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Alternative Theorems and Necessary Optimality Conditions for Directionally Differentiable Multiobjective Programs

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In this paper we study, in a unified way, some alternative theorems that involve linear and sublinear functions between finite dimensional spaces and a convex set, and we propose several generalizations of them. These theorems are applied to obtain, under different constraint qualifications, several necessary conditions for a point to be Pareto optimum, both Fritz John and Kuhn-Tucker type, in multiobjective programming problems which are defined by directionally differentiable functions and which include three types of constraints: inequality, equality and set constraints. In particular, these necessary conditions are applicable to convex programs and to differentiable programs.

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1. Introduction

Alternative theorems are indispensable tools in mathematical programming since they allow the transformation of inequality systems, of a difficult direct approach, into equality systems.

The first alternative theorems come from the beginning of the century and from then numerous generalizations have been proposed until today. For example, see Mangasarian [10] for a study of the classic theorems. These deal with a finite number of linear functions from $\mathbb{R}^n$ to $\mathbb{R}$, and from there, the generalizations have included a finite or infinite number of convex or with some type of generalized convexity functions and even multifunctions and increasingly abstract spaces. See Jeyakumar [7] for a review.

These theorems are used in the optimization theory to obtain necessary conditions, in terms of Lagrange multipliers (dual form), so that a point will be an optimum for a mathematical programming problem. Usually a necessary condition expressed through the incompatibility of a system of equations and inequations, formed with the directional derivatives of the functions involved in the problem (primal form), is transformed by an

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alternative theorem in the checking of the existence of some multipliers, or what is the same, in the checking of the compatibility of a system of equations, whose verification is usually much more simple. Hence the great relevance of these theorems.

In this work the results obtained are not directly applied to the functions of the problem, but to their directional derivatives, that is why, in the systems considered, the inequalities are given by sublinear (positively homogeneous and convex) functions and the equalities, by linear functions.

After introducing the notations in Section 2, we study, in Section 3, various generalizations of some of the classic alternative theorems, working with linear or sublinear functions in spaces of finite dimension and substituting the gradient by the subdifferential of the Convex Analysis. The emphasis has been put in a unified treatment of the different situations, including a convex set constraint and giving a differentiated treatment to the equality constraints, which could be treated as two inequalities in some occasions, but in others we obtain advantages from specific treatment, otherwise we would obtain trivial or inapplicable results. The equality constraints considered are given by linear functions, sufficient in many cases, since they are referred to derivatives in the applications.

Finally, in Section 4, the obtained alternative theorems are applied to the demonstration of different necessary optimality conditions for multiobjective programs (in finite dimension) with directionally differentiable functions (not necessarily differentiable neither convex). This permits us to generalize, for example, the results of Singh [16] and of Giorgi and Guerraggio [3], that deal with differentiable functions, and also several results on convex programs, in particular those of Kanniappan [8] and Islam [6].

2. Notations

Let x and y be two points of $\mathbb{R}^n$. Throughout this paper, we shall use the following notations.

$$x \leq y \text{ if } x_i \leq y_i, i = 1, \dots, n; \quad x < y \text{ if } x_i < y_i,\ i = 1, \dots, n.$$

Let S be a subset of $\mathbb{R}^n$, as usual, $\operatorname{cl} S$, $\operatorname{int} S$, $\operatorname{ri} S$, $\operatorname{co} S$, $\operatorname{aff} S$, $\operatorname{cone} S$, $\operatorname{lin} S$, will denote the closure, interior, relative interior, convex hull, affine hull, generated cone and linear span by S, respectively. $B(x_0, \delta)$ is the open ball centered at x_0 and radius $\delta > 0$.

Given a point $x_0 \in S$ and a function $f : \mathbb{R}^n \to \mathbb{R}^p$, the following multiobjective optimization problem is considered

$$\operatorname{Min}\{f(x) :\ x \in S\}.$$

It is said that x_0 is to be a weak Pareto minimum if there exists no $x \in S$ such that $f(x) < f(x_0)$. The point x_0 is to be a local weak Pareto minimum, written $x_0 \in \operatorname{LWMin}(f, S)$, if the previous condition is verified on $S \cap B(x_0, \delta)$, for some $\delta > 0$. The usual notion of Pareto minimum is also used, will be denoted $\operatorname{Min}(f, S)$.

The following cones (Definition 2.1) and directional derivatives (Definition 2.2) are considered.

Definition 2.1. Let $S \subset \mathbb{R}^n$, $x_0 \in \operatorname{cl} S$,

(*a*) The tangent cone (or contingent cone) to S at x_0 is
$T(S, x_0) = \{v \in \mathbb{R}^n :\ \exists t_k > 0,\ \exists x_k \in S,\ x_k \to x_0 \text{ such that } t_k(x_k - x_0) \to v\}$.

(*b*) The cone of linear directions (or radial tangent cone) is
$Z(S, x_0) = \{v \in \mathbb{R}^n : \exists \delta > 0 \text{ such that } x_0 + tv \in S \ \forall t \in (0, \delta]\}$.

(*c*) The cone of sequential linear directions (or sequential radial tangent cone, Penot [12, Definition 2.3]) is
$Z_s(S, x_0) = \{v \in \mathbb{R}^n : \exists t_k \to 0^+ \text{ such that } x_0 + t_k v \in S \ \forall k \in \mathbb{N}\}$.

We have that $Z(S, x_0) \subset Z_s(S, x_0) \subset T(S, x_0)$.

Let $D \subset \mathbb{R}^n$, the polar cone to D is $D^* = \{v \in \mathbb{R}^n : \langle v, d\rangle \leq 0 \ \forall d \in D\}$ and the strict polar cone is $D^{s-} = \{v \in \mathbb{R}^n : \langle v, d\rangle < 0 \ \forall d \in D, \ d \neq 0\}$. If D is a subspace, then $D^* = D^\perp$, orthogonal subspace to D. The normal cone to S at x_0 is the polar of tangent cone: $N(S, x_0) = T(S, x_0)^*$. If S is a convex set, one has that $N(S, x_0) = (S - x_0)^*$.

Definition 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}^p, \ x_0, v \in \mathbb{R}^n$.

(*a*) The Dini derivative (or directional derivative) of f at x_0 in the direction v is
$Df(x_0, v) = \lim\limits_{t\to 0^+} [f(x_0 + tv) - f(x_0)]/t$.

(*b*) The Hadamard derivative of f at x_0 in the direction v is
$df(x_0, v) = \lim\limits_{(t,u)\to(0^+,v)} [f(x_0 + tu) - f(x_0)]/t$.

(*c*) f is Dini differentiable or directionally differentiable (resp. Hadamard differentiable) at x_0 if its Dini derivative (resp. Hadamard derivative) exists in all directions.

The following properties hold:

- If f is Fréchet differentiable at x_0, with Fréchet derivative $\nabla f(x_0)$, then $df(x_0, v) = \nabla f(x_0)v$.
- If $df(x_0, v)$ exists, then also $Df(x_0, v)$ exists and they are equal.
- $Df(x_0, v)$ (resp. $df(x_0, v)$) is the vector of components $Df_i(x_0, v)$ (resp. $df_i(x_0, v)$), $i = 1, \ldots, p$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Dini differentiable function at x_0. The concept of subdifferential is well known (see Penot [12]).

Definition 2.3. The Dini subdifferential of f at x_0 is

$$\partial_D f(x_0) = \{\xi \in \mathbb{R}^n : \langle \xi, v\rangle \leq Df(x_0, v) \ \forall v \in \mathbb{R}^n\}.$$

If $Df(x_0, v)$ is a convex function in v, then its subdifferential (in the Convex Analysis sense) at $v = 0$ exists, and it is denoted $\partial Df(x_0, \cdot)(0)$. This is a nonempty, compact and convex set of $\mathbb{R}^n$ and the following asserts are true:

$$\partial_D f(x_0) = \partial Df(x_0, \cdot)(0),$$

$$Df(x_0, v) = \text{Max}\{\langle \xi, v\rangle : \ \xi \in \partial_D f(x_0)\}.$$

If $Df(x_0, v)$ is not convex, then $\partial_D f(x_0)$ can be empty.

A function f whose Dini derivative $Df(x_0, \cdot)$ is convex was called by Pshenichnyi quasidifferentiable at x_0. This concept was later extended by Demyanov (see Demyanov and Rubinov [2]).

3. Alternative Theorems

In this section, several generalizations of some alternative theorems are demonstrated. In particular, we extend the classic theorems of Gordan and Motzkin [10] and the results of Robinson [14, Theorem 3], and Ishizuka and Shimizu [5, Lemma 2]. In the first place, the case of equalities is considered and, then, we work with both equalities and inequalities. The proof of the following Lemma 3.1 is omitted because it is very simple.

