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Tangential Regularity in the Space of Directional-Morphological Transitions*

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Dedicated to Jean-Pierre Aubin on the occasion of his 65th birthday.

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In this paper the space of directional-morphological transitions is analyzed and a tangential regularity result is established for the subset $\mathcal{V}$. This subset arises in a natural way when one considers viability problems where both, trajectories and constraints, are evolving in time governed by differential and morphological equations, respectively. Furthermore a control problem associated with this kind of evolutionary-morphological systems is studied.

Keywords: Tangential regularity, morphological equations, differential inclusions, constrained control systems

1. Introduction

Mutational Analysis is an extension of the usual differential calculus to metric spaces, which works in spaces without linear (vectorial) structure. It was introduced by J.-P. Aubin in the nineties (see [2] and the references therein), mainly with the aim to describe the evolution of moving sets by means of a suitable notion of "velocity" (properly, "set of velocities") of a time-evolving set. Motivations to develop this research program come from dynamical models in economy, image processing, shape optimization, visual control or propagation of fronts, among others (see, for instance, [2], [9], [10], [13] and also [7], where a similar theory, without locally compactness assumptions, is developed). The basic idea is to endow the metric space with a family of generalized directions (that are called *transitions* in this framework) which allows to define the *mutation* of a map. This concept is the key to introduce a kind of differential equations, called *mutational equations*, to describe evolution phenomena in metric spaces. When we consider additional constraints on such evolutions (mutational viability), the notion of tangent set arises in a natural way in this framework.

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This paper is devoted to prove a regularity result in the mutational space $(X\times\mathcal{K}(X),\ X\times \mathrm{LIP}(X,X))$, where X is a finite dimensional vector space, $\mathcal{K}(X)$ is the space of its nonempty compact subsets and $\mathrm{LIP}(X,X)$ denotes the family of all the bounded Lipschitz set-valued maps with convex compact values. Indeed, we will show that

$$\forall\ (x,K)\in\mathcal{V},\qquad \operatorname*{Liminf}_{\mathcal{V}\ni(z,M)\to(x,K)} T_{\mathcal{V}}(z,M)=C_{\mathcal{V}}(x,K),$$

where $\mathcal{V}:=\{(x,K)\in X\times\mathcal{K}(X): x\in K\}$, $T_{\mathcal{V}}(x,K)$ denotes the contingent (or Bouligand-Severi) transition set and $C_{\mathcal{V}}(x,K)$ is the circatangent (or Clarke) transition set. Furthermore, this result provides a sufficient condition of regularity (or sleekness) for $\mathcal{V}$, which is the key to obtain the existence of viable solution to the control system

$$\left.\begin{array}{l} x'(t)=f(t,x(t),K(t),u(t))\\ \overset{\circ}{K}(t)\ni\Phi(t,x(t),K(t),u(t))\\ u(t)\in U(x(t),K(t)) \end{array}\right\}$$

under the state (viability) constraint $x(t)\in K(t)$.

The paper is organized as follows. In Section 2 we recall and discuss basical topics on Mutational Analysis and introduce two technical lemmas that we use in the next section to give a proof of Theorem 3.1, linking the lower limit of tangent transition sets and the Clarke tansition set in general mutational spaces. Section 4 is devoted to state several results describing tangent sets to $\mathcal{V}$ and also to prove the announced tangential regularity result, Theorem 4.6. Finally, in the last section we investigate the existence of solutions to a control problem with viability constraints on the state, whose dynamics is described by a joint evolutionary-morphological system.

2. Mutational Analysis

In this section we will recall some basic topics on Mutational Analysis. We refer to [2] for details.

