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# Separability of $H$-Convex Sets

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We consider the problem of $H$-separability for two $H$-convex sets $A, B \subset \mathbb{R}^n$. There are two types of $H$-separability. The first one, called "strict $H$-separability", is the separation (in the usual sense) of the sets $A$ and $B$ by an $H$-convex hyperplane. The second one ("weak $H$-separability") means to look for an $H$-convex half-space $P$ such that $A$ is situated in $P$, whereas $B$ has no point in common with the interior of $P$. We give necessary and sufficient conditions for both these types of $H$-separability; the results are connected with $H$-convexity of the Minkowski sum of $H$-convex sets, see [7]. Some examples illustrate the obtained results.

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## 1. Introduction

The separation of convex sets and, in particular, of convex cones is an important tool and field in classical and applied convexity. In almost every book on convexity one can find extra paragraphs on this topic, cf. [10], §1.7, [17], Part II, [12], §3, [11], Section 4, [18], §2.4, [8], §8, and [15], §1.3. Important fields of application are Mathematical Programming and Control Theory (see [16], §3.3 and [4]) as well as Convex Analysis (cf. [14], §11, and [13], §9). It is our aim to extend known separation theorems to a natural generalization of the usual convexity notion.

Let $H$ be a *non-onesided subset* of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, i.e., $H$ is not contained in a closed half-sphere of $\mathbb{S}^{n-1}$. A closed half-space $P \subset \mathbb{R}^n$ is said to be $H$-*convex* if its unit outward normal belongs to $H$. Furthermore, a subset $Q$ of $\mathbb{R}^n$ is said to be an $H$-*convex set* if it is representable as the intersection of a (finite or infinite) family of $H$-convex half-spaces. The class of $H$-convex sets was introduced in [1]. For example, there are many applications of $H$-convexity to problems of Combinatorial Geometry; see [2], [3], [8], Chapter III, [5], [9], and the recent publications [6] and [7]. Results on $H$-convex sets usually extend classical observations on sets which are convex in the common sense, since the latter ones are described by the special case $H = \mathbb{S}^{n-1}$. Here we present some new results on $H$-convexity in view of *separation theory.*

First we introduce the definitions of $H$-separability and give some illustrating examples.

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As usual, we use the abbreviations int and bd for *interior* and *boundary*, respectively.

**Definition 1.1.** Let $Q_1, Q_2 \subset \mathbb{R}^n$ be two nonempty sets without common interior points, i.e., $(\operatorname{int} Q_1) \cap (\operatorname{int} Q_2) = \emptyset$. Let, furthermore, $H \subset \mathbb{S}^{n-1}$ be a non-onesided vector system. We say that the sets $Q_1$, $Q_2$ are *strictly $H$-separable* if there exists an $H$-convex hyperplane $\Gamma$ such that $Q_1 \subset P_1$ and $Q_2 \subset P_2$, where $P_1, P_2$ are the two closed half-spaces defined by the hyperplane $\Gamma$.

**Definition 1.2.** Let $Q_1, Q_2 \subset \mathbb{R}^n$ be two nonempty sets satisfying again $(\operatorname{int} Q_1) \cap (\operatorname{int} Q_2) = \emptyset$. Let, furthermore, $H \subset \mathbb{S}^{n-1}$ be a non-onesided vector system. We say that the set $Q_1$ *is weakly $H$-separable from* $Q_2$ if there exists an $H$-convex half-space $P$ such that $Q_1 \subset P$ and $Q_2 \cap (\operatorname{int} P) = \emptyset$.

It is clear that if $Q_1$ and $Q_2$ are strictly $H$-separable, then $Q_1$ is weakly $H$-separable from $Q_2$ and, at the same time, $Q_2$ is weakly $H$-separable from $Q_1$. Nevertheless, if $Q_1$ is weakly $H$-separable from $Q_2$ and $Q_2$ is weakly $H$-separable from $Q_1$, we cannot conclude that $Q_1$ and $Q_2$ are strictly $H$-separable (see Example 1.6 below).