Lemma 3.1. *Let B be a nonempty set and C a cone of $\mathbb{R}^n$ with $0 \in C$. Then*

$$0 \notin B + C \quad \Leftrightarrow \quad (-B) \cap C = \emptyset \quad \Leftrightarrow \quad 0 \notin B \text{ and } (-\operatorname{cone} B) \cap C = \{0\}.$$

Lemma 3.2. *Let $Q \subset \mathbb{R}^n$ be a convex set with $0 \in Q$ and $h : \mathbb{R}^n \to \mathbb{R}^r$ linear with component functions h_k, $k \in K = \{1, \dots, r\}$ given by $h_k(u) = \langle c_k, u \rangle$. Suppose the affine hull of Q is given by $\operatorname{aff} Q = \{x \in \mathbb{R}^n : \ \langle d_j, x \rangle = 0, \ j = 1, \dots, l\}$, being $d_1, \dots, d_l$ linearly independent. Consider the following propositions:*

(h1) $0 \in \sum_{k=1}^r \nu_k c_k + N(Q, 0)$, $\nu \in \mathbb{R}^r \quad \Rightarrow \quad \nu = 0$.
(h2) $\sum_{k=1}^r \nu_k h_k(u) \geq 0 \ \forall u \in Q$, $\nu \in \mathbb{R}^r \quad \Rightarrow \quad \nu = 0$.
(h3) *(h3.1)* $c_1, \dots, c_r, d_1, \dots, d_l$ *are linearly independent and (h3.2)* $\operatorname{Ker} h \cap \operatorname{ri} Q \neq \emptyset$.
(h4) *(h4.1)* $c_1, \dots, c_r$ *are linearly independent and (h4.2)* $\operatorname{lin}\{c_k : \ k \in K\} \cap N(Q, 0) = \{0\}$.
(h5) $0 \in \operatorname{int} h(Q)$.
(h6) $\operatorname{Ker} h \cap \operatorname{ri} Q \neq \emptyset$.
(h7) $0 \in \operatorname{ri} h(Q)$.
(h8) $\operatorname{lin}\{c_k : \ k \in K\} \cap (\operatorname{ri} Q)^{s-} = \emptyset$.

Then

(i) (h6) to (h8) are equivalent.
(ii) (h1) to (h5) are equivalent.
(iii) Each proposition (h1) to (h5) implies (h6), (h7) and (h8).
(iv) If $\operatorname{int} h(Q) \neq \emptyset$ then
  1) (h4.1) holds.
  2) (h1) to (h8) are equivalent.

Proof. (i) $(h7) \Rightarrow (h8)$. Suppose that $(h8)$ is false and take $\mu \in \operatorname{lin}\{c_k : \ k \in K\} \cap (\operatorname{ri} Q)^{s-}$. Then $\mu = \sum_{k=1}^r \nu_k c_k$ and $\langle \sum_{k=1}^r \nu_k c_k, q \rangle < 0 \ \forall q \in \operatorname{ri} Q \setminus \{0\}$.
Let $\varphi : \mathbb{R}^r \to \mathbb{R}$ be the linear application defined by $\varphi(y) = \langle \nu, y \rangle$ and $\psi = \varphi \circ h$, which is given by $\psi(x) = \langle \mu, x \rangle$. One has

$$\psi(q) < 0 \quad \forall q \in \operatorname{ri} Q \setminus \{0\}. \tag{1}$$

By convexity of Q, $Q \subset \operatorname{cl} \operatorname{ri} Q$, and by the continuity of ψ, $\psi(q) \leq 0 \ \forall q \in Q$. Thus $\varphi(y) \leq 0 \ \forall y \in h(Q)$. Therefore, $y = 0$ is a maximum of the convex function φ on the convex $h(Q)$. By hypothesis, $0 \in \operatorname{ri} h(Q)$. But if a convex function reaches its maximum at a relative interior point of its domain, then the function is constant (Rockafellar [15, Theorem 32.1]). It follows that $\varphi(y) = 0 \ \forall y \in h(Q)$, that is, $\varphi(h(q)) = 0 \ \forall q \in Q$, which contradicts (1).
$(h6) \Leftrightarrow (h7)$. It is clear if we take into account that $\operatorname{ri} h(Q) = h(\operatorname{ri} Q)$ [15, Theorem 6.6].
Not $(h6) \Rightarrow$ Not $(h8)$. As $\operatorname{Ker} h \cap \operatorname{ri} Q = \emptyset$ and $\operatorname{Ker} h$ is a convex cone, using Theorems

11.3 and 11.7 in [15], $\operatorname{Ker} h$ and Q are separated properly by a hyperplane M through the origin. That is, there exists $\mu \in \mathbb{R}^n$, $\mu \neq 0$ such that $M = \operatorname{Ker}\langle \mu, \cdot \rangle$ and

$$\langle \mu, q \rangle \leq 0 \leq \langle \mu, x \rangle \quad \forall q \in Q,\ \forall x \in \operatorname{Ker} h.$$

Hence $\mu \in -(\operatorname{Ker} h)^* = (\operatorname{Ker} h)^\perp = \operatorname{lin}\{c_k : \ k \in K\}$. It follows that $\operatorname{Ker} h \subset M$, and as M separates properly to Q and $\operatorname{Ker} h$, there exists $q_0 \in Q$ such that $\langle \mu, q_0 \rangle < 0$. As a matter of fact

$$\langle \mu, q \rangle < 0 \quad \forall q \in \operatorname{ri} Q, \tag{2}$$

because if for some $q_1 \in \operatorname{ri} Q$, $\langle \mu, q_1 \rangle = 0$, then the convex function $\langle \mu, \cdot \rangle$ has a maximum on Q at q_1 and, therefore, it is constant on Q, this is $\langle \mu, q \rangle = 0\ \forall q \in Q$, which contradicts that M separates properly.

Accordingly, from (2), $\mu \in \operatorname{lin}\{c_k : \ k \in K\} \cap (\operatorname{ri} Q)^{s-}$.

(ii) $(h1) \Leftrightarrow (h2)$. This is obvious.

$(h1) \Rightarrow (h4)$. Let us prove the linear independence, since the second part is obvious. Let $\sum_{k=1}^r \nu_k c_k = 0$. Taking $d = 0 \in N(Q, 0)$ one has $\sum_{k=1}^r \nu_k c_k + d = 0$. By $(h1)$, $\nu = 0$.

$(h4) \Rightarrow (h3)$. Let us observe in the first place that $(h4.2) \Rightarrow (h8)$ because $(\operatorname{ri} Q)^{s-} \subset Q^* \setminus \{0\}$. Now, by (i), $(h8) \Leftrightarrow (h6){=}(h3.2)$. We have to prove $(h3.1)$.

Let $\sum_{k=1}^r \nu_k c_k + \sum_{j=1}^l \alpha_j d_j = 0$. As $Q \subset \operatorname{aff} Q = (\operatorname{lin}\{d_1, \ldots, d_l\})^\perp$ results that

$$N(\operatorname{aff} Q, 0) = (\operatorname{aff} Q)^\perp = \operatorname{lin}\{d_1, \ldots, d_l\} \subset N(Q, 0).$$

Hence, $d = \sum_{j=1}^l \alpha_j d_j = -\sum_{k=1}^r \nu_k c_k \in \operatorname{lin}\{c_k : \ k \in K\} \cap N(Q, 0)$; by $(h4.2)$, $d = 0$. Since the vectors d_j are linearly independent, $\alpha = 0$, and by $(h4.1)$, $\nu = 0$.

$(h3) \Rightarrow (h2)$. Suppose that the hypothesis of $(h2)$ holds. Thus
$\langle \mu, u \rangle = \langle \nu, h(u) \rangle \geq 0\ \forall u \in Q$, being $\mu = \sum_{k=1}^r \nu_k c_k$. Take $q \in \operatorname{Ker} h \cap \operatorname{ri} Q$. It is verified that the concave function $\langle \mu, \cdot \rangle$ has a minimum of value 0 on Q at $q \in \operatorname{ri} Q$. Hence it is constant on Q, then $\langle \mu, u \rangle = 0\ \forall u \in Q$ and, therefore, $\langle \mu, u \rangle = 0\ \forall u \in \operatorname{aff} Q$. Thereby $\mu \in (\operatorname{aff} Q)^\perp = \operatorname{lin}\{d_1, \ldots, d_l\}$. Consequently, $\mu = \sum_{j=1}^l \alpha_j d_j = \sum_{k=1}^r \nu_k c_k$, taking into account $(h3.1)$, $\alpha = 0$ and $\nu = 0$.

The equivalence of $(h2)$ with $(h5)$ is proved in a similar way to the equivalence of (b) with (e) in Theorem 3.9.

(iii) It is obvious since $(h3.2){=}(h6)$.

(iv) 1) $\operatorname{int} h(Q) \neq \emptyset \Rightarrow (h4.1)$. In fact, the hypothesis implies that $h(\mathbb{R}^n) = \mathbb{R}^r$, that is, h has rank r. The conclusion follows from observing that the vectors $c_1, \ldots, c_r$ are the rows of the matrix of h in the canonical bases.

2) It is immediate since $(h7)$ becomes $(h5)$. □

Lemma 3.3. *If (h6), (h7) or (h8) is verified then* $\operatorname{lin}\{c_k : \ k \in K\} + N(Q, 0)$ *is closed.*

Proof. Since $\operatorname{ri} \operatorname{Ker} h = \operatorname{Ker} h$, one has by hypothesis $(h6)$, $\operatorname{ri} \operatorname{Ker} h \cap \operatorname{ri} Q \neq \emptyset$. By Corollary 23.8.1 in [15], it follows

$$N[\operatorname{Ker} h \cap Q, 0] = N[\operatorname{Ker} h, 0] + N(Q, 0) = \operatorname{lin}\{c_k : \ k \in K\} + N(Q, 0),$$

which is closed. □

Let us observe that $(h4.2)$ is a form more interesting than $(h8)$, since $(\operatorname{ri} Q)^{s-}$, which is, in general, neither open nor closed, is substituted by $N(Q,0)$, which is closed.