2.1. Transitions on metric spaces

Let (E,d) be a metric space. A continuous map $\vartheta:[0,1]\times E\longrightarrow E$ is said to be a *transition* if it satisfies

(H-1) $\vartheta(0,x)=x,\ \forall\ x\in E$

(H-2) $\forall\ t\in[0,1[,\ \forall\ x\in E,\ \lim_{h\to0^+}\dfrac{d\left(\vartheta(t+h,x),\ \vartheta(h,\vartheta(t,x))\right)}{h}=0$

(H-3) $\alpha(\vartheta):=\max\left(\sup_{x\neq y}\left(\limsup_{h\to0^+}\dfrac{d\left(\vartheta(h,x),\ \vartheta(h,y)\right)-d(x,y)}{h\,d(x,y)}\right),\ 0\right)<\infty$

(H-4) $\beta(\vartheta):=\sup_{x\in E}\left(\limsup_{h\to0^+}\dfrac{d\left(\vartheta(h,x),\ x\right)}{h}\right)<\infty.$

It is clear that $\mathbf{1}(h,x):=x$ is a transition on E. It is called the *neutral transition.* Given two transitions ϑ,υ on the metric space E, (H-4) ensures that

$$d_\Lambda(\vartheta,\ \upsilon):=\sup_{x\in E}\left(\limsup_{h\to0^+}\frac{d\left(\vartheta(h,x),\ \upsilon(h,x)\right)}{h}\right)<\infty. \tag{1}$$

Thus, by identifying transitions with $d_\Lambda(\vartheta, \upsilon) = 0$, we have that d_Λ defines a distance in the family of all the transitions on E. Finally, we say that $(E, \Theta(E))$ is a (complete) mutational space if E is a (complete) metric space and $\Theta(E)$ is a subset of transitions which contains the neutral one and is closed in $\mathcal{C}([0,1] \times E; E)$ with respect to the distance d_Λ.

It must be noted that pointwise convergence in (H-2) is actually uniform on compact subintervals of $[0, 1[$, as the next lemma shows.

Lemma 2.1. *Let $\vartheta \in \Theta(E)$, $K \subset E$ be a compact and $0 < T < 1$. For any $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$d(\vartheta(t+h, x), \vartheta(h, \vartheta(t, x))) < h\varepsilon, \quad 0 \le h \le \eta \tag{2}$$

for all $t \in [0, T]$, $x \in K$.

Proof. Given $\varepsilon > 0$, by (H-2), for any $t \in [0, T]$ and any $x \in K$, we have $\eta_{t,x} > 0$ such that

$$d(\vartheta(t+h, x), \vartheta(h, \vartheta(t, x))) < h\varepsilon, \quad 0 \le h \le \eta_{t,x}. \tag{3}$$

Furthermore, since ϑ is continuous, one can find $\delta_{t,x,h} > 0$ such that

$$d(\vartheta(s+h', z), \vartheta(h', \vartheta(s, z))) < h'\varepsilon \tag{4}$$

whenever $s \in]t - \delta_{t,x,h}, t + \delta_{t,x,h}[\cap [0, T]$, $z \in B_{\delta_{t,x,h}}(x) \cap K$ and $h' \in J_{t,x,h}$, where $J_{t,x,h} :=]h - \delta_{t,x,h}, h + \delta_{t,x,h}[\cap [0, \eta_{t,x}]$. Clearly, the family of open subsets $\{J_{t,x,h} : 0 \le h \le \eta_{t,x}\}$, is a covering of the compact $[0, \eta_{t,x}]$. Thus there are a finitely many points $h_1, \ldots, h_r$ such that $[0, \eta_{t,x}] \subset J_{t,x,h_1} \cup \cdots \cup J_{t,x,h_r}$ and taking $\delta_{t,x} := \min_{1 \le j \le r} \delta_{t,x,h_j}$ we can write

$$d(\vartheta(s+h, z), \vartheta(h, \vartheta(s, z))) < h\varepsilon, \quad 0 \le h \le \eta_{t,x} \tag{5}$$

for any $s \in I_{t,x} :=]t - \delta_{t,x}, t + \delta_{t,x}[\cap [0, T]$ and $z \in \Omega_{t,x} := B_{\delta_{t,x}}(x) \cap K$. We will use again the classical compactness argument. In fact the family of open sets $\{I_{t,x} \times \Omega_{t,x} : (t, x) \in [0, T] \times K\}$ is a covering of the compact $[0, T] \times K$. Thus we can obtain a finite subcovering

$$[0, T] \times K \subset \bigcup_{1 \le j \le n} I_{t_j, x_j} \times \Omega_{t_j, x_j}$$

and setting $\eta := \min_{1 \le j \le n} \eta_{t_j, x_j}$ the lemma is proved. □

Moreover, (H-3) and (H-4) provide estimates for transitions, as next lemma shows. Its proof easily follows from Lemma 1.1.3 in [2] and $\beta(\vartheta)$-Lipschitzianity of transitions with respect to the first variable (see Lemma 1.2 in [12]).