**Example 1.3.** Let $Q_1, Q_2$ be two closed balls in $\mathbb{R}^n$ which are tangent to each other at their common boundary point $a$, i.e., $Q_1 \cap Q_2 = \{a\}$. Denote by $\Gamma$ their tangential hyperplane at the point $a$ and by $p_1, p_2$ the unit normal vectors of $\Gamma$ with $p_1 = -p_2$. Denote by $H$ the set $\mathbb{S}^{n-1} \setminus \{p_1, p_2\}$. Then both the sets $Q_1, Q_2$ are $H$-convex and they are separable in usual sense, but they are not $H$-separable (neither in the strict sense nor in the weak sense). This occurs since the set $H$ is not closed. $\square$

**Example 1.4.** Under the notation of the previous example we put $H = \mathbb{S}^{n-1} \setminus \{p_1\}$, assuming that the ball $Q_1$ is contained in the closed half-space $P = \{x : \langle p_1, x - a\rangle \leq 0\}$. The $H$-convex sets $Q_1, Q_2$ are not strictly $H$-separable. At the same time, $Q_1$ is weakly $H$-separable from $Q_2$ (since $Q_1 \subset P$ and $Q_2 \cap (\operatorname{int} P) = \emptyset$), but $Q_2$ is not weakly $H$-separable from $Q_1$. $\square$

**Example 1.5.** In Example 1.4 the set $H$ is not closed. We give an example with analogous properties for a closed set $H$. Let $Q_1 \subset \mathbb{R}^n$ be an $n$-dimensional simplex and let $H$ be the set of unit outward normals of its facets. The set $H$ consists of $n+1$ vectors and is closed. Moreover, $Q_1$ is $H$-convex. We choose a relatively interior point $a$ of a facet $F$ of the simplex $Q_1$ and denote by $b$ the vertex opposite to the facet $F$ in $Q_1$, and by $p$ the unit outward normal of the facet $F \subset Q_1$. The half-space $P = \{x : \langle p, x - a\rangle \leq 0\}$ is $H$-convex. Furthermore, denote by $t$ the translation by the vector $a - b$ and put $Q_2 = t(Q_1)$. The sets $Q_1$ and $Q_2$ have no common interior points in $\mathbb{R}^n$ and are separable in the usual sense. Furthermore, the $H$-convex sets $Q_1, Q_2$ are not strictly $H$-separable. At the same time, the set $Q_1$ is weakly $H$-separable from $Q_2$ (since $Q_1 \subset P$ and $Q_2 \cap \operatorname{int} P = \emptyset$), but $Q_2$ is not weakly $H$-separable from $Q_1$. $\square$

**Example 1.6.** Let $Q_1 \subset \mathbb{R}^n$ be an $n$-dimensional simplex and let $H$ be the set of unit outward normals of its facets. The set $H$ consists of $n+1$ vectors and is closed. Let $a_1, a_2$ be two vertices of $Q_1$. Denote by $t$ the translation by the vector $a_2 - a_1$ and put $Q_2 = t(Q_1)$. The sets $Q_1$ and $Q_2$ have no common interior points in $\mathbb{R}^n$ and are separable in the usual sense. Furthermore, the $H$-convex sets $Q_1, Q_2$ are not strictly $H$-separable (since there is no $H$-convex hyperplane). But every of the sets $Q_1, Q_2$ is weakly $H$-separable from the other one. Indeed, denote by $S_1$ the facet of the simplex $Q_1$ which is opposite to its vertex $a_1$, and by $S_2$ the facet of the simplex $Q_2$ which is opposite to its vertex $t(a_2)$. Let $p_i$ be the unit outward normal of the facet $S_i$ of the simplex $Q_i$, and $P_i$ be the half-space $\{x : \langle p_i, x - a_2\rangle \le 0,\ i = 1, 2\}$. Then the $H$-convex half-space $P_i$ contains the simplex $Q_i$, and $\operatorname{int} P_i$ has no common points with the other simplex. $\square$

The above examples show that the situations "$Q_1$ and $Q_2$ are strictly $H$-separable", "$Q_1$ is weakly $H$-separable from $Q_2$", and "$Q_2$ is weakly $H$-separable from $Q_1$" are in general pairwise different for two sets $Q_1, Q_2$ with $(\operatorname{int} Q_1) \cap (\operatorname{int} Q_2) = \emptyset$. Moreover, the situation "every of the two sets $Q_1, Q_2$ is weakly $H$-separable from the other one" does not mean that $Q_1$ and $Q_2$ are strictly $H$-separable.

## 2. Sufficient conditions for $H$-separability

In [7] we describe all non-onesided vector systems $H \subset \mathbb{S}^{n-1}$ such that the Minkowski sum of any two $H$-convex sets in $\mathbb{R}^n$ also is $H$-convex. We recall the solution of that problem.