Let us prove with an example that in $(h8)$ of Lemma 3.2 $(\operatorname{ri} Q)^{s-}$ cannot be substituted, in general, by Q^* (Example 3.4(a)), either by Q^{s-}, or by $\operatorname{ri}(Q^*)$ (Example 3.4(b)) and that the converse of (iii) is false. Also it is easy to prove that $(h4.1)$ is neither a necessary nor sufficient condition so that $(h6)$ to $(h8)$ are verified.

Example 3.4.

(a) Let $h : \mathbb{R}^3 \to \mathbb{R}^2$ given by $h(x_1, x_2, x_3) = (x_1, x_2)$ and $Q = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0,\ x_3 \geq 0\}$. It is easy to prove the following results.

1. $L = \operatorname{lin}\{c_k : \ k \in K\} = \operatorname{lin}\{(1,0,0),\ (0,1,0)\} = \{x : \ x_3 = 0\}$, $\operatorname{Ker} h = \{x : \ x_1 = 0,\ x_2 = 0\}$.
2. $h(Q) = \{y = (y_1, y_2) \in \mathbb{R}^2 : \ y_1 = 0\}$, $\operatorname{ri} Q = \{x : \ x_1 = 0,\ x_3 > 0\}$, $Q^* = N(Q,0) = \{x : \ x_2 = 0,\ x_3 \leq 0\}$, $\operatorname{ri}(Q^*) = \{x : \ x_2 = 0,\ x_3 < 0\}$, $Q^{s-} = \emptyset$.
3. $(h7)$ in Lemma 3.2 holds, and logically its equivalents $(h6)$ and $(h8)$. But, $(1,0,0) \in L \cap Q^* \neq \{0\}$, this means that $(h4.2)$ does not hold. Then, the converse of (iii) is false. Let us observe that $\operatorname{int} h(Q) = \emptyset$.

(b) We change the set Q. It will be now $Q = \{x : \ x_1 = 0,\ x_2 \geq 0,\ x_3 \geq 0\}$. It is proved without difficulty:

4. $h(Q) = \{y : \ y_1 = 0,\ y_2 \geq 0\}$, $\operatorname{ri} Q = \{x : \ x_1 = 0,\ x_2 > 0,\ x_3 > 0\}$, $Q^* = \{x : \ x_2 \leq 0,\ x_3 \leq 0\}$, $\operatorname{ri}(Q^*) = Q^{s-} = \{x : \ x_2 < 0,\ x_3 < 0\}$, $(\operatorname{ri} Q)^{s-} = Q^* \setminus \{x : \ x_2 = 0,\ x_3 = 0\}$.
5. $(h7)$ in Lemma 3.2 does not hold, neither does $(h8)$, obviously. However, $L \cap Q^{s-} = \emptyset$ and $L \cap \operatorname{ri}(Q^*) = \emptyset$. Note that $c_1,\ c_2$ are linearly independent, therefore, this is not a sufficient condition so that $(h6)$ to $(h8)$ are fulfilled.

(c) In part (a) we change h. Now $h(x_1, x_2, x_3) = (x_1, x_2, x_1 + x_2)$. 3 of part (a) holds since the sets of 1 and 2 are all maintained except $h(Q)$ which now is $h(Q) = \{y \in \mathbb{R}^3 : \ y_1 = 0,\ y_2 = y_3\}$ and one has that $0 \in \operatorname{ri} h(Q)$, that is, $(h6)$, $(h7)$ and $(h8)$ are fulfilled. The vectors $c_1 = (1,0,0)$, $c_2 = (0,1,0)$, $c_3 = (1,1,0)$ are linearly dependent, hence, this is not a necessary condition so that $(h6)$ to $(h8)$ are fulfilled.

Notice that Theorem 21.2 of Rockafellar [15] is not applicable to the situation described by Lemma 3.2 since there are no strict inequality constraints. Example 3.4(a) above shows that the two incompatible alternatives of the aforesaid theorem would be verified if it were applicable. In fact, the vector $u = (0,0,1) \in \operatorname{ri} Q$ verifies the system $h_1(u) \leq 0,\ h_2(u) \leq 0$ and considering $\lambda_1 = 1,\ \lambda_2 = 0$ we have the inequality $\lambda_1 h_1(u) + \lambda_2 h_2(u) \geq 0\ \forall u \in Q$.

Theorem 3.5. *Let $f_1, \ldots, f_p$ be sublinear functions from $\mathbb{R}^n$ to $\mathbb{R}$ with $p \geq 1$, $f = (f_1, \ldots, f_p)$, $h : \mathbb{R}^n \to \mathbb{R}^r$ linear given by $h = (h_1, \ldots, h_r)$ being $h_k(u) = \langle c_k, u\rangle$, $k \in K = \{1, \ldots, r\}$, and $Q \subset \mathbb{R}^n$ a convex set with $0 \in Q$. Consider the following propositions:*

(a) $0 \in \sum_{i=1}^p \lambda_i \partial f_i(0) + \sum_{k=1}^r \nu_k c_k + N(Q,0),\ \lambda \geq 0$ *implies* $\lambda = 0$.

(b)
$$\sum_{i=1}^{p} \lambda_i f_i(u) + \sum_{k=1}^{r} \nu_k h_k(u) \geq 0 \quad \forall u \in Q,\ \lambda \geq 0 \tag{3}$$
implies $\lambda = 0$.

(c) *There exists* $v \in \mathbb{R}^n$ *such that* $f(v) < 0,\ h(v) = 0,\ v \in Q$.

(d) $0 \notin \mathrm{co}(\cup_{i=1}^{p} \partial f_i(0)) + \mathrm{lin}\{c_k : \ k \in K\} + N(Q,0)$.

Then

(i) *(a), (b) and (d) are equivalent.*
(ii) *(c) ⇒ (b).*
(iii) *If* $0 \in \mathrm{ri}\, h(Q)$, *then (b) ⇒ (c) and, consequently, the four are equivalent.*

Proof. $(b) \Rightarrow (a)$. Suppose that the hypothesis of (a) holds. Then there exist $a_i \in \partial f_i(0)$ and $d \in N(Q,0)$ such that

$$\sum_{i=1}^{p} \lambda_i a_i + \sum_{k=1}^{r} \nu_k c_k + d = 0. \tag{4}$$

As $f_i(u) \geq \langle a_i, u\rangle \ \forall u \in \mathbb{R}^n$ and $0 \geq \langle d, u\rangle \ \forall u \in Q$, it follows:
$\sum_{i=1}^{p} \lambda_i f_i(u) + \sum_{k=1}^{r} \nu_k h_k(u) \geq \sum_{i=1}^{p} \lambda_i \langle a_i, u\rangle + \sum_{k=1}^{r} \nu_k \langle c_k, u\rangle + \langle d, u\rangle = 0 \ \forall u \in Q$.
By (b), $\lambda = 0$.
$(a) \Rightarrow (b)$. Suppose that (3) is verified. Then, $u = 0$ is a minimum of the convex function $\varphi = \sum_{i=1}^{p} \lambda_i f_i + \sum_{k=1}^{r} \nu_k h_k$ on Q. Hence,

$$0 \in \partial\varphi(0) + N(Q,0) = \textstyle\sum_{i=1}^{p} \lambda_i \partial f_i(0) + \sum_{k=1}^{r} \nu_k c_k + N(Q,0).$$

By (a), $\lambda = 0$.
Not (a) ⇔ Not (d). Suppose that (4) holds for some $\lambda \geq 0$, $\lambda \neq 0$, $a_i \in \partial f_i(0)$, $d \in N(Q,0)$. We can assume that $\sum_{i=1}^{p} \lambda_i = 1$, otherwise we just divide by $\sum_{i=1}^{p} \lambda_i > 0$. Hence, (4) means that $0 \in \mathrm{co}(\cup_{i=1}^{p} \partial f_i(0)) + \mathrm{lin}\{c_k : \ k \in K\} + N(Q,0)$. The converse is now obvious.
$(c) \Rightarrow (b)$. Suppose that (3) holds for some $\lambda \neq 0$ and let v be a vector satisfying (c). Then $\sum_{i=1}^{p} \lambda_i f_i(v) + \sum_{k=1}^{r} \nu_k h_k(v) < 0$ in contradiction with what is obtained in (3) taking $u = v$. Hence $\lambda = 0$.
(iii) Suppose that (c) is not fulfilled. Using Lemma 3.2(i) on has $\mathrm{Ker}\, h \cap \mathrm{ri}\, Q \neq \emptyset$ and then we can apply [15, Theorem 21.2] to f, h and $-h$, resulting that there exist $(\lambda, \alpha, \beta) \in \mathbb{R}^p \times \mathbb{R}^r \times \mathbb{R}^r$ such that $(\lambda, \alpha, \beta) \geq 0$, $\lambda \neq 0$ and

$$\textstyle\sum_{i=1}^{p} \lambda_i f_i(u) + \sum_{k=1}^{r} \alpha_k h_k(u) - \sum_{k=1}^{r} \beta_k h_k(u) \geq 0 \ \forall u \in Q,$$

therefore, (b) is not verified taking $\nu_k = \alpha_k - \beta_k$. □

The following proposition shows the connection with subsequent theorems. Corollary 3.7 simplifies this theorem when there are no linear constraints.