Lemma 2.2. *Let $\vartheta \in \Theta(E)$. Then for any $x, y \in E$ and any $0 < h < 1$:*

$$d(\vartheta(h, x), \vartheta(h, y)) - d(x, y) \le d(x, y)\left(e^{\alpha(\vartheta)h} - 1\right), \tag{6}$$

and also

$$d(\vartheta(h, x), x) \le h\beta(\vartheta). \tag{7}$$

2.2. Directional and morphological transitions

Let X be a finite dimensional vector space (we keep this notation throughout the rest of the paper) endowed with the usual Euclidean norm denoted by $|\cdot|$. Given an arbitrary vector (*direction*) $v \in X$, we define the transition

$$\vartheta_v(h, x) := x + hv \tag{8}$$

which clearly satisfies $\alpha(\vartheta_v) = 0$ and $\beta(\vartheta_v) = |v|$. Furthermore, for any $v, u \in X$, $d_\Lambda(\vartheta_v, \vartheta_u) = |v - u|$. Thus, identifying any direction v with its associated directional transition ϑ_v, we can regard X as a family of transitions. This allows to consider X as a mutational space.

On the other hand, let $\mathcal{K}(X)$ be the family of all the nonempty compact subsets of X equipped with the Hausdorff distance,

$$\mathbf{dl}(K, M) := \max(\mathbf{e}(K, M), \mathbf{e}(M, K)) \tag{9}$$

where $\mathbf{e}(K, M) := \sup_{x\in K} d_M(x) := \sup_{x\in K} \inf_{y\in M} |x - y|$ is the excess of K over M. The metric space $(\mathcal{K}(X), \mathbf{dl})$ is separable and complete, and closed balls

$$B_X(K, \delta) := \{M \in \mathcal{K}(X) : \mathbf{dl}(M, K) \le \delta\}$$

are compact (see [2], [8]). Let us consider the family $\mathrm{LIP}(X, X)$ of all the bounded Lipschitz set-valued maps with compact convex values, equipped with the uniform Hausdorff distance:

$$\mathbf{dl}_\infty(\Phi, \Psi) := \sup_{x\in X} \mathbf{dl}(\Phi(x), \Psi(x)), \qquad \Phi, \Psi \in \mathrm{LIP}(X, X). \tag{10}$$

Associated with any $\Phi \in \mathrm{LIP}(X, X)$, for every $x \in X$, we consider the (nonempty) set $\mathcal{S}_\Phi(x) \subset \mathcal{AC}(0, \infty; X)$ of all the solutions of the Cauchy problem

$$\left.\begin{array}{l} x'(t) \in \Phi(x(t)), \\ x(0) = x \end{array}\right\} \tag{11}$$

Then for any $h \in [0, 1]$, $K \in \mathcal{K}(X)$, we define the map

$$\vartheta_\Phi(h, K) := \{x(h) : x(\cdot) \in \mathcal{S}_\Phi(x),\ x \in K\} \tag{12}$$

taking the compact K into the reachable set from K by Φ at time h. This map is a transition, called *morphological transition*, on the metric space $(\mathcal{K}(X), \mathbf{dl})$, with

$$\alpha(\vartheta_\Phi) \le \|\Phi\|_\Lambda := \sup_{x\ne z} \frac{\mathbf{dl}(\Phi(x), \Phi(z))}{|x - z|} \tag{13}$$

and

$$\beta(\vartheta_\Phi) \le \|\Phi\|_\infty := \sup_{x\in X}\left(\sup_{y\in\Phi(x)} |y|\right) \tag{14}$$

(see [2] for details). Identifying any map Φ with the transition ϑ_Φ, and since the inequality

$$d_\Lambda(\vartheta_\Phi, \vartheta_\Psi) \le \mathbf{dl}_\infty(\Phi, \Psi) \tag{15}$$

holds, we get the complete mutational space $(\mathcal{K}(X), \mathrm{LIP}(X,X))$, called the space of morphological transitions or *morphological space* over X.

