possible confusion with the Morrey quasiconvexity, which is a concept quite different and also very commonly used in the calculus of variations (see for instance Dacorogna [8] for further details).

We consider functionals $F : L^\infty_n \times \mathcal{F} \to \overline{\mathbf{R}}$; our goal is to show that under suitable conditions they can be written in a supremal form

$$F(u,B) = \mu\text{-}\sup\big\{f\big(x,u(x)\big) : x \in B\big\} \tag{4}$$

for a suitable supremand f. We first show that if a representation in the form (4) exists then it is unique.

Proposition 2.3. *Let f and g be two supremands. Then the inequality*

$$\mu\text{-}\sup\{f(x,u(x)): x \in B\} \leq \mu\text{-}\sup\{g(x,u(x)): x \in B\} \tag{5}$$

for every $u \in L^\infty_n$ and $B \in \mathcal{B}_n$ implies that for μ - a.e. $x \in \Omega$ we have

$$f(x,s) \leq g(x,s) \qquad \forall s \in \mathbf{R}^n .$$

Proof. Let $\{\Omega_k\}$ be a sequence in $\mathcal{F}$ such that $\Omega_k \uparrow \Omega$ and $\mu(\Omega_k) < +\infty$, and for every integer k set

$$f_k = f \wedge k , \qquad g_k = g \wedge k .$$

In order to achieve the proof it is clearly enough to show that for every $k \in \mathbf{N}$ and for μ - a.e. $x \in \Omega_k$ we have

$$f_k(x,s) \leq g_k(x,s) \qquad \text{for all } s \in \mathbf{R}^n \text{ with } |s| \leq k. \tag{6}$$

For every $k \in \mathbf{N}$, $\varepsilon > 0$, $x \in \Omega$ we set

$$\begin{aligned} S_k^\varepsilon &= \{(x,s) \in \Omega_k \times \mathbf{R}^n : |s| \leq k,\ f_k(x,s) > g_k(x,s) + \varepsilon\} \\ S_k^\varepsilon(x) &= \{s \in \mathbf{R}^n : (x,s) \in S_k^\varepsilon\} \\ \Omega_k^\varepsilon &= \{x \in \Omega : S_k^\varepsilon(x) \neq \emptyset\} . \end{aligned}$$

Since f and g are $\mathcal{F} \otimes \mathcal{B}_n$-measurable, the set S_k^ε belongs to $\mathcal{F} \otimes \mathcal{B}_n$, thus the Aumann measurable selection theorem (see [7] Theorems III.22 and III.23, and [5] Theorem 2.1) applies and we have that $\Omega_k^\varepsilon \in \mathcal{F}$ and that there exists an $\mathcal{F}$-measurable selection $u_k^\varepsilon : \Omega_k^\varepsilon \to \mathbf{R}^n$ such that

$$u_k^\varepsilon(x) \in S_k^\varepsilon(x) \qquad \text{for every } x \in \Omega_k^\varepsilon .$$

Setting $u_k^\varepsilon(x) = 0$ on $\Omega \setminus \Omega_k^\varepsilon$ we obtain that $u_k^\varepsilon \in L^\infty_n$, $|u_k^\varepsilon| \leq k$, and

$$f_k(x,u_k^\varepsilon(x)) > g_k(x,u_k^\varepsilon(x)) + \varepsilon \qquad \text{for every } x \in \Omega_k^\varepsilon . \tag{7}$$

Hence

$$g_k(x,u_k^\varepsilon(x)) < k \qquad \text{for every } x \in \Omega_k^\varepsilon \tag{8}$$

so that

$$g_k(x,u_k^\varepsilon(x)) = g(x,u_k^\varepsilon(x)) \qquad \text{for every } x \in \Omega_k^\varepsilon . \tag{9}$$