Proposition 3.6. *In the hypotheses of Theorem 3.5, if (d) holds and* $\mathrm{Ker}\, h \cap \mathrm{ri}\, Q \neq \emptyset$, *then*

$$D := \mathrm{cone}\,\mathrm{co}(\cup_{i=1}^{p} \partial f_i(0)) + \mathrm{lin}\{c_k : \ k \in K\} + N(Q,0) \text{ is closed.}$$

Proof. By Lemma 3.1, proposition (d) is equivalent to

$$(e) \begin{cases} (e1) \ 0 \notin \mathrm{co}(\cup_{i=1}^{p} \partial f_i(0)) \\ (e2) \ [-\mathrm{cone}\,\mathrm{co}(\cup_{i=1}^{p} \partial f_i(0))] \cap [\mathrm{lin}\{c_k : \ k \in K\} + N(Q,0)] = \{0\}, \end{cases}$$

but $(e1)$, by [4, Proposition 1.4.7, Chap. 3], implies that cone co$(\cup_{i=1}^{p}\partial f_i(0))$ is closed and by Lemma 3.3, $\text{lin}\{c_k : k \in K\} + N(Q,0)$ is closed. Taking into account $(e2)$, by [15, Corollary 9.1.3], we deduce that D is closed. □

Corollary 3.7. *Let $f_1, \ldots, f_p$ be sublinear functions from $\mathbb{R}^n$ to $\mathbb{R}$, $f = (f_1, \ldots, f_p)$ and $Q \subset \mathbb{R}^n$ a convex set which contains to 0. The following propositions are equivalent:*

(a) $0 \in \sum_{i=1}^{p} \lambda_i \partial f_i(0) + N(Q,0), \ \lambda \geq 0 \ \Rightarrow \ \lambda = 0.$
(b) $\sum_{i=1}^{p} \lambda_i f_i(u) \geq 0 \ \forall u \in Q, \ \lambda \geq 0 \ \Rightarrow \ \lambda = 0.$
(c) *There exists $v \in \mathbb{R}^n$ such that $f(v) < 0, \ v \in Q$.*
(d) $0 \notin \text{co}(\cup_{i=1}^{p}\partial f_i(0)) + N(Q,0).$

If some of the conditions (a)-(d) with $Q = \mathbb{R}^n$ is satisfied, we will say that the sets $\partial f_1(0), \ldots, \partial f_p(0)$ are positively linearly independent.

This corollary is a generalization of the Gordan alternative theorem [10] since we obtain it if $Q = \mathbb{R}^n$ and if $\partial f_i(0) = \{a_i\}$, this is, if $f_i(u) = \langle a_i, u\rangle$ is linear.

Remark 3.8. It can be seen in Theorem 3.5 (with $Q = \mathbb{R}^n$, and thus, propositions (a) to (d) are equivalent) a characterization of the compatibility of a system with infinite equations through the compatibility of infinite systems with a finite number of equations. In fact, proposition (c) is equivalent to
(c') There exists a solution $u \in \mathbb{R}^n$ of the system

$$\begin{cases} \text{Max}_{a_i \in A_i} \langle a_i, u\rangle < 0, \ i = 1, \ldots, p \\ \langle c_k, u\rangle = 0, \ k = 1, \ldots, r \end{cases} \tag{5}$$

being $A_i = \partial f_i(0)$, since $f_i(u) = \text{Max}\{\langle a_i, u\rangle : a_i \in A_i\}$.
Proposition (a) can be formulated as:
(a') For every $a_i \in A_i, \ i = 1, \ldots, p$, there exist no $(\lambda, \nu) \in \mathbb{R}^p \times \mathbb{R}^r, \ \lambda \geq 0, \ \lambda \neq 0$ such that

$$\sum_{i=1}^{p} \lambda_i a_i + \sum_{k=1}^{r} \nu_k c_k = 0,$$

and, by the classic Motzkin alternative theorem, this is equivalent to
(a'') For every $a_i \in A_i, \ i = 1, \ldots, p$, there exists a solution $u \in \mathbb{R}^n$ of the system

$$\begin{cases} \langle a_i, u\rangle < 0, \ i = 1, \ldots, p \\ \langle c_k, u\rangle = 0, \ k = 1, \ldots, r. \end{cases} \tag{6}$$

(c') expresses that a system of infinite equations has a solution and it is equivalent to (a''), that expresses that the infinite finite systems of type (6) have a solution.

If in the equivalence $(c') \Leftrightarrow (a'')$ the equality constraints are removed, we obtain Theorem 2.1 in Wang, Dong and Liu [18].

Note that every solution u of (5) it is also of (6), but not inversely. The solutions of (6) depend on the election on $a_1, \ldots, a_p$.

The results obtained in Lemma 3.2 and Theorem 3.5 are combined in the following theorem.

Theorem 3.9. *In the hypotheses of Theorem 3.5, assume that the affine hull to Q is* $\operatorname{aff} Q = \{x \in \mathbb{R}^n : \langle d_j, x\rangle = 0,\ j = 1, \ldots, l\}$, *being $d_1, \ldots, d_l$ linearly independent. The following propositions are equivalent:*

(a) $0 \in \sum_{i=1}^p \lambda_i \partial f_i(0) + \sum_{k=1}^r \nu_k c_k + N(Q, 0),\ \lambda \geq 0 \quad \Rightarrow \quad \lambda = 0,\ \nu = 0.$

(b) $\sum_{i=1}^p \lambda_i f_i(u) + \sum_{k=1}^r \nu_k h_k(u) \geq 0\ \forall u \in Q,\ \lambda \geq 0 \quad \Rightarrow \quad \lambda = 0,\ \nu = 0.$

(c) *(c1) $c_1, \ldots, c_r, d_1, \ldots, d_l$ are linearly independent and (c2) there exists $v \in \mathbb{R}^n$ such that*
$$f(v) < 0,\ h(v) = 0,\ v \in \operatorname{ri} Q.$$

(d) *(d1) $c_1, \ldots, c_r$ are linearly independent, (d2)* $\operatorname{lin}\{c_k : k \in K\} \cap N(Q, 0) = \{0\}$ *and (d3)* $0 \notin \operatorname{co}(\cup_{i=1}^p \partial f_i(0)) + \operatorname{lin}\{c_k : k \in K\} + N(Q, 0)$.

(e) $0 \in \operatorname{int}[(f \times h)(Q) + \mathbb{R}^p_+ \times \{0_r\}]$.

(f) $(f \times h)(\operatorname{cone} Q) + \mathbb{R}^p_+ \times \{0_r\} = \mathbb{R}^p \times \mathbb{R}^r$.

For the demonstration of this theorem we need the following lemma.

Lemma 3.10.
$$(c2) \quad \Leftrightarrow \quad \begin{cases} (c') \text{ there exists } w \in \mathbb{R}^n \text{ such that } h(w) = 0,\ w \in \operatorname{ri} Q \\ (c'') \text{ there exists } v \in \mathbb{R}^n \text{ such that } f(v) < 0,\ h(v) = 0,\ v \in Q. \end{cases}$$

Proof. It is clear that $(c2) \Rightarrow (c')$ and (c''). For the other implication let w and v be vectors satisfying (c') and (c''), respectively, and $v_\lambda = \lambda v + (1 - \lambda) w$. By linearity of h, $h(v_\lambda) = 0$, and by [4, Lemma 2.1.6, Chap. 2], we deduce that $v_\lambda \in \operatorname{ri} Q\ \forall \lambda \in [0, 1)$. As $\lim_{\lambda \to 1^-} v_\lambda = v$, by the continuity of f, $f(v_\lambda) < 0$ for λ near enough 1 and, consequently, all these vectors v_λ verify $(c2)$. □

Proof of Theorem 3.9. One has, obviously, the following equivalences: $(a) \Leftrightarrow (h1)$ and $(3.5a)$, $(b) \Leftrightarrow (h2)$ and $(3.5b)$ and $(d) \Leftrightarrow (h4)$ and $(3.5d)$, where $(h1)$ denotes the same proposition of Lemma 3.2, $(3.5a)$ denotes proposition (a) of Theorem 3.5, etc. By Lemma 3.2(ii) and Theorem 3.5(i), it follows the equivalence of (a), (b) and (d).
By Lemma 3.10 one has that $(c2) \Leftrightarrow (h3.2)$ and $(3.5c)$, and as $(c1){=}(h3.1)$ it follows
$$(c) \quad = \quad (c1) \text{ and } (c2) \quad \Leftrightarrow \quad (h3.1) \text{ and } (h3.2) \text{ and } (3.5c).$$

By Theorem 3.5(ii), $(3.5c) \Rightarrow (3.5a)$; by Lemma 3.2, $(h3.1)$ and $(h3.2) \Leftrightarrow (h1)$ and by Theorem 3.5(iii), $(h3.2)$ and $(3.5a) \Rightarrow (3.5c)$. Hence, $(h3.1)$ and $(h3.2)$ and $(3.5c) \Leftrightarrow (h1)$ and $(3.5a) \Leftrightarrow (a)$. Consequently, $(c) \Leftrightarrow (a)$.
Let us prove that $(b) \Rightarrow (e)$. Let
$$\begin{aligned} A &= (f \times h)(Q) + \mathbb{R}^p_+ \times \{0_r\} \\ &= \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^r : \exists u \in Q \text{ such that } f(u) \leq x,\ h(u) = y\}. \end{aligned}$$

A is a non-empty convex set and verifies that $\operatorname{int} A \neq \emptyset$. In fact, if $\operatorname{int} A = \emptyset$, A is contained in a hyperplane, that is, there exist $(\lambda, \nu) \in \mathbb{R}^p \times \mathbb{R}^r \setminus \{(0, 0)\}$ such that $A \subset \operatorname{Ker}\langle(\lambda, \nu), (\cdot, \cdot)\rangle$. Therefore, $\langle \lambda, f(u)\rangle + \langle \nu, h(u)\rangle = 0\ \forall u \in Q$ and, by (b), $(\lambda, \nu) = (0, 0)$, which is a contradiction.
Let us prove that $0 \in \operatorname{int} A$. Otherwise, as $0 \in A$, there exists a supporting hyperplane to A at 0 and by a similar reasoning to the previous one it also results a contradiction.