The mutational space $(X \times \mathcal{K}(X), X \times \mathrm{LIP}(X,X))$, that we call *directional-morphological space*, is the cartesian product of the previously described spaces. Therefore, associated with any transition $\vartheta \in X \times \mathrm{LIP}(X,X)$, there are a vector $v \in X$ and a morphological map Φ such that

$$\forall\ (x,K) \in X \times \mathcal{K}(X), \quad \vartheta(h,(x,K)) = (x + hv, \vartheta_\Phi(h,K)).$$

For simplicity we write $\vartheta = (v, \Phi)$.

3. Tangent transition sets

The natural concept of tangent direction to a set at a point can be adapted in the framework of metric spaces by using "tangent transitions". Then given a subset $M \subset E$, with $(E, \Theta(E))$ a mutational space, the *contingent transition set* to M at a point $x \in M$, with respect to the family of transitions $\Theta(E)$, is defined by

$$T_M(x) := \left\{ \vartheta \in \Theta(E) : \liminf_{h \to 0^+} \frac{d_M(\vartheta(h,x))}{h} = 0 \right\} \tag{16}$$

and the *Clarke tangent (or circatangent) transition set* is

$$C_M(x) := \left\{ \vartheta \in \Theta(E) : \frac{d_M(\vartheta(h,x'))}{h} \to 0 \quad \text{as } h \to 0^+,\ M \ni x' \to x \right\} \tag{17}$$

where $M \ni x' \to x$ means $x' \to x$ with $x' \in M$.

Clearly, $\mathbf{1} \in C_M(x) \subset T_M(x)$, which implies that these sets are always nonempty. It is also clear that above definitions coincide with the usual ones when E is a normed space and $\Theta(E) = E$ is the family of directional transitions.

Another notion of tangent set in mutational spaces can be found in [12], where the *sequential tangent set* is defined as

$$T_M^\diamondsuit(x) := \left\{ \vartheta \in \Theta(E) : \exists\, h_n \to 0^+,\ \vartheta_n \to \vartheta \quad \text{such that } \vartheta_n(h_n, x) \in M \right\}. \tag{18}$$

This set is nonempty ($\mathbf{1} \in T_M^\diamondsuit(x)$) and closed, and it is (in general) strictly contained in the contingent transition set (see [12]). However, when E is a normed space and one considers directional transitions, both tangent sets $T_M(x)$ and $T_M^\diamondsuit(x)$ coincide.

Given a closed subset $M \subset E$, the lower limit (in the sense of Painlevé-Kuratowski) of the tangent transition sets $T_M(y)$, as $y \to x$, is always contained in the Clarke tangent transition set to M at x. This result can be deduced from Theorem 1.5.6 in [2], nevertheless we present here a different proof based on Lemma 2.1 and estimates provided by Lemma 2.2.

Theorem 3.1. *Let $(E, \Theta(E))$ be a mutational space, satisfying that all the closed balls in E are compact. Given a nonempty closed subset $M \subset E$, then*

$$\forall\ x \in M, \quad \operatorname*{Liminf}_{M \ni y \to x} T_M(y) \subset C_M(x). \tag{19}$$

Proof. Let ϑ be a transition in the lower limit of the tangent transition sets:

$$\vartheta \in \operatorname*{Liminf}_{M \ni y \to x} T_M(y) \;\Leftrightarrow\; \lim_{M \ni y \to x} d_{T_M(y)}(\vartheta) = 0.$$

Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that $T_M(y) \cap B_\varepsilon(\vartheta) \neq \emptyset$, whenever $y \in B_\delta(x) \cap M$, that is, there exists $\vartheta_y \in T_M(y)$ satisfying $d_\Lambda(\vartheta_y, \vartheta) < \varepsilon$.