From (7), (8), (9) we obtain for every $x \in \Omega_k^\varepsilon$

$$g(x,u_k^\varepsilon(x)) + \varepsilon = g_k(x,u_k^\varepsilon(x)) + \varepsilon < f_k(x,u_k^\varepsilon(x)) \leq f(x,u_k^\varepsilon(x)) \tag{10}$$

and taking the μ-sup in Ω_k^ε

$$\begin{aligned} \mu\text{-}\sup\{g(x,u_k^\varepsilon(x)) + \varepsilon : x \in \Omega_k^\varepsilon\} &\leq \mu\text{-}\sup\{f(x,u_k^\varepsilon(x)) : x \in \Omega_k^\varepsilon\} \\ &\leq \mu\text{-}\sup\{g(x,u_k^\varepsilon(x)) : x \in \Omega_k^\varepsilon\} , \end{aligned}$$

where the last inequality follows from assumption (5). Therefore, since by (8) and (9) the function $g(x,u_k^\varepsilon(x))$ is bounded from above, we deduce that $\mu(\Omega_k^\varepsilon) = 0$. Setting $N_k = \bigcup\{\Omega_k^\varepsilon : \varepsilon > 0\}$ we then have $\mu(N_k) = 0$ and

$$f_k(x,s) \leq g_k(x,s)$$

for every $x \in \Omega_k \setminus N_k$ and for every $s \in \mathbf{R}^n$ with $|s| \leq k$, which proves (6). □

Corollary 2.4. *Let f and g be two supremands such that*

$$\mu\text{-}\sup\big\{f\big(x,u(x)\big):\, x\in B\big\}=\mu\text{-}\sup\big\{g\big(x,u(x)\big):\, x\in B\big\}$$

for every $u\in L^\infty_n$ and $B\in\mathcal{B}_n$. Then f and g are equivalent, in the sense that for μ - a.e. $x\in\Omega$ we have

$$f(x,s)=g(x,s)\qquad \forall s\in\mathbf{R}^n\ .$$

A key tool we shall use is a modification of the Moreau-Yosida transform (Pasch-Hausdorff envelope, according to the terminology of [9], Chapter 9). More precisely, the following approximation result holds.

Proposition 2.5. *Let (X,d) be a metric space, let $F:X\to[0,+\infty]$ be a lower semicontinuous function, and let $L:\mathbf{R}\to\mathbf{R}$ be an increasing function such that*

$$L(0)=0\ ,\qquad L(t)>0\quad \textit{for every } t>0\ . \tag{11}$$

If we set for every $\lambda>0$

$$F_\lambda(x)=\inf\big\{F(y)\vee\lambda L\big(d(x,y)\big):\, y\in X\big\} \tag{12}$$

then we have

$$F(x)=\sup\big\{F_\lambda(x):\,\lambda>0\big\}\quad \textit{for every } x\in X\ .$$

Proof. Fix an element $x\in X$; by taking $y=x$ in the definition of $F_\lambda(x)$ we obtain the inequality

$$F_\lambda(x)\le F(x)\ .$$

Let now $t<F(x)$; since F is lower semicontinuous there exists $\delta>0$ such that

$$t<\inf\{F(y):\, y\in X,\ d(x,y)<\delta\}\ ,$$

and using (11) there also exists a number $\lambda>0$ such that $\lambda L(\delta)>t$. For every $y\in X$ with $d(x,y)<\delta$ we have

$$F(y)\vee\lambda L\big(d(x,y)\big)\ge F(y)>t$$

whereas for every $y\in X$ with $d(x,y)\ge\delta$ we have

$$F(y)\vee\lambda L\big(d(x,y)\big)\ge\lambda L(\delta)>t\ .$$

Thus $F_\lambda(x)\ge t$ and, since t was arbitrary, the inequality

$$F(x)\le\sup\big\{F_\lambda(x):\,\lambda>0\big\}$$

is proved. □

Proposition 2.6. *In the case $L(t)=t$ the functional F_λ defined in* (12) *satisfies the condition*

$$F_\lambda(x)\le F_\lambda(y)+\lambda d(x,y)\qquad \textit{for every } x,y\in X\ . \tag{13}$$

Therefore F_λ turns out to be λ-Lipschitz continuous whenever F is not identically $+\infty$.