$(e) \Rightarrow (b)$. Suppose that the hypothesis of (b) holds, this is that $\langle \lambda, f(u) \rangle + \langle \nu, h(u) \rangle \geq 0 \ \forall u \in Q$. Then $\langle \lambda, x \rangle + \langle \nu, y \rangle \geq 0 \ \forall (x, y) \in A$. Hence, the concave function $\langle (\lambda, \nu), (\cdot, \cdot) \rangle$ has a minimum on the convex A at $(0, 0) \in \operatorname{int} A$ (from hypothesis). Therefore, this function is constant on A, namely, $\langle \lambda, x \rangle + \langle \nu, y \rangle = 0 \ \forall (x, y) \in A$. Since $\operatorname{int} A \neq \emptyset$, it follows that $(\lambda, \nu) = (0, 0)$.
Finally, $(e) \Leftrightarrow (f)$. By [15, Corollary 6.4.1], condition (e) is equivalent to say that for all $u \in \mathbb{R}^p \times \mathbb{R}^r$ there exists $t > 0$ such that $0 + tu \in A$, this means, $\operatorname{cone} A = \mathbb{R}^p \times \mathbb{R}^r$. Now, it is immediate that

$$\operatorname{cone} A = \operatorname{cone}[(f \times h)(Q) + \mathbb{R}^p_+ \times \{0_r\}] = (f \times h)(\operatorname{cone} Q) + \mathbb{R}^p_+ \times \{0_r\},$$

with what one has the equivalence of (e) and (f). □

If some of the six equivalent conditions of Theorem 3.9 with $Q = \mathbb{R}^n$ is verified, we will say $\{\partial_D f_i(0) : \ i = 1, \ldots, p\}$ is posindependent of $\{c_1, \ldots, c_r\}$, or, that $f_1, \ldots, f_p$ are posindependent of $h_1, \ldots, h_r$. This notion generalizes that of positive-linearly independent vectors in Qi and Wei [13, Definition 2.1].

Theorem 3.10 extends Theorem 3 in Robinson [14] who supposes f linear and only considers the propositions (b), (c) and (e).

Remark 3.11.

(1) Taking into account Lemmas 3.2 and 3.10, condition (c) can be expressed:
(c) $0 \in \operatorname{int} h(Q)$ and there exists $v \in \mathbb{R}^n$ such that $f(v) < 0, \ h(v) = 0, \ v \in Q$.

(2) When f is linear, condition (e) becomes the so-called Robinson constraint qualification [14], (f) becomes Zowe-Kurcyusz constraint qualification [19] and, if furthermore $Q = \mathbb{R}^n$, (c) becomes the classic Mangasarian-Fromovitz constraint qualification [11].

Finally we approach the Motzkin alternative theorem in the most general situation that includes inequality constraints, both strict and not strict, equality constraints and convex set constraint.

Theorem 3.12. *Let $f_1, \ldots, f_p, g_1, \ldots, g_m$ be sublinear functions from $\mathbb{R}^n$ to $\mathbb{R}$ with $p \geq 1$ and $m \geq 0$, $f = (f_1, \ldots, f_p)$, $g = (g_1, \ldots, g_m)$, $h : \mathbb{R}^n \to \mathbb{R}^r$ linear given by $h = (h_1, \ldots, h_r)$, $h_k(u) = \langle c_k, u \rangle$, $k \in K = \{1, \ldots, r\}$, and $Q \subset \mathbb{R}^n$ convex subset with $0 \in Q$. Consider the following propositions:*

(a) $0 \in \sum_{i=1}^p \lambda_i \partial f_i(0) + \sum_{j=1}^m \mu_j \partial g_j(0) + \sum_{k=1}^r \nu_k c_k + N(Q, 0), \ (\lambda, \mu) \geq 0 \ \ \Rightarrow \ \ \lambda = 0.$

(b) $\sum_{i=1}^p \lambda_i f_i(u) + \sum_{j=1}^m \mu_j g_j(u) + \sum_{k=1}^r \nu_k h_k(u) \geq 0 \ \forall u \in Q, \ (\lambda, \mu) \geq 0 \ \ \Rightarrow \ \ \lambda = 0.$

(c) *There exists $v \in \mathbb{R}^n$ such that $f(v) < 0, \ g(v) \leq 0, \ h(v) = 0, \ v \in Q$.*

(d) $0 \notin \operatorname{co}(\cup_{i=1}^p \partial f_i(0)) + \operatorname{cone} \operatorname{co}(\cup_{j=1}^m \partial g_j(0)) + \operatorname{lin}\{c_k : \ k \in K\} + N(Q, 0).$

Then

(i) *(a), (b) and (d) are equivalent.*

(ii) *(c) ⇒ (b).*

(iii) *If the condition*
(c′) there exists $w \in \mathbb{R}^n$ such that $g(w) < 0, \ h(w) = 0, \ w \in \operatorname{ri} Q$
holds, then (b) ⇒ (c) and, consequently, the four are equivalent.

Proof. (i) and (ii) are proved in a similar way to (i) and (ii) of Theorem 3.5.
(iii) Suppose that (c) is not true. Then there exists no $v \in \mathbb{R}^n$ such that

$$f(v) < 0,\ g(v) < 0,\ h(v) = 0,\ v \in Q.$$

By Theorem 3.5(iii), (which is applicable since by (c'), $\operatorname{Ker} h \cap \operatorname{ri} Q \neq \emptyset$, and by Lemma 3.2$(i)$, $0 \in \operatorname{ri} h(Q)$), there exist $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that $(\lambda, \mu) \geq 0,\ (\lambda, \mu) \neq 0$ and

$$\sum_{i=1}^{p} \lambda_i f_i(u) + \sum_{j=1}^{m} \mu_j g_j(u) + \sum_{k=1}^{r} \nu_k h_k(u) \geq 0 \quad \forall u \in Q. \tag{7}$$

By (b), $\lambda = 0$. Let w be a vector verifying (c'), then $\sum_{j=1}^{m} \mu_j g_j(w) + \sum_{k=1}^{r} \nu_k h_k(w) < 0$, in contradiction with which is obtained applying (7) to $u = w$ (with $\lambda = 0$). □

If in this theorem we take $m = 0$, we obtain Theorem 3.5.

In the next theorem it is proved that the implication $(b) \Rightarrow (c)$ of Theorem 3.12 is also verified with other different conditions to (iii) in the aforesaid theorem.

Theorem 3.13. *In the hypotheses of Theorem 3.12, assume that some of the equivalent conditions (a), (b), (d) of that theorem holds.*
If $D = \operatorname{cone} \operatorname{co}(\cup_{j=1}^{m} \partial g_j(0)) + \operatorname{lin}\{c_k : \ k \in K\} + N(Q, 0)$ *is closed and* $\operatorname{cone} Q$ *is closed, then (c) holds, so the four are equivalent.*

Proof. Let $C = \operatorname{co}(\cup_{i=1}^{p} \partial f_i(0))$. C is a non-empty compact convex set and D is a closed convex cone. By assumption (d), $0 \notin C + D$, and by Lemma 3.1, $C \cap (-D) = \emptyset$. By the strong separation theorem, there exist $v \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle v, x \rangle < \alpha < \langle v, y \rangle \quad \forall x \in C,\ \forall y \in -D. \tag{8}$$

Taking $y = 0 \in -D$ it follows that $\alpha < 0$. If for some $y \in -D$ we have $\langle v, y \rangle < 0$, then $\lim_{t \to +\infty} \langle v, ty \rangle = -\infty$ with $ty \in -D$ and (8) does not hold. Hence, $\langle v, y \rangle \geq 0\ \forall y \in -D$, or rather, $\langle v, y \rangle \leq 0\ \forall y \in D$. In particular:

1) $\forall b_j \in \partial g_j(0)$ one has $\langle v, b_j \rangle \leq 0$. As $g_j(v) = \operatorname{Max}\{\langle b_j, v \rangle : \ b_j \in \partial g_j(0)\}$, it follows

$$g_j(v) \leq 0, \quad \text{for } j = 1, \ldots, m. \tag{9}$$

2) $\langle v, c_k \rangle \leq 0$ and $\langle v, -c_k \rangle \leq 0$, hence

$$\langle v, c_k \rangle = 0 \quad \forall k \in K. \tag{10}$$

3) $\langle v, d \rangle \leq 0\ \forall d \in Q^*$, therefore $v \in Q^{**} = \operatorname{cl} \operatorname{cone} Q$ and since $\operatorname{cone} Q$ is closed,

$$v \in \operatorname{cone} Q. \tag{11}$$

From (7), $\langle v, x \rangle < \alpha < 0\ \forall x \in C$. In particular, $\forall a_i \in \partial f_i(0)\ \langle v, a_i \rangle < 0$. Thus,

$$f_i(v) < 0, \quad \text{for } i = 1, \ldots, p. \tag{12}$$

From (11) there exist $t > 0$, $u \in Q$ such that $v = tu$ (since $v \neq 0$) and substituting v by tu in (9), (10) and (12), these equations are verified for u in the place of v, this means that u is solution of system (c). □

According to Theorem 3.5 and Proposition 3.6 if $0 \in \mathrm{ri}\, h(Q)$ and there exists $v \in \mathbb{R}^n$ such that $g(v) < 0$, $h(v) = 0$, $v \in Q$, then D is closed and, if furthermore $\mathrm{cone}\, Q$ is closed, Theorem 3.13 can be applied, but in this case we get no advantage since Theorem 3.12(*iii*) is applicable (by Lemma 3.10) with slightly weaker hypotheses (it is not required that $\mathrm{cone}\, Q$ be closed). Nevertheless, there are, obviously, cases in which Theorem 3.13 is applicable and Theorem 3.12 is not applicable as in Example 3.4(b) considering, for example, $f(x_1, x_2, x_3) = -x_3$.