For any fixed $y \in B_\delta(x) \cap M$, let us consider the map

$$g_y(t) := d_M(\vartheta(t, y)), \quad t \in [0, 1]. \tag{20}$$

Clearly, $|g_y(t) - g_y(s)| \le d(\vartheta(t, y), \vartheta(s, y))$. Moreover, by assuming that $t > s$, we can write

$$\begin{aligned} d(\vartheta(t, y), \vartheta(s, y)) \;\le\; & d(\vartheta(s + (t - s), y), \vartheta(t - s, \vartheta(s, y)) \\ & + d(\vartheta(t - s, \vartheta(s, y)), \vartheta(s, y)). \end{aligned} \tag{21}$$

Since ϑ is continuous, for every fixed $0 < T < 1$, $\vartheta([0, T], \overline{B}_\delta(x) \cap M)$ is a compact subset of E and, by virtue of Lemmas 2.1 and 2.2, there exists $\eta > 0$ such that

$$d(\vartheta(s + (t - s), y), \vartheta(t - s, \vartheta(s, y)) < t - s \tag{22}$$

and

$$d(\vartheta(t - s, \vartheta(s, y)), \vartheta(s, y)) \le (t - s)\beta(\vartheta) \tag{23}$$

for any $s, t \in [0, T]$ with $0 < t - s < \eta$, and any $y \in B_\delta(x) \cap M$. Combining (21)-(23) we obtain

$$|g_y(t) - g_y(s)| \le (1 + \beta(\vartheta))\, |t - s| \tag{24}$$

whenever $s, t \in [0, T]$ with $|s - t| < \eta$, which implies that $g_y(\cdot)$ is locally Lipschitz. Even more, we have actually shown that the Lipschitz constant of $g_y(\cdot)$ is the same whenever $y \in B_\delta(x) \cap M$. In addition, since $g_y(\cdot)$ is locally Lipschitz it is almost everywhere differentiable on $]0, 1[$. Let us now estimate the derivative $g_y'(t)$, for every $t \in\,]0, T[$ such that $g_y(\cdot)$ is differentiable at such a point:

$$\begin{aligned} g_y(t + h) - g_y(t) \;&=\; d_M(\vartheta(t + h, y)) - d_M(\vartheta(t, y)) \\ &\le\; d(\vartheta(t + h, y), \hat{z}) - d(\vartheta(t, y), z_t) \end{aligned} \tag{25}$$

where $z_t \in \Pi_M(\vartheta(t, y))$, the set of projectors of $\vartheta(t, y)$ onto M, and $\hat{z} \in M$ is an arbitrary point. Furthermore, by the triangular inequality,

$$\begin{aligned} & d(\vartheta(t + h, y), \hat{z}) - d(\vartheta(t, y), z_t) \\ \le\; & d(\vartheta(t + h, y), \vartheta(h, \vartheta(t, y))) + d(\vartheta(h, \vartheta(t, y)), \hat{z}) - d(\vartheta(h, \vartheta(t, y)), \vartheta(h, z_t)) \\ & \quad + d(\vartheta(h, \vartheta(t, y)), \vartheta(h, z_t)) - d(\vartheta(t, y), z_t) \\ \le\; & h\varepsilon + d(\vartheta(h, z_t), \hat{z}) + d(\vartheta(t, y), z_t)\left(e^{\alpha(\vartheta)h} - 1\right) \end{aligned} \tag{26}$$

whenever $0 < h < \eta_\varepsilon$, where the constant $\eta_\varepsilon > 0$ is provided by Lemma 2.1 applied to the compact $\overline{B}_\delta(x)$, and we have also used (6). Let us now estimate the distance between x and z_t:

$$\begin{aligned} d(x, z_t) \;&\le\; d(x, y) + d(y, z_t) \le d(x, y) + d(y, \vartheta(t, y)) + d(\vartheta(t, y), z_t) \\ &\le\; d(x, y) + 2d(\vartheta(t, y), y) \end{aligned} \tag{27}$$