Proof. Take $x, y \in X$ and $\varepsilon > 0$, and let $w \in X$ be such that

$$F_\lambda(y) \geq F(w) \vee \lambda d(y, w) - \varepsilon \,.$$

Since for every $a, b, c \in \mathbf{R}$

$$a \vee b \leq a \vee c + |b - c| \,,$$

we have

$$\begin{aligned} F_\lambda(x) &\leq F(w) \vee \lambda d(x, w) \\ &\leq F(w) \vee \lambda d(y, w) + \lambda |d(x, w) - d(y, w)| \\ &\leq F_\lambda(y) + \varepsilon + \lambda d(x, y) \,, \end{aligned}$$

which immediately gives (13). □

We finally need the following Radon-Nikodym type result, recently obtained by Barron, Cardaliaguet, Jensen in [1].

Theorem 2.7. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with μ non-negative and σ-finite. Let $F : \mathcal{F} \to \overline{\mathbf{R}}$ be a mapping such that*

(a) $F\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \bigvee_{i=1}^{\infty} F(A_i)$ *whenever $A_i \in \mathcal{F}$,*

(b) $F(A) = F(B)$ *for every $A, B \in \mathcal{F}$ such that $\mu(A \triangle B) = 0$.*

Then there exists a μ-measurable function $f : \Omega \to \overline{\mathbf{R}}$ such that

$$F(A) = \mu\text{-}\sup\{f(x) \colon x \in \mathcal{A}\} \,.$$

Remark 2.8. The previous statement slightly differs from the original one of [1], since our F may take infinite values, but we can reduce our case to the original one by considering the functional

$$G(A) = \arctan F(A) \,.$$

3. The representation result

In this section we show that any mapping $F : L^\infty_n \times \mathcal{F} \to \overline{\mathbf{R}}$ satisfying a certain set of assumptions may actually be written as a supremal functional

$$F(u, B) = \mu\text{-}\sup\big\{f\big(x, u(x)\big) \colon x \in B\big\} \tag{14}$$

for a suitable normal supremand f. The mappings F we consider satisfy the conditions:

(*locality*) $F(u, A) = F(v, B)$ for all $u, v \in L^\infty_n$ and $A, B \in \mathcal{F}$ with $u = v$ μ - a.e. on B and $\mu(A \triangle B) = 0$; (15)

(*supremality*) $F(u, \bigcup_{n=1}^{\infty} A_i) = \bigvee_{n=1}^{\infty} F(u, A_i)$ for all $u \in L^\infty_n$ and $A_i \in \mathcal{F}$; (16)

(*lower semicontinuity*) for every $B \in \mathcal{F}$ the mapping $F(\cdot, B)$ is strongly lower semicontinuous in $L^\infty_n(B)$. (17)

Remark 3.1. It is easy to see that the supremality condition (16) is equivalent to the following one:

$$(18)\begin{cases} i)\ (\textit{monotonicity})\ F(u,A)\le F(u,B) \quad \text{for all } u\in L^\infty_n \text{ and } A,B\in\mathcal{F} \text{ with } A\subset B;\\ ii)\ (\textit{supremality on disjoint sets})\ F(u,\bigcup_{n=1}^\infty A_i)=\bigvee_{n=1}^\infty F(u,A_i) \quad \text{for all } u\in L^\infty_n\\ \qquad \text{and } A_i\in\mathcal{F} \text{ with } A_i\cap A_j=\emptyset \text{ when } i\ne j. \end{cases}$$

Indeed the implication (16) $\Rightarrow$ (18) is straightforward. For the opposite implication, given $u\in L^\infty_n$ and $A_i\in\mathcal{F}$ for every $i\in\mathbf{N}$, set $E_i=A_i\setminus(\bigcup_{j<i}A_j)$: then

$$F(u,\bigcup_{i=1}^\infty A_i)=F(u,\bigcup_{i=1}^\infty E_i)=\bigvee_{i=1}^\infty F(u,E_i)\le\bigvee_{i=1}^\infty F(u,A_i)\ .$$

The opposite inequality is a consequence of the monotonicity condition (18) i).

The main result of this paper is the following.