This theorem generalizes Lemma 2 in Ishizuka and Shimizu [5] which does not include equality constraints or constraint set. This theorem also generalizes the classic Motzkin alternative theorem, because if $f_i(u) = \langle a_i, u\rangle$ and $g_j(u) = \langle b_j, u\rangle$ are linear and $Q = \mathbb{R}^n$, then $\mathrm{cone}\,\mathrm{co}\{b_j : \ j = 1, \ldots, m\} + \mathrm{lin}\{c_k : \ k \in K\}$ is closed.

4. Optimality conditions

In this section, as an application of the results of the previous section, we obtain both Fritz John and Kuhn-Tucker type necessary optimality conditions for multiobjective optimization problems. These problems are defined by Dini or Hadamard differentiable functions with convex derivative in the direction, and they include three types of constraints: inequality, equality and set constraints. For this purpose, we need a constraint qualification of extended Abadie type, if the objective function is Hadamard differentiable, or one of extended Zangwill type, if only it is Dini differentiable. To obtain the Kuhn-Tucker type conditions we need the addition of other regularity conditions. If there are no equality constraints (Theorem 4.8) the extended qualifications (Abadie or Zangwill) are not necessary. These results can be applied to study the differentiable programs and the convex programs, so that the results of Singh [16] and of Giorgi and Guerraggio [3], for differentiable programs, and those of Kanniappan [8], Islam [6] and Kouada [9], for convex programs, are just particular cases.

Consider the following multiobjective optimization problem

$$(\mathrm{P}) \quad \mathrm{Min}\{f(x) : \ x \in S \cap Q\},$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$, $S = \{x \in \mathbb{R}^n : \ g(x) \le 0, \ h(x) = 0\}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^r$ and Q is an arbitrary set of $\mathbb{R}^n$.

Let f_i, $i \in I = \{1, \ldots, p\}$, g_j, $j \in J = \{1, \ldots, m\}$, h_k, $k \in K = \{1, \ldots, r\}$ be the component functions of f, g and h, respectively. Given $x_0 \in S$, the set of active indexes at x_0 is $J_0 = \{j \in J : \ g_j(x_0) = 0\}$. The sets defined by the constraints g and h are denoted, respectively, $G = \{x \in \mathbb{R}^n : \ g(x) \le 0\}$, $H = \{x \in \mathbb{R}^n : \ h(x) = 0\}$, accordingly, $S = G \cap H$, and the set of points "better" than x_0 is $F = \{x \in \mathbb{R}^n : \ f(x) \le f(x_0)\}$.

Hereafter we will suppose that the involved functions are Dini differentiable at x_0. We will consider the following cones:

$$\begin{aligned}
&C_0(G) = \{v \in \mathbb{R}^n : \ Dg_j(x_0, v) < 0 \ \forall j \in J_0\},\\
&C(G) = \{v \in \mathbb{R}^n : \ Dg_j(x_0, v) \le 0 \ \forall j \in J_0\},\\
&C_0(S) = C_0(G) \cap \mathrm{Ker}\, Dh(x_0, \cdot), \ C(S) = C(G) \cap \mathrm{Ker}\, Dh(x_0, \cdot),\\
&C_0(F) = \{v \in \mathbb{R}^n : \ Df_i(x_0, v) < 0 \ \forall i \in I\},\\
&C(F) = \{v \in \mathbb{R}^n : \ Df_i(x_0, v) \le 0 \ \forall i \in I\}.
\end{aligned}$$

($C(S)$ is called the linearized cone). Let us point out that the active inequality constraints take part with "$<$" or "$\leq$", according to be C or C_0 and those of equality, with "$=$", and that the point x_0 is omitted to cut it short.

It is assumed that the functions $g_j,\ j \in J \setminus J_0$ are continuous at x_0.

Theorem 4.1. *Let $x_0 \in S \cap Q$ be a feasible point of problem (P) and suppose the following:*

(a) $T(Q, x_0)$ *is convex.*
(b) $g_j,\ j \in J_0$ *are Dini differentiable at x_0 with convex derivative at x_0 and h is Dini differentiable at x_0 with linear derivative given by $Dh_k(x_0, \cdot) = \langle c_k, \cdot \rangle,\ k \in K$.*
(c) *The extended Abadie constraint qualification (EACQ) is verified:*

$$C(S) \cap T(Q, x_0) \subset T(S \cap Q, x_0).$$

(d) $x_0 \in \mathrm{LWMin}(f, S \cap Q)$.
(e) *f is Hadamard differentiable at x_0 with convex derivative at x_0.*

Then

(i) *There exist $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that $(\lambda, \mu) \geq 0,\ (\lambda, \mu, \nu) \neq 0$ and*

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_D f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial_D g_j(x_0) + \sum_{k=1}^{r} \nu_k c_k + N(Q, x_0),$$

$$\mu_j g_j(x_0) = 0,\ j = 1, \ldots, m.$$

(ii) *If $\mathrm{lin}\{c_k :\ k \in K\} + N(Q, x_0)$ is closed, then (i) is satisfied with $(\lambda, \mu) \neq 0$.*
(iii) *If*

$$\operatorname{cone}\operatorname{co}(\cup_{j \in J_0} \partial_D g_j(x_0)) + \mathrm{lin}\{c_k :\ k \in K\} + N(Q, x_0) \text{ is closed,} \tag{13}$$

then (i) is satisfied with $\lambda \neq 0$.

Proof. It is known that if the objective function is Hadamard differentiable and (d) holds, then $T(S \cap Q, x_0) \cap C_0(F) = \emptyset$. From here, by the extended Abadie constraint qualification, it follows that there exists no $v \in \mathbb{R}^n$ such that

$$\begin{cases} Df_i(x_0, v) < 0\ \forall i \in I \\ Dg_j(x_0, v) \leq 0\ \forall j \in J_0 \\ Dh_k(x_0, v) = 0\ \forall k \in K \\ v \in T(Q, x_0). \end{cases} \tag{14}$$

Therefore, we also find the incompatibility in $v \in \mathbb{R}^n$ of the systems

$$(a) \begin{cases} Df_i(x_0, v) < 0\ \forall i \in I \\ Dg_j(x_0, v) < 0\ \forall j \in J_0 \\ Dh_k(x_0, v) = 0\ \forall k \in K \\ v \in T(Q, x_0) \end{cases} \qquad (b) \begin{cases} Df_i(x_0, v) < 0\ \forall i \in I \\ Dg_j(x_0, v) < 0\ \forall j \in J_0 \\ Dh_k(x_0, v) = 0\ \forall k \in K \\ v \in \operatorname{ri} T(Q, x_0). \end{cases} \tag{15}$$

By Theorem 3.9 applied to system (15)(b), part (i) is obtained (taking $\mu_j = 0$, for $j \in J \setminus J_0$, as usual).
Parts (ii) and (iii) are deduced, respectively, by Theorem 3.13 (with $m = 0$) applied to system (15)(a) and by Theorem 3.13 applied to system (14). □

Part (iii) of this theorem generalizes Theorem 3.1 of Singh [16] (which is identical to Theorem 2 of Wang [17]), valid for differentiable functions and without constraint set ($Q = \mathbb{R}^n$), and therefore $\partial_D f_i(x_0) = \{\nabla f_i(x_0)\}$, $\partial_D g_j(x_0) = \{\nabla g_j(x_0)\}$, $c_k = \nabla h_k(x_0)$ and condition (i) becomes the one used by Singh. Condition (13) is always satisfied, since

$$\begin{aligned}\operatorname{cone}\operatorname{co}(\cup_{j\in J_0}\partial_D g_j(x_0)) + \operatorname{lin}\{c_k :\ k \in K\} \\ = \operatorname{cone}\operatorname{co}\{\nabla g_j(x_0) :\ j \in J_0\} + \operatorname{lin}\{\nabla h_k(x_0) :\ k \in K\}\end{aligned}$$

is a polyhedral convex set and therefore, it is closed. Condition (c) in this case becomes $C(S) \subset T(S, x_0)$, which is the constraint qualification used by Singh.

Remark 4.2.

(1) If

$$0 \in \operatorname{ri} Dh(x_0, \cdot)(T(Q, x_0)), \tag{16}$$

then (i) is satisfied with $(\lambda, \mu) \neq 0$. This follows from Lemma 3.3 (applied to the convex $T(Q, x_0)$) and part (ii).

(2) If the constraint qualification

$$C_0(S) \cap \operatorname{ri} T(Q, x_0) \neq \emptyset, \tag{17}$$

is satisfied, then (13) holds by Theorem 3.5 and Proposition 3.6. Therefore, (13) is more general than (17), but this is simpler to check.

(3) Notice that hypotheses of part (ii) can be false and (13) can be true though. Therefore these criteria are of independent application. On the other hand, (17) implies (16) by Lemma 3.2.

The following example shows that, in fact, there are situations in which part (iii) is applicable but (17) is not.