By (7), for any $0 < t < T$ and any $y \in B_\delta(x) \cap M$, $d(\vartheta(t,y),y) < t\beta(\vartheta)$. Then, if $y \in B_{\delta/2}(x)$ and $0 < t < \min(T, \delta/4\beta(\vartheta))$, by virtue of (27) we can conclude that $z_t \in B_\delta(x) \cap M$ and, therefore, there will exist $\vartheta_{z_t} \in T_M(z_t) \cap B_\varepsilon(\vartheta)$. Let us pick $\hat{z} \in \Pi_M(\vartheta_{z_t}(h,z_t))$. Clearly,

$$\begin{aligned} d(\vartheta(h,z_t),\hat{z}) &\leq d(\vartheta(h,z_t),\vartheta_{z_t}(h,z_t)) + d(\vartheta_{z_t}(h,z_t),\hat{z}) \\ &\leq h\varepsilon + d_M(\vartheta_{z_t}(h,z_t)) \end{aligned} \tag{28}$$

if $0 < h < \rho$, where this constant comes from the very definition of the distance between transitions. Inserting (28) into (26) we obtain

$$\frac{g_y(t+h)-g_y(t)}{h} \leq 2\varepsilon + \frac{d_M(\vartheta_{z_t}(h,z_t))}{h} + d(\vartheta(t,y),z_t)\left(\frac{e^{\alpha(\vartheta)h}-1}{h}\right) \tag{29}$$

and taking upper limit as $h \to 0^+$, since $\vartheta_{z_t} \in T_M(z_t)$, we get

$$g_y'(t) \leq 2\varepsilon + d(\vartheta(t,y),z_t)\alpha(\vartheta) \tag{30}$$

whenever $0 < t < \min(T, \delta/4\beta(\vartheta))$ and $y \in B_{\delta/2}(x) \cap M$. Moreover,

$$d(\vartheta(t,y),z_t) \leq d(\vartheta(t,y),y) \leq t\beta(\vartheta) \tag{31}$$

and then

$$g_y'(t) \leq 2\varepsilon + t\beta(\vartheta)\alpha(\vartheta) \leq 3\varepsilon \tag{32}$$

if $0 < t < \hat{\eta} := \min(T, \delta/4\beta(\vartheta), \varepsilon/(\alpha(\vartheta)+1)\beta(\vartheta))$. Since the above inequality is satisfied for almost every $0 < t < \hat{\eta}$, integrating in (32) we obtain $g_y(t) \leq 3\varepsilon t$, which implies that

$$\frac{d_M(\vartheta(t,y))}{t} \leq 3\varepsilon, \quad 0 < t < \hat{\eta}, \quad y \in B_{\delta/2}(x) \cap M.$$

Hence it follows that $\vartheta \in C_M(x)$ and the proof is complete. □

Definition 3.2. When the lower limit in (19) coincides with the contingent transition set $T_M(x)$, then the set M is said to be *sleek* or *tangent regular* at $x \in M$.

4. Tangential regularity in $X \times \mathcal{K}(X)$

This section is devoted to investigate the properties of tangent transitions to

$$\mathcal{V} := \{(x,K) \in X \times \mathcal{K}(X) : x \in K\}. \tag{33}$$

This set arises in a natural way when one considers viability trajectories. Its contingent transitions were first investigated in [2].

4.1. Characterization of tangents to $\mathcal{V}$

We start by showing that the contingent transition set to $\mathcal{V}$ at any point (x,K) can be described in terms of tangent vectors to K at x. Although this result appears in [2] (Theorem 4.2.2) as a particular case of a more general statement (Theorem 4.2.4), in this paper we shall provide a direct proof, which can be easily adapted to obtain an analogous representation for the Clarke tangent transition set. With this aim we need estimates for solutions of differential inclusions associated with morphological maps. Proofs of next lemmas easily come from Gronwall's inequality.

Lemma 4.1. *Let $x_n(\cdot)$ be a sequence of solutions of the differential inclusion $x' \in \Phi(x)$, with $\Phi \in \mathrm{LIP}(X,X)$ such that $x_n(0) \to \hat{x}_0$. Given a constant $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ and n_ε such that, for any $n \geq n_\varepsilon$,*

$$\sup_{0\leq t\leq\delta_\varepsilon} |x_n(t) - \hat{x}_0| \leq \varepsilon. \tag{34}$$

Lemma 4.2. *Given $\Phi \in \mathrm{LIP}(X,X)$ and $x \in X$, we have that*

$$x(t) - x \in \Phi(x)t + \frac{\|\Phi(x)\|}{\|\Phi\|_\Lambda}\left(e^{\|\Phi\|_\Lambda t} - \|\Phi\|_\Lambda t - 1\right) B_X \tag{35}$$

for any $x(\cdot) \in \mathcal{S}_\Phi(x)$, where $\|\Phi(x)\| := \sup_{z\in\Phi(x)} |z|$.