Theorem 3.2. *Let $F: L^\infty_n\times\mathcal{F}\to\overline{\mathbf{R}}$ be a mapping satisfying* (15)–(17)*. Then there exists a normal supremand f such that the representation formula* (14) *holds. Moreover, this supremand f is unique.*

Proof. The proof will be achieved in several steps.

Step 1. By considering the mapping

$$(u,B)\mapsto\frac{1}{2}+\frac{1}{\pi}\arctan F(u,B)\ ,$$

we may assume that F takes its values in the bounded interval $[0,1]$.

Step 2. For every $\lambda>0$ we consider the approximating functional F_λ given by

$$F_\lambda(u,B)=\inf\big\{F(v,B)\vee\lambda\|u-v\|_{L^\infty_n(B)}\colon v\in L^\infty_n(B)\big\}\ .$$

It is easy to check that the mappings F_λ still satisfy assumptions (15) – (17). In fact (15) is trivial; in order to prove property (16), due to Remark 3.1, we will prove the equivalent condition (18). Condition (18) i) is trivial, so we only prove condition (18) ii).

Let $u\in L^\infty_n$ and $A_i\in\mathcal{F}$ with $A_i\cap A_j=\emptyset$ for every $i\ne j$; setting $A=\bigcup_{i=1}^\infty A_i$ we have

$$F(v,A)\vee\lambda\|u-v\|_{L^\infty_n(A)}\ge F(v,A_i)\vee\lambda\|u-v\|_{L^\infty_n(A_i)}$$

for every $v\in L^\infty_n$, so

$$F_\lambda(u,A)\ge F_\lambda(u,A_i)\qquad\forall i\in\mathbf{N}\ :$$

this implies

$$F_\lambda(u,A)\ge\bigvee_{i=1}^\infty F(u,A_i). \tag{19}$$

On the other hand, for every $\varepsilon > 0$ there exists $v_{i,\varepsilon} \in L^\infty_n(A_i)$ such that

$$\lambda\|u - v_{i,\varepsilon}\|_{L^\infty_n(A_i)} \vee F(v_{i,\varepsilon}, A_i) \leq F_\lambda(u, A_i) + \varepsilon\ ;$$

remark that

$$\lambda\|v_{i,\varepsilon}\|_{L^\infty_n(A_i)} \leq \lambda\|u\|_{L^\infty_n(A_i)} + \lambda\|u - v_{i,\varepsilon}\|_{L^\infty_n(A_i)} \leq \lambda\|u\|_{L^\infty_n(A_i)} + F_\lambda(u, A_i) + \varepsilon\ ,$$

thus if we define $z_\varepsilon = v_{i,\varepsilon}$ in A_i we have $z_\varepsilon \in L^\infty_n(A)$ and

$$\begin{aligned} \lambda\|u - z_\varepsilon\|_{L^\infty_n(A)} \vee F(z_\varepsilon, A) &= \bigvee_{i=1}^{\infty} \big(\lambda\|u - z_\varepsilon\|_{L^\infty_n(A_i)} \vee F(z_\varepsilon, A_i)\big) \\ &\leq \bigvee_{i=1}^{\infty} \big(F_\lambda(u, A_i) + \varepsilon\big) = \bigvee_{i=1}^{\infty} F_\lambda(u, A_i) + \varepsilon\ , \end{aligned}$$

whence

$$F_\lambda(u, A) \leq \bigvee_{i=1}^{\infty} F(u, A_i) + \varepsilon$$

for every $\varepsilon > 0$. Together with (19) this implies condition (18) ii).

The lower semicontinuity (17) follows from the fact that, by Proposition 2.6, the functionals F_λ are λ-Lipschitz continuous, that is

$$F_\lambda(u, B) \leq F_\lambda(v, B) + \lambda\|u - v\|_{L^\infty_n(B)} \qquad \text{for every } u, v \in L^\infty_n,\ B \in \mathcal{F}. \tag{20}$$

Step 3. For every $\lambda > 0$ and $u \in L^\infty_n$ the set function $F_\lambda(u, \cdot)$ satisfies the assumptions of Theorem 2.7, so that

$$F_\lambda(u, B) = \mu\text{-}\sup\big\{h_{\lambda,u}(x) \colon x \in B\big\} \qquad \text{for every } B \in \mathcal{F} \tag{21}$$

for a suitable measurable function $h_{\lambda,u}$.