Example 4.3. In $\mathbb{R}^2$, let $x_0 = (0, 0)$, $f(x, y) = -2y$, $g_1(x, y) = -2x$ and g_2 the support function of the set $B = \{(x, y) :\ (x-2)^2 + y^2 \leq 2,\ y \geq 0\}$, that is to say

$$g_2(x, y) = \begin{cases} 2x + \sqrt{2x^2 + 2y^2} & \text{if} \quad y \geq 0 \\ 2x + \sqrt{2x^2} & \text{if} \quad y < 0. \end{cases}$$

Obviously $Dg_2(x_0, v) = g_2(v)$, $\partial_D g_2(x_0) = B$.

The feasible set is $G = \{(0, y) :\ y \leq 0\}$. The point x_0 is an (absolute) minimum of f on G. We have: $C_0(G) = \emptyset$ (hence, (17) is false), $C(G) = G$, $T(G, x_0) = G = C(G)$ and, therefore, the Abadie constraint qualification holds at x_0. Furthermore, condition (13) holds, because

$\operatorname{cone}\operatorname{co}(\cup_{j\in J_0}\partial_D g_j(x_0)) = \operatorname{cone}\operatorname{co}(B \cup \{(-2, 0)\}) = \{(x, y) :\ y \geq 0\}$ is closed.

Hence, Theorem 4.1(iii) is applicable. Concretely the conclusion is satisfied with $(\lambda, \mu_1, \mu_2) = (1, 1, 2)$ and the element $b = (1, 1) \in B$ (there are infinite solutions).

When the constraint set Q is not present (or if Q is a closed convex set) and the functions are continuously Fréchet differentiable, no regularity condition is required in obtaining the usual Fritz John conditions. However, under the hypotheses of Theorem 4.1, the following example shows that (EACQ) cannot be eliminated to obtain part (i).

Example 4.4. Let $h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x, y) = \begin{cases} -y^2 & \text{if } x = 0,\ y < 0 \\ x + x^2 & \text{otherwise,} \end{cases}$$

$x_0 = (0, 0)$ and $f(x, y) = y$. It is clear that x_0 is a local minimum of f on $H := h^{-1}(0) = \{(0, y) :\ y \geq 0\} \cup \{(x, y) :\ x = -1\}$. However, the Fritz John conditions

$$\lambda \nabla f(x_0) + \nu \nabla h(x_0) = (0, 0) \quad \text{with } (\lambda, \nu) \neq (0, 0),\ \lambda \geq 0,$$

are not satisfied. Notice that (EACQ) is not verified: $\operatorname{Ker} \nabla h(x_0) \not\subset T(H, x_0)$.

Similar conditions to (EACQ) are used by other authors. For example, Bender, who considers Hadamard differentiable functions with linear derivative, uses the following [1, Condition (3)]:

$$\operatorname{Ker} Dh(x_0, \cdot) \cap T(Q, x_0) \subset T(H \cap Q, x_0). \tag{18}$$

If the inequality constraints are not considered, (EACQ) becomes (18), but if they are present, there is not implications between (EACQ) and (18). Thus, if we incorporate the constraint $g(x, y) = -y$ in the example 4.4, then (EACQ) holds but (18) is not verified. On the other hand, if we consider $h(x, y) = y$, $g(x, y) = -y - x^3$ and $Q = \{(x, y) :\ y \geq 0\}$ then though (18) holds, (EACQ) is not true.

It is still possible to hold up the conclusions of Theorem 4.1 if f is only Dini differentiable. In return, the constraint qualification must be more restrictive.

Theorem 4.5. *Suppose that conditions (a), (b), (d) of Theorem 4.1 are verified and the following ones:*
(c) The extended Zangwill constraint qualification is satisfied:

$$\operatorname{cl} Z_s(S \cap Q, x_0) = C(S) \cap T(Q, x_0).$$

(e) f is Dini differentiable at x_0 with convex derivative.
Then the conclusions are the same (i)-(iii) that in Theorem 4.1.

Proof. It is known ([12, Proposition 4.1]) that if f is Dini differentiable and (d) is verified, then

$$Z_s(S \cap Q, x_0) \cap C_0(F) = \emptyset.$$

Therefore, $\operatorname{cl} Z_s(S \cap Q, x_0) \cap \operatorname{int} C_0(F) = \emptyset$. As $Df(x_0, \cdot)$ is continuous, $C_0(F)$ is open, hence, $\operatorname{cl} Z_s(S \cap Q, x_0) \cap C_0(F) = \emptyset$, and taking into account (c) we deduce that $C(S) \cap T(Q, x_0) \cap C_0(F) = \emptyset$, this means that system (14) is incompatible. From here, it is continued just as in the proof of Theorem 4.1. □

Remark 4.6.

(1) Notice that if the extended Zangwill constraint qualification holds, also the extended Abadie constraint qualification holds, since $Z_s(S \cap Q, x_0) \subset T(S \cap Q, x_0)$.

(2) In Theorems 4.1 and 4.4 can be used any convex subcone $T_1(Q)$ of $T(Q, x_0)$ instead of the $T(Q, x_0)$ itself, in whose case the normal cone $N(Q, x_0)$ would be substituted by the polar cone $T_1(Q)^*$, condition (c) of Theorem 4.1 would be $C(S) \cap T_1(Q) \subset T(S \cap Q, x_0)$ and condition (c) of Theorem 4.5 would be $C(S) \cap T_1(Q) \subset \operatorname{cl} Z_s(S \cap Q, x_0)$. Of course, the best results are obtained choosing the largest convex subcone.

The following proposition provides us with a sufficient condition for the extended Zangwill constraint qualification to hold.

Proposition 4.7. *If Q is a convex set, g_j, $j \in J_0$ are Dini differentiable at x_0 with convex derivative, $Z_s(H, x_0) = \operatorname{Ker} Dh(x_0, \cdot)$ and $C_0(S) \cap \operatorname{ri}\operatorname{cone}(Q - x_0) \neq \emptyset$, then the extended Zangwill constraint qualification holds.*

Proof. For the set G, we have $C_0(G) \subset Z(G, x_0) \subset C(G)$. Therefore,

$$C_0(G) \cap \operatorname{Ker} Dh(x_0, \cdot) \subset Z(G, x_0) \cap Z_s(H, x_0) \subset Z_s(G \cap H, x_0) \subset C(G) \cap \operatorname{Ker} Dh(x_0, \cdot),$$

hence $C_0(S) \subset Z_s(S, x_0) \subset C(S)$, so

$$C_0(S) \cap \operatorname{cone}(Q - x_0) \subset Z_s(S, x_0) \cap Z(Q, x_0) \subset Z_s(S \cap Q, x_0) \subset C(S) \cap T(Q, x_0).$$

The conclusion follows by taking closure, since $\operatorname{cl}[C_0(S) \cap \operatorname{cone}(Q - x_0)] = C(S) \cap T(Q, x_0)$ by [4, Proposition 2.1.10, Chap. 3]. $\square$

In the following theorem, equality constraints are removed.

Theorem 4.8. *Let $x_0 \in G \cap Q$ and suppose the following:*

(a) $T_1(Q)$ *is a convex subcone of* $T(Q, x_0)$.
(b) $x_0 \in \operatorname{LWMin}(f, G \cap Q)$.

Then

(i) *If f and g_j, $j \in J_0$ are Hadamard differentiable at x_0 with convex derivative, then there exist $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^m$, $(\lambda, \mu) \neq 0$ such that*

$$\left.\begin{array}{l} (\lambda, \mu) \geq 0, \\ 0 \in \sum_{i=1}^p \lambda_i \partial_D f_i(x_0) + \sum_{j=1}^m \mu_j \partial_D g_j(x_0) + T_1(Q)^*, \\ \mu_j g_j(x_0) = 0, \ j = 1, \dots, m. \end{array}\right\} \tag{19}$$

(ii) *If f and g_j, $j \in J_0$ are Dini differentiable at x_0 with convex derivative and $T_1(Q) \subset Z_s(Q, x_0)$, then (19) holds with $(\lambda, \mu) \neq 0$.*
(iii) *If, moreover,*

$$C_0(G) \cap T_1(Q) \neq \emptyset, \tag{20}$$

then (19) holds with $\lambda \neq 0$.