**Definition 2.1.** Let $H \subset \mathbb{S}^{n-1}$ be a non-onesided vector system. We say that the system $H$ is *$M$-complete* if for every $m$ vectors $e_1, ..., e_m \in H$, each $m - 1$ of which are linearly independent, the following condition holds: if a unit vector $p \in \mathbb{R}^n$ is representable as a positive linear combination of the vectors $e_1, ..., e_l$, where $1 < l < m - 1$, and, at the same time, $p$ is representable as a positive linear combination of the other vectors $e_{l+1}, ..., e_m$, then $p \in H$.

In [7] we prove the following result:

**Theorem 2.2.** *Assume that a non-onesided vector system $H \subset \mathbb{S}^{n-1}$ is $M$-complete. Then the Minkowski sum of every two $H$-convex sets is itself an $H$-convex set.*

Using this result, we now obtain some sufficient conditions for $H$-separability.

**Theorem 2.3.** *Let $Q_1, Q_2 \subset \mathbb{R}^n$ be two nonempty sets without common interior points, i.e., $(\operatorname{int} Q_1) \cap (\operatorname{int} Q_2) = \emptyset$. Let, furthermore, $H \subset \mathbb{S}^{n-1}$ be a non-onesided vector system that is $M$-complete. If $Q_1$ is $H$-convex and $Q_2$ is $(-H)$-convex, then $Q_1$ is weakly $H$-separable from $Q_2$.*

**Proof.** Consider the set $K = Q_1 - Q_2$. By the condition of Theorem 2.3, the origin 0 is not an interior point of the set $K$ (otherwise $(\operatorname{int} Q_1) \cap (\operatorname{int} Q_2)$ is nonempty). Moreover, the set $(-Q_2)$ is $H$-convex, and consequently the set $K = Q_1 - Q_2$ is $H$-convex, too (since $H$ is $M$-complete). Since $0 \notin \operatorname{int} K$, there is an $H$-convex half-space $P$ with boundary hyperplane through 0 such that $P \supset K$. Denote by $p$ the unit outward normal of the half-space $P$; then $p \in H$.

Let $N \subset \mathbb{R}^n$ be a convex set and $\Pi \subset \mathbb{R}^n$ be a half-space with unit outward normal $p$. We say that $\Pi$ is a *quasi-support half-space* of $N$ if $\Pi \supset N$ and, moreover, no half-space $-\lambda p + \Pi$ contains $N$ if $\lambda > 0$ (we note that if $N$ is compact, then $\Pi$ is a usual support half-space of the set $N$).

Since $P \supset K = Q_1 - Q_2$, for any point $x \in Q_2$ we have $P \supset Q_1 - x$, i.e., the half-space $x + P$ with the outward normal $p$ contains $Q_1$. Consequently there exists a half-space $\Pi_1$ with the outward normal $p$ that is a quasi-support half-space of $Q_1$. Analogously we conclude that there exists a half-space $\Pi_2$ with the outward normal $p$ that is a quasi-support half-space of $-Q_2$. We have $\Pi_i = -\lambda_i p + P$, $i = 1, 2$. Moreover, $\lambda_1$ (respectively, $\lambda_2$) is the maximal real number for which $-\lambda_1 p + P \supset Q_1$ (respectively, $-\lambda_2 p + P \supset -Q_2$). The half-space $\Pi_1 + \Pi_2$ is the quasi-support half-space of $Q_1 - Q_2$ with the outward normal $p$, and consequently $\Pi_1 + \Pi_2 \subset P$. This implies $\lambda_1 + \lambda_2 \geq 0$. Hence $\lambda_2 \geq -\lambda_1$, and therefore the half-space $\lambda_1 p + P$ contains the set $-Q_2$, i.e., the half-space $-\lambda_1 p - P$ contains the set $Q_2$.