More precisely, following the proof of Theorem 3.5 in [1], for every $u \in L^\infty_n$ and for every $\lambda > 0$, one may construct the measure

$$\nu_{\lambda,u}(A) = \inf\left\{\sum_{i=1}^{\infty} F_\lambda(u, A_i)\mu(A_i) \colon \bigcup_{i=1}^{\infty} A_i = A,\ A_i \in \mathcal{F}\right\}\ . \tag{22}$$

This is showed in [1] to be a non-negative measure which is absolutely continuous with respect to μ and so, for every $\lambda > 0$ and $u \in L^\infty_n$, there exists a measurable function $h_{\lambda,u} : \Omega \to [0, +\infty]$ such that

$$\nu_{\lambda,u}(A) = \int_A h_{\lambda,u}\, d\mu$$

and that (21) holds.

Step 4. Set for every $\lambda > 0$

$$f_\lambda(x, s) = h_{\lambda,s}(x) \qquad \text{for every } x \in \Omega,\ s \in \mathbf{R}^n\ .$$

The function $f_\lambda(x,s)$ is non-negative, measurable with respect to x and λ-Lipschitz continuous with respect to s. Indeed, from (20) and (22) it follows that

$$\nu_{\lambda,s}(A) \le \nu_{\lambda,t}(A) + \lambda|t-s|\mu(A) \ ,$$

so that

$$\int_A f_\lambda(x,s)d\mu(x) \le \int_A f_\lambda(x,t)d\mu(x) + \lambda \int_A |t-s|d\mu$$

for every $A \in \mathcal{F}$, which implies

$$f_\lambda(x,s) \le f_\lambda(x,t) + \lambda|t-s|$$

for μ - a.e. $x \in \Omega$ and for every $s,t \in \mathbf{R}^n$.

Now we shall prove that

$$F_\lambda(u,B) = \mu\text{-}\sup\big\{f_\lambda\big(x,u(x)\big) \colon x \in B\big\} \tag{23}$$

for every $B \in \mathcal{F}$ and for every $u \in L^\infty_n$. If u is a simple function, that is

$$u = \sum_{i=1}^N c_i \mathbf{1}_{B_i} \quad \text{with} \quad B_i \in \mathcal{F},\ \bigcup_{i=1}^N B_i = \Omega,\ B_i \cap B_j = \emptyset,\ c_i \in \mathbf{R}^n \ ,$$

then using the conditions (15) and (16) satisfied by F_λ we obtain for every $B \in \mathcal{F}$

$$\begin{aligned} F_\lambda(u,B) &= F_\lambda\Big(u, \bigcup_{i=1}^N (B\cap B_i)\Big) = \bigvee_{i=1}^N F_\lambda\Big(\sum_{i=1}^N c_i\mathbf{1}_{B_i}, B\cap B_i\Big) \\ &= \bigvee_{i=1}^N F_\lambda(c_i, B\cap B_i) = \bigvee_{i=1}^N \mu\text{-}\sup\big\{f_\lambda(x,c_i) \colon x \in B\cap B_i\big\} \\ &= \bigvee_{i=1}^N \mu\text{-}\sup\big\{f_\lambda\big(x,u(x)\big) \colon x \in B\cap B_i\big\} = \mu\text{-}\sup\big\{f_\lambda\big(x,u(x)\big) \colon x \in B\big\} \ . \end{aligned}$$

If $u \in L^\infty_n$, there exists a sequence $\{u_k\}_{k\in\mathbf{N}}$ of simple functions such that $\|u_k - u\|_{L^\infty_n} \to 0$. Using the λ-Lipschitz continuity of F_λ and of $f_\lambda(x,\cdot)$, we have

$$\begin{aligned} F_\lambda(u,B) &= \lim_{k\to\infty} F_\lambda(u_k,B) \\ &= \lim_{k\to\infty} \mu\text{-}\sup\big\{f_\lambda\big(x,u_k(x)\big) \colon x \in B\big\} \\ &= \mu\text{-}\sup\big\{f_\lambda\big(x,u(x)\big) \colon x \in B\big\} \ . \end{aligned}$$

Step 5. Remark that if $\lambda \le \nu$ then $f_\lambda(x,s) \le f_\nu(x,s)$ for μ - a.e. $x \in \Omega$ and for every $s \in \mathbf{R}^n$.