Proof. (*i*) Since f is Hadamard differentiable and (*b*) holds, as it has been pointed out in the proof of Theorem 4.1,

$$T(G \cap Q, x_0) \cap C_0(F) = \emptyset. \tag{21}$$

Let us show that

$$C_0(G) \cap T(Q, x_0) \subset T(G \cap Q, x_0). \tag{22}$$

In fact, if $v \in C_0(G) \cap T(Q, x_0)$, then there exist sequences $x_n \in Q$, $t_n \to 0^+$ such that $\dfrac{x_n - x_0}{t_n} = v_n \to v$. As g_j, $j \in J_0$ are Hadamard differentiable, one has

$$dg_j(x_0, v) = \lim_{n\to\infty} \frac{g_j(x_0 + t_n v_n) - g_j(x_0)}{t_n} = \lim_{n\to\infty} \frac{g_j(x_n)}{t_n} < 0,\ \forall j \in J_0,$$

because $v \in C_0(G)$. Hence, for n large enough, $g_j(x_n) < 0$. For $j \in J \setminus J_0$, by the continuity of g_j at x_0, also it is $g_j(x_n) < 0$, for n large. Therefore, $x_n \in G \cap Q$ and consequently $v \in T(G \cap Q, x_0)$.
From (21), taking into account (22) and hypothesis (a), it follows that $C_0(G) \cap T_1(Q) \cap C_0(F) = \emptyset$, that is, there exists no $v \in \mathbb{R}^n$ such that

$$Df(x_0, v) < 0,\ Dg_j(x_0, v) < 0 \quad \forall j \in J_0,\ v \in T_1(Q).$$

By Corollary 3.7 one has the conclusion.
(ii) Since f is Dini differentiable and (c) holds, as has been pointed out in the proof of Theorem 4.5,

$$Z_s(G \cap Q, x_0) \cap C_0(F) = \emptyset. \tag{23}$$

On the other hand, since $C_0(G) \subset Z(G, x_0)$, we deduce that

$$C_0(G) \cap Z_s(Q, x_0) \subset Z(G, x_0) \cap Z_s(Q, x_0) \subset Z_s(G \cap Q, x_0). \tag{24}$$

From (23), taking into account (24) and hypothesis of (ii), it follows that $C_0(G) \cap T_1(Q) \cap C_0(F) = \emptyset$. From here we would continue as in part (i) above.
(iii) By reduction to the absurd, suppose that $\lambda = 0$. Then we have

$$0 \in \textstyle\sum_{j\in J_0} \mu_j \partial_D g_j(x_0) + T_1(Q)^*,\ \mu \geq 0,\ \mu \neq 0.$$

By Corollary 3.7, $C_0(G) \cap T_1(Q) = \emptyset$, which contradicts (20). □

Remark 4.9.

(1) If Q is a convex set, the sharpest results in Theorem 4.8 are obtained for $T_1(Q) = \operatorname{cone}(Q - x_0)$, with what $T_1(Q)^* = N(Q, x_0)$ and (20) is equivalent to
$$C_0(G) \cap (Q - x_0) \neq \emptyset.$$
(2) The condition obtained in (i) is Fritz John type. If we wish one of Kuhn-Tucker type, a constraint qualification can be used, as for example $C_0(G) \cap T_1(Q) \neq \emptyset$, but in this case, if $T_1(Q) \subset Z_s(Q, x_0)$, it is preferable to use (iii) which is less restrictive.

Part (i) generalizes Theorem 5 in Giorgi and Guerraggio [3] in which the functions f and g are differentiable. Parts (i) and (ii) generalize Theorem 3.2 in Kanniappan [8] and Theorem 3.1 in Islam [6] in which it is supposed that the functions f and g are convex on $\mathbb{R}^n$ and Q is convex. In fact, if a function is convex, it is Dini differentiable with convex derivative, and in addition, it is locally Lipschitz, with what it is also Hadamard differentiable, and (i) and (ii) can be applied by taking $T_1(Q) = \operatorname{cone}(Q - x_0)$ (the results of these authors are obtained since, for a convex real function φ, $\partial_D \varphi(x_0) = \partial \varphi(x_0)$).

Theorems 3.4 and 3.2 of the same authors can be deduced from part (iii). These theorems state: "*If f and g are convex, Q is convex, $x_0 \in \operatorname{Min}(f, G \cap Q)$ and the Slater constraint qualification (SCQ) holds, that is, for each $i = 1, \ldots, p$ there exists x_i such that $f_k(x_i) < f_k(x_0)$ $\forall k \neq i$, $g_j(x_i) < 0$ $\forall j \in J$, $x_i \in Q$, then* (19) *holds with $\lambda > 0$.*" In fact, let us observe in the first place that if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex and $\varphi(x) < \varphi(x_0)$ for some $x, x_0 \in \mathbb{R}^n$, then

$$D\varphi(x_0, x - x_0) < 0. \tag{25}$$

By convexity,

$$\varphi(x_0 + t(x - x_0)) \leq \varphi(x_0) + t(\varphi(x) - \varphi(x_0)) \ \forall t \in (0, 1).$$

Hence, $\dfrac{\varphi(x_0 + t(x - x_0)) - \varphi(x_0)}{t} \leq \varphi(x) - \varphi(x_0) < 0$. Taking the limit when $t \to 0^+$, (25) is obtained.

Therefore, applying (25), by (SCQ), for each $i = 1, \ldots, p$ there exists $v = x_i - x_0$ such that

$$Df_k(x_0, v) < 0 \ \forall k \neq i, \ Dg_j(x_0, v) < 0 \ \forall j \in J_0, \ v \in Q - x_0. \tag{26}$$

Theorem 4.6(iii) (with $T_1(Q) = \operatorname{cone}(Q - x_0)$) can be applied, since any Pareto minimum is a weak local Pareto minimum, resulting (19) with $\lambda \neq 0$. If $p = 1$, one has $\lambda > 0$. If $p \geq 2$, suppose that some $\lambda_i = 0$ (with some $\lambda_j \neq 0$, since $\lambda \neq 0$). By Corollary 3.7 (applied to the $p - 1$ functions $Df_k(x_0, \cdot)$, $k \neq i$ and to $Dg_j(x_0, \cdot)$, $j \in J_0$) there exists no vector v satisfying system (26), in contradiction to the existence of the solution $v = x - x_i$. Therefore, $\lambda > 0$.

Notice that in (SCQ) it is sufficient to require the condition $g_j(x_i) < 0$ for every $j \in J_0$ (instead of for all $j \in J$). Observe also that if the weak Slater constraint qualification holds: there exists $x \in Q$ such that $g_j(x) < 0 \ \forall j \in J_0$, then (19) holds with $\lambda \neq 0$. Finally, notice that at the same time it has been proved that if (SCQ) holds, every Pareto minimum is a solution for a scalarized problem, $\operatorname{Min}\{\langle \lambda, f(x) \rangle : \ x \in G \cap Q\}$, with $\lambda > 0$ ([9, Theorem 5]).

The next lemma provides us with a simple expression for the tangent cone to a set which is intersection of two sets: the first convex and the second is defined by quasiconvex or quasilinear functions. This expression will allow us to obtain an optimality criterion with this type of functions. Let us recall these concepts previously.

Let $\Gamma \subset \mathbb{R}^n$ be a convex set, $\varphi : \Gamma \to \mathbb{R}$, $x_0 \in \Gamma$. φ is quasiconvex at x_0 on Γ if

$$\forall x \in \Gamma, \ \varphi(x) \leq \varphi(x_0) \ \Rightarrow \ \varphi(\lambda x + (1 - \lambda) x_0) \leq \varphi(x_0) \ \forall \lambda \in (0, 1).$$

φ is quasilinear at x_0 if φ and $-\varphi$ are quasiconvex at x_0.

Lemma 4.10. *Suppose that $Q \subset \mathbb{R}^n$ is a convex set and g_j, $j \in J_0$ are quasiconvex at x_0 on a neighborhood of x_0 and h is quasilinear at x_0 on a neighborhood of x_0. Then*

$$\operatorname{cl} Z(S \cap Q, x_0) = T(S \cap Q, x_0).$$

Proof. Let $B(x_0, \delta)$ be a neighborhood of x_0 on which g_j, $j \in J_0$ are quasiconvex at x_0 and h is quasilinear. Hence, $\forall x \in B(x_0, \delta)$ if $g_j(x) \leq g_j(x_0)$, then

$$g_j(\lambda x + (1 - \lambda) x_0) \leq g_j(x_0) \ \forall \lambda \in [0, 1].$$

Therefore, if $x \in G \cap B(x_0, \delta)$ we derive $g_j(\lambda x + (1 - \lambda) x_0) \leq 0 \ \forall \lambda \in [0, 1]$, $\forall j \in J_0$. If $j \in J \setminus J_0$, by the continuity of g_j, there exists a neighborhood $B(x_0, \delta_1)$ such that $g_j(x) < 0 \ \forall x \in B(x_0, \delta_1)$, $\forall j \in J \setminus J_0$. Taking $\delta_0 = \operatorname{Min}\{\delta, \delta_1\}$, both conditions are verified and, therefore $[x_0, x] \subset G \ \forall x \in G \cap B_0$, being $B_0 = B(x_0, \delta_0)$. Similarly, since

h and $-h$ are quasiconvex at x_0, $[x_0, x] \subset H$. Hence $[x_0, x] \subset S \; \forall x \in S \cap B_0$. As Q is convex, one has $[x_0, x] \subset S \cap Q \; \forall x \in S \cap Q \cap B_0$. Hence, $v = x - x_0 \in Z(S \cap Q, x_0)$. It follows

$$\operatorname{cone}(S \cap Q \cap B_0 - x_0) \subset Z(S \cap Q, x_0) \subset T(S \cap Q, x_0).$$

Now, $T(S \cap Q, x_0) = T(S \cap Q \cap B_0, x_0) \subset \operatorname{cl}\operatorname{cone}(S \cap Q \cap B_0 - x_0) \subset \operatorname{cl} Z(S \cap Q, x_0)$. In conclusion, by closedness of the tangent cone,

$$T(S \cap Q, x_0) = \operatorname{cl} Z(S \cap Q, x_0) = \operatorname{cl}\operatorname{cone}(S \cap Q \cap B_0 - x_0).$$

□

Theorem 4.11. *In the hypotheses of Lemma 4.10. Suppose that conditions (b), (c) and (d) of Theorem 4.1 are verified and that f is Dini differentiable at x_0 with convex derivative. Then the conclusions are the same (i)-(iii) that in Theorem 4.1.*

Proof. By Lemma 4.10, condition (c) becomes the extended Zangwill constraint qualification, i.e., condition (c) of Theorem 4.5, and it is enough to apply this theorem. □

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