Finally, denote by $\Gamma$ the boundary hyperplane of the half-space $P$; the hyperplane $\Gamma$ contains the origin. By construction, the half-space $\Pi_1$ contains the set $Q_1$. We note that the boundary of $\Pi_1$ has the representation $(\mathrm{bd}\Pi_1) = -\lambda_1 p + \Gamma$. Moreover, the half-space $-\lambda_1 p - P$ with outward normal $-p$ contains the set $Q_2$ and has the same boundary $-\lambda_1 p + \Gamma$ (we remark that $0 \in \Gamma$ and therefore $-\Gamma = \Gamma$). Consequently, the open half-space $(\mathrm{int}\, \Pi_1) = \mathbb{R}^n \setminus (-\lambda_1 p - P)$ has no common points with $Q_2$. Thus $\Pi_1 \supset Q_1$ and $(\mathrm{int}\, \Pi_1) \cap Q_2 = \emptyset$. Since the unit outward normal $p$ of $\Pi_1$ belongs to $H$ we conclude that the sets $Q_1$ and $Q_2$ are weakly $H$-separable. $\square$

**Theorem 2.4.** *Let $Q_1, Q_2 \subset \mathbb{R}^n$ be two nonempty sets without common interior points, i.e., $(\mathrm{int}\, Q_1) \cap (\mathrm{int}\, Q_2) = \emptyset$. Let, furthermore, $H \subset \mathbb{S}^{n-1}$ be a non-onesided vector system that is $M$-complete and symmetric with respect to the origin. If $Q_1$ and $Q_2$ are $H$-convex, then they are strictly $H$-separable.*

**Proof.** Since the set $H$ is symmetric with respect to the origin, we conclude that the $H$-convex set $Q_2$ is also $(-H)$-convex. Moreover, since the vector system $H \subset \mathbb{S}^{n-1}$ is $M$-complete, we conclude by Theorem 2.3 above that the $H$-convex set $Q_1$ is weakly $H$-separable from the $(-H)$-convex set $Q_2$. In other words, there exists an $H$-convex half-space $\Pi$ such that $Q_1 \subset \Pi$ and $Q_2$ has no common points with the open half-space $\mathbb{R}^n \setminus \Pi$. This means that $Q_2$ is contained in the other closed half-space $\Pi'$ with the same boundary $\Gamma = (\mathrm{bd}\, \Pi) = (\mathrm{bd}\, \Pi')$. Denote by $p \in H$ the unit outward normal of the half-space $\Pi$. Since the set $H$ is symmetric with respect to the origin and $p \in H$, we conclude that $-p \in H$, i.e., *both* closed half-spaces $\Pi$ and $\Pi'$ are $H$-convex, and hence the hyperplane $\Gamma = \Pi \cap \Pi'$ is $H$-convex. Since the hyperplane $\Gamma$ separates the sets $Q_1$ and $Q_2$, we conclude that $Q_1$ and $Q_2$ are strictly $H$-separable. $\square$

## 3. Examples

Simple examples show that the sufficient conditions for $H$-separability given in Theorems 2.3 and 2.4 are not necessary.

**Example 3.1.** In Theorem 2.3 two sets $Q_1$ and $Q_2$ with $(\mathrm{int}\, Q_1) \cap (\mathrm{int}\, Q_2) = \emptyset$ satisfy the following conditions:

(A) the set $H \subset \mathbb{S}^{n-1}$ is $M$-complete;
(B) the set $Q_1$ is $H$-convex;
(C) the set $Q_2$ is $(-H)$-convex.

Theorem 2.3 affirms that the disjunction of the conditions (A), (B), (C) is sufficient for weak $H$-separability of $Q_1$ from $Q_2$. To show that this condition is not necessary, we describe situations when $Q_1$ is weakly $H$-separable from $Q_2$, but at least one of the conditions (A), (B), (C) fails. Indeed, in Example 1.5 above the set $Q_1$ is weakly $H$-separable from $Q_2$ and the conditions (A), (B) hold; nevertheless condition (C) fails. $\square$

**Example 3.2.** Let $T \subset \mathbb{R}^n$ be an $n$-dimensional simplex and $\{e_0, e_1, ..., e_n\}$ be the system of all unit outward normals of its facets. We add two vectors $-e_0$ and $-e_1$ and denote by $H \subset \mathbb{S}^{n-1}$ the obtained system that consists of $n+3$ vectors. The system $H$ is not $M$-complete. Indeed, denote by $c$ the centroid of the simplex $T$ and by $P_1$ the half-space $\{x : \langle -e_0, x - c\rangle \le 0\}$. The set $Q_1 = T \cap P$ is $H$-convex. Let $P_2$ be the support half-space of $T$ with the outward normal $-e_1$, and $a_1$ be the corresponding support vertex, i.e., $(\operatorname{bd} P_2) \cap T = \{a_1\}$. Denote by $Q_2$ the set $(c + a_1) - T$. Then $c \in Q_1 \cap Q_2$ and $(\operatorname{int} Q_1) \cap (\operatorname{int} Q_2) = \emptyset$, i.e., the hyperplane $\operatorname{bd} P_1$ separates the bodies $Q_1$ and $Q_2$. The set $Q_2$ is $(-H)$-convex. Since the $H$-convex half-space $P$ contains $Q_1$ and, moreover, $Q_2 \cap (\operatorname{int} P) = \emptyset$, we conclude that $Q_1$ is weakly $H$-separable from $Q_2$. The conditions (B) and (C) indicated in the previous example hold, but (A) fails. $\square$