Therefore, if we define

$$f(x,s) = \sup\big\{f_\lambda(x,s) \colon \lambda > 0\big\} \ ,$$

the function f turns out to be $\mathcal{F} \otimes \mathcal{B}_n$-measurable and lower semicontinous on $\mathbf{R}^n$, and so f is a normal supremand. By Proposition 2.5 and (23), we finally obtain

$$F(u,B) = \sup\big\{F_\lambda(u,B) \colon \lambda > 0\big\} = \sup\big\{f\big(x,u(x)\big) \colon x \in B\big\}$$

for every $B \in \mathcal{F}$, which concludes the proof of the existence of a normal supremand f for which (14) holds. The uniqueness of f easily follows from Corollary 2.4. □

Remark 3.3. A careful inspection of the proofs of the results above shows that the euclidean space $\mathbf{R}^n$ can be replaced by any separable metric space (X,d). Indeed, in this case we still have that the class of measurable functions defined on Ω and with countable values in X is dense in $L^\infty_\mu(\Omega;X)$. Moreover, it is easy to see that all arguments still hold if instead of $L^\infty_\mu(\Omega;X)$ we consider the larger space $L^0_\mu(\Omega;X)$ of all measurable functions from Ω into X, endowed with the uniform convergence

$$u_k \to u \text{ uniformly in } L^0_\mu(\Omega;X) \qquad \Longleftrightarrow \qquad \mu\text{-}\sup\big\{d\big(u_k(x),u(x)\big) \colon x \in \Omega\big\} \to 0.$$

Now we show by a counterexample that the representation result of Theorem 3.2 may fail if we drop the lower semicontinuity assumption (17). The example is similar to the one constructed in [5] for the case of functionals defined on L^p (see also [4], Section 2.5).

We consider $\Omega =]0,1[$, $\mathcal{F}$ the σ-field of Lebesgue measurable subsets of Ω and μ the Lebesgue measure. We shall give an example of a functional $F : L^\infty \times \mathcal{F} \to [0,1]$ with the following properties:

(i) F satisfies the locality condition (15);
(ii) F satisfies the supremality condition (16);
(iii) for every $B \in \mathcal{F}$ with $\mu(B) > 0$ and $u \in L^\infty$ we have $F(0,B) = 0$ and $F(u,B) \geq 0$;
(iv) F does not admit any supremal representation, that is, for every supremand f there exist $u \in L^\infty$ and $B \in \mathcal{F}$ such that

$$F(u,B) \neq \mu\text{-}\sup\big\{f\big(x,u(x)\big) \colon x \in B\big\}\ .$$

For every $u \in L^\infty$, let $Tu : \Omega \to [0,1]$ be defined by

$$Tu(x) = \begin{cases} 1 \text{ if } \mu\big(\{y \in \Omega \colon u(y) = u(x)\}\big) = 0 \\ 0 \text{ otherwise,} \end{cases}$$

and let $F : L^\infty \times \mathcal{F} \to [0,1]$ be the functional defined by

$$F(u,B) = \mu\text{-}\sup\big\{T\big(u(x)\big) \colon x \in B\big\}\ .$$

Theorem 3.4. *The functional F verifies the properties (i), (ii), (iii), (iv) listed above.*