**Example 3.3.** In Theorem 2.4 two sets $Q_1$ and $Q_2$ with $(\operatorname{int} Q_1) \cap (\operatorname{int} Q_2) = \emptyset$ satisfy the following conditions:

- (A′) the set $H \subset \mathbb{S}^{n-1}$ is $M$-complete;
- (B′) the set $H$ is symmetric with respect to the origin;
- (C′) the sets $Q_1$ and $Q_2$ are $H$-convex.

Theorem 2.4 affirms that the disjunction of the conditions (A′), (B′), (C′) is sufficient for strict $H$-separability of $Q_1$ and $Q_2$. To show that this condition is not necessary, we describe situations when $Q_1$ is weakly $H$-separable from $Q_2$, but at least one of the conditions (A′), (B′), (C′) fails. Indeed, in the notation of the previous example the bodies $Q_1$ and $Q_2'$ are separable by the hyperplane $\Gamma = (\operatorname{bd} P)$. Since $e_0, -e_0 \in H$, the hyperplane $\Gamma$ is $H$-convex and hence the bodies $Q_1$ and $Q_2'$ are $H$-separable. At the same time condition (C′) holds, but both the conditions (A′) and (B′) fail. $\square$

**Example 3.4.** Now we give an example which shows that without the assumption of $M$-completeness of the set $H \subset \mathbb{S}^{n-1}$ the sufficient conditions given in Theorems 2.3 and 2.4 fail. Consider in $\mathbb{R}^3$ the vector system

$$H = \left\{ \frac{1}{\sqrt{2}}(\mathrm{e}_1 \pm \mathrm{e}_3), \frac{1}{\sqrt{2}}(\mathrm{e}_2 \pm \mathrm{e}_3) \right\}$$

(all combinations of signs), where $\{e_1, e_2, e_3\}$ is an orthonormal basis. Taking the points $a(0,1,0)$, $b(0,-1,0)$, $c(1,0,-1)$, $d(-1,0,-1)$, it is easily shown that the tetrahedra

$$Q_1 = \operatorname{conv}\{a, b, c, d\}, \quad Q_2 = e_3 + Q_1$$

are $H$-convex. Denoting by $P$ the plane $\{x : \langle e_3, x\rangle = 0\}$, we note that $P \cap Q_1$ is a segment parallel to $e_2$ and $P \cap Q_2$ is a segment parallel to $e_1$, both these segments having their midpoints at the origin. Thus the only plane that separates (in the usual sense) $Q_1$ and $Q_2$ is $P$. But this plane *is not $H$-convex*, and hence the conclusion of Theorem 2.3 fails. This occurs since the conditions (B) and (C) indicated in Example 3.1 hold, but condition (A) fails, i.e., the system $H$ *is not $M$-complete*. Moreover, the conclusion of Theorem 2.4 also fails; this occurs since the conditions (B′) and (C′) indicated in Example 3.3 hold, but condition (A′) fails, i.e., the system $H$ *is not $M$-complete.* Indeed, the polytope $N = Q_1 + (-Q_2)$ is not $H$-convex (although both $Q_1$ and $-Q_2$ are $H$-convex), since $N$ contains the facet $F = \operatorname{conv}\{a, b, e_3 + c, e_3 + d\}$ which is a parallelogram with the outward normal $e_3$ not belonging to $H$. □

## 4. Necessary conditions for $H$-separability

In the above Theorems 2.3 and 2.4 some *sufficient* conditions for $H$-separability are given. Moreover, the above Examples show that these sufficient conditions are not necessary. Nevertheless, as we show below, if weak $H$-separability (respectively, strict $H$-separability) holds for *any* sets $Q_1, Q_2 \subset \mathbb{R}^n$ under suitable conditions, then the sufficient conditions given in Theorems 2.3 and 2.4 are also *necessary.*