Proof. Using Theorem 3.1 of [5], we know that T is locally defined. Moreover, if $|B| = 0$, then $F(u,B) = -\infty$ for every $u \in L^\infty$, thus property (i) is satisfied. Properties (ii), (iii) are very easy. Let us prove property (iv); we argue by contradiction: suppose that there exists a supremand f such that

$$F(u,B) = \mu\text{-}\sup\big\{f\big(x,u(x)\big) \colon x \in B\big\}$$

for every $u \in L^\infty$ and $B \in \mathcal{F}$. Following step by step the proof of Theorem 3.1 in [5], we obtain that $F(s,B) = 1$ for every $B \in \mathcal{F}$. But, by definition of F, it is $F(s,B) = 0$ for every $B \in \mathcal{F}$, which gives the contradiction. □

Remark 3.5. It is easy to verify that the functional F above is not semicontinuous with respect to the strong topology of L^∞. In fact, for example, we may consider $u(x) = x$ on $]0,1[$ and a sequence $\{u_k\}$ of simple functions that converges uniformly to u. For every simple function

$$v = \sum_{i=1}^N c_i \mathbf{1}_{B_i} \quad \text{with} \quad B_i \in \mathcal{F},\ \bigcup_{i=1}^N B_i = \Omega,\ B_i \cap B_j = \emptyset,\ c_i \in \mathbf{R}\,,$$

and using (i), (ii), (iii) we obtain

$$F(v,\Omega) = \bigvee_{i=1}^N F(v, B_i) = \bigvee_{i=1}^N F(c_i \mathbf{1}_{B_i}, B_i) = \bigvee_{i=1}^N F(c_i, B_i) = 0\,,$$

so $F(u_k,\Omega) = 0$ for every $k \in \mathbf{N}$, while $F(u,\Omega) = 1$.

4. The weak* l.s.c. case

In this section we consider mappings $F : L^\infty_n \times \mathcal{F} \to \overline{\mathbf{R}}$ satisfying (15), (16) and the following condition:

(weak lower semicontinuity) for every $B \in \mathcal{F}$* the mapping $F(\cdot, B)$ is weakly* lower semicontinuous in $L^\infty_n(B)$. (24)

Assumption (24) is obviously stronger than (17) and so, by Theorem 3.2, such mappings F can actually be written as supremal functionals

$$F(u,B) = \mu\text{-}\sup\big\{f\big(x,u(x)\big) : x \in B\big\} \tag{25}$$

for a suitable normal supremand f.

Theorem 4.1. *Let $F : L^\infty_n \times \mathcal{F} \to \overline{\mathbf{R}}$ be a mapping which satisfies assumptions* (15), (16), (24)*, and let μ be a nonatomic measure. Then there exists a level convex normal supremand f such that the representation formula* (25) *holds.*

In order to prove Theorem 4.1, we state the following result (see Lemma 2.9 in [5]):

Proposition 4.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, where μ is a non-negative, σ-finite, non atomic measure. Then, for every $\lambda \in]0,1[$ there exists a net $(1_{A_i})_{i\in I}$ with $A_i \in \mathcal{F}$, converging to the constant function λ in the weak* topology of L^∞.*

Remark 4.3. If $L^1(\Omega;\mu)$ is separable, which happens when the σ-algebra $\mathcal{F}$ is generated by a countable family $\{E_n\}_{n\in\mathbf{N}}$, then Proposition 4.2 holds with a sequence instead of a net. As a consequence, in this case Theorem 4.1 still holds if we assume, instead of (24), the weaker condition

(sequential weak lower semicontinuity) for every $B \in \mathcal{F}$* the mapping $F(\cdot, B)$ is sequentially weakly* lower semicontinuous in $L^\infty_n(B)$. (26)

Proof of Theorem 4.1. Let $u, v \in L_n^\infty$ and let $\lambda \in (0,1)$. Then we have

$$F(\lambda u + (1-\lambda)v, \Omega) \le F(u,\Omega) \vee F(v,\Omega) \,. \tag{27}$$

In fact, by Proposition 4.2, there exists a net $(A_i)_{i\in I}$ of elements of $\mathcal{F}$ such that the net $(u1_{A_i})_{i\in I}$ converges to λu and $(v1_{\Omega\setminus A_i})_{i\in I}$ converges to $(1-\lambda)v$ in the weak* topology of $L_n^\infty(\Omega)$. Now, for every $x \in \Omega$ and $S \in \mathbf{R}^{2n}$, $S = (s_1, s_2)$, we define

$$\begin{aligned} h_\lambda(x,S) &= f(x, \lambda s_1 + (1-\lambda)s_2) \\ g(x,S) &= f(x,s_1) \vee f(x,s_2) \,. \end{aligned}$$