**Theorem 4.1.** *Let $H \subset \mathbb{S}^{n-1}$ be a non-onesided vector system. Suppose that for any two nonempty sets $Q_1, Q_2 \subset \mathbb{R}^n$ without common interior points such that $Q_1$ is $H$-convex and $Q_2$ is $(-H)$-convex, the set $Q_1$ is weakly $H$-separable from $Q_2$. Then the vector system $H$ is $M$-complete.*

**Proof.** Assume that $H$ is not $M$-complete, i.e., there are $m$ vectors $e_1, ..., e_m$ in $H$, each $m-1$ of them being linearly independent, such that a unit vector $p \notin H$ is representable as a positive linear combination of the vectors $e_1, ..., e_l$, $1 < l < m-1$, and, at the same time, $p$ is representable as a positive linear combination of the other vectors $e_{l+1}, ..., e_m$. Denote by $P_1, ..., P_m \subset \mathbb{R}^n$ the half-spaces with respective unit outward normals $e_1, ..., e_m$ and boundary hyperplanes through the origin. Denote by $L_1, ..., L_m$ their boundary hyperplanes. The intersection $Q_1 = P_1 \cap ... \cap P_l$ is a convex cone with apex face $K = L_1 \cap ... \cap L_l$, and we have $\dim K = n - l$ (since the vectors $p_1, ..., p_l$ are linearly independent). Analogously, the intersection $Q' = P_{l+1} \cap ... \cap P_m$ is a convex cone with apex face $K' = L_{l+1} \cap ... \cap L_m$, and we have $\dim K' = n - (m - l)$. We remark that the intersection $K \cap K'$ has dimension $n - (m - 1)$, and hence the vector sum $K + K'$ has dimension $n - 1$, i.e., $K + K'$ is the hyperplane through the origin with normal vector $p$. Since both the cones $Q_1, Q'$ are $H$-convex, we have that the cone $Q_2 = -Q'$ is $(-H)$-convex. Moreover, $Q_1$ and $Q_2$ have no common interior point. By assumption, $Q_1$ is weakly $H$-separable from $Q_2$, i.e., there is a closed $H$-convex half-space $P$ such that $P \supset Q_1$ and $(\operatorname{int} P) \cap Q_2 = \emptyset$. But this is contradictory, since the only half-space $P$ satisfying $P \supset Q_1$ and $(\operatorname{int} P) \cap Q_2 = \emptyset$ is the half-space with boundary hyperplane through the origin and outward normal $p$, but this half-space is not $H$-convex, since $p \notin H$. Thus the assumption that $H$ is not $M$-complete yields a contradiction. □

We remark that in the above proof the bodies $Q_1$, $Q_2$ are *unbounded.* But it is also possible to change the construction in such a manner that $Q_1$, $Q_2$ will be compact. Indeed, since $H$ is non-onesided, we may choose in $H$ some vectors $g_1, ..., g_k$ such that the vector system $e_1, ..., e_m, g_1, ..., g_k$ is non-onesided. Hence, intersecting the sets $Q_1$ and $Q'$ (considered in the previous proof) with some hyperplanes having outward normals $g_1, ..., g_k$, we can obtain from $Q_1$ and $Q'$ *compact* convex bodies with the same properties. Thus in the statement of Theorem 4.1 it is possible to require that weak separability holds only for *compact* sets $Q_1$ and $Q_2$.

**Theorem 4.2.** *Let $H \subset \mathbb{S}^{n-1}$ be a non-onesided vector system symmetric with respect to the origin. Suppose that any two nonempty $H$-convex sets $Q_1, Q_2 \subset \mathbb{R}^n$ without common* *interior points are strictly $H$-separable. Then the vector system $H$ is $M$-complete.*

**Proof.** Assume that $H$ is not $M$-complete and consider the same bodies $Q_1$ and $Q'$ as in the previous proof. Then the only hyperplane that separates $Q_1$ and $Q'$ is $(\mathrm{bd}P)$. But this hyperplane is not $H$-convex, contradicting the assumption. $\square$

As above, in the statement of Theorem 4.2 it is possible to require that strict separability holds only for *compact* sets $Q_1$ and $Q_2$.

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