Inequality (27) and Proposition 2.3 imply that there exists a μ - negligibile set $N_\lambda \in \mathcal{F}$ such that

$$h_\lambda(x,S) \le g(x,S) \qquad \forall x \in \Omega \setminus N_\lambda,\ \forall S \in \mathbf{R}^{2n} \,.$$

Therefore, if $N = \bigcup \{N_\lambda \colon \lambda \in \mathbf{Q} \cap]0,1[\}$, we obtain

$$f(x, \lambda s_1 + (1-\lambda)s_2) \le f(x,s_1) \vee f(x,s_2) \qquad \forall x \in \Omega \setminus N,\ \forall S \in \mathbf{R}^{2n},\ \forall \lambda \in \mathbf{Q} \cap]0,1[\,.$$

By Theorem 15, $f(x,\cdot)$ is lower semicontinuous on $\mathbf{R}^n$ for μ - a.e. $x \in \Omega$ and so

$$f(x, \lambda s_1 + (1-\lambda)s_2) \le f(x,s_1) \vee f(x,s_2)$$

for every $S = (s_1, s_2) \in \mathbf{R}^{2n}$, for every $\lambda \in]0,1[$ and for every $x \in \Omega \setminus N$. This implies that for every $t \in \mathbf{R}$ the level set $\{s \in \mathbf{R}^n \colon f(x,s) \le t\}$ is convex. □

Remark 4.4. If f is a level convex normal supremand and $F(u,B)$ is a functional of the form (25), then F satisfies the lower semicontinuity condition (24). Indeed it is sufficient to prove that for every $c \in \mathbf{R}$ the set

$$K_c = \{u \in L_n^\infty(B) \colon F(u,B) \le c\}$$

is closed in the weak* topology of $L_n^\infty(B)$. Now, a function u belongs to K_c if and only if $f(x,u(x)) \le c$ for μ - a.e. $x \in B$, which turns out to be equivalent to $u(x) \in E(x,c)$ for μ - a.e. $x \in B$, being $E(x,c) = \{s \in \mathbf{R}^n \colon f(x,s) \le c\}$. Since f is a level convex normal supremand the sets $E(x,c)$ are closed and convex for μ - a.e. $x \in B$. Then the function

$$\phi(x,z) = \chi_{E(x,c)}(z) = \begin{cases} 0 & \text{if } z \in E(x,c) \\ +\infty & \text{otherwise} \end{cases}$$

is $\mathcal{F} \otimes \mathcal{B}_n$-measurable and $\phi(x,\cdot)$ is convex and lower semicontinuous for μ - a.e. $x \in B$. Then the functional

$$G(u) = \int_B \phi(x,u(x))\, d\mu = \begin{cases} 0 & \text{if } u \in K_c \\ +\infty & \text{otherwise} \end{cases}$$

turns out to be weakly* lower semicontinuous in $L_n^\infty(B)$ (see for instance Theorem 2.3.1 of [4]), which gives the weak* closedness of K_c.

Remark 4.5. If the measure μ has some atoms, even though the functional F satisfies (15), (16) and (24), but the normal supremand f is not necessarily a level convex function. For example, take $\Omega = \mathbf{R}$, $\mu = \delta_0$ and

$$F(u,B) = \delta_0\text{-}\sup\big\{f\big(x,u(x)\big) : x \in B\big\} = \begin{cases} f\big(u(0)\big) & \text{if } 0 \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then a sequence $\{u_n\}_{n\in\mathbf{N}}$ converges to u in the weak* topology of $L^\infty_n(\delta_0)$ if and only if $u_n(0)$ converges to $u(0)$. It is now sufficient that the supremand f is lower semicontinuous to obtain the weak* lower semicontinuity of the functional F.

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