Theorem 4.3. *Let X be a finite dimensional vector space and $\mathcal{K}(X)$ be the metric space of all its compact subsets equipped with the Hausdorff metric. Then for any $(x,K) \in \mathcal{V}$,*

$$T_\mathcal{V}(x,K) = \{(v,\Phi) \in X \times \mathrm{LIP}(X,X) : v \in \Phi(x) + T_K(x)\}. \tag{36}$$

Proof. Let $\vartheta \in T_\mathcal{V}(x,K)$. From the very definition of the tangent transition set there exist sequences $h_n \to 0^+$, $\varepsilon_n \to 0^+$ and $(z_n, C_n) \in \mathcal{V}$ such that

$$d(\vartheta(h_n,(x,K)),(z_n,C_n)) \leq \varepsilon_n h_n \tag{37}$$

or equivalently,

$$|x + h_n v - z_n| \leq \varepsilon_n h_n, \quad \text{and} \quad \mathbf{dl}(C_n, \vartheta_\Phi(h_n,K)) \leq \varepsilon_n h_n \tag{38}$$

$\vartheta = (v,\Phi)$ being. Second inequality yields $C_n \subset \vartheta_\Phi(h_n,K) + \varepsilon_n h_n B_X$, where B_X is the unit closed ball in X, which implies that there exists a map $x_n(\cdot) \in \mathcal{S}_\Phi(K)$ satisfying

$$\varepsilon_n h_n \geq |z_n - x_n(h_n)| = \left| z_n - x_n(0) - \int_0^{h_n} x_n'(s)\,ds \right|. \tag{39}$$

Since $x_n(0) \in K$ and this set is compact, there will exist a subsequence, again denoted $x_n(0)$ for simplicity, convergent to a point $\hat{x} \in K$. By Lemma 4.1, given $\eta > 0$, we have

$$\Phi(x_n(s)) \subset \Phi(\hat{x}) + \eta\|\Phi\|_\Lambda B_X \tag{40}$$

if $0 < s < h_n$, for n great enough, also by using that Φ is Lipschitz. Hence

$$\frac{1}{h_n}\int_0^{h_n} x_n'(s)\,ds \in \frac{1}{h_n}\int_0^{h_n} \Phi(x_n(s))\,ds \subset \Phi(\hat{x}) + \eta\|\Phi\|_\Lambda B_X \tag{41}$$

and applying again that sequences in a compact set have convergent subsequences, now to the compact $\Phi(\hat{x}) + \eta\|\Phi\|_\Lambda B_X$, we obtain $y_\eta \in \Phi(\hat{x}) + \eta\|\Phi\|_\Lambda B_X$ such that

$$\frac{1}{h_n}\int_0^{h_n} x_n'(s)\,ds \to y_\eta \tag{42}$$

as $n \to \infty$. On the other hand, setting

$$\omega_n := (x_n(0) - x)/h_n, \tag{43}$$

we can write $x + h_n\omega_n \in K$ and, moreover,

$$\begin{aligned}
|v - y_\eta - \omega_n| &= \frac{|x + h_n v - x_n(0) - h_n y_\eta|}{h_n} \\
&\leq \frac{|x + h_n v - z_n|}{h_n} + \frac{|z_n - x_n(0) - h_n y_\eta|}{h_n} \\
&\leq \frac{|x + h_n v - z_n|}{h_n} + \frac{|z_n - x_n(h_n)|}{h_n} + \left|\frac{1}{h_n}\int_0^{h_n} x_n'(s)\,ds - y_\eta\right| \\
&\leq 2\varepsilon_n + \left|\frac{1}{h_n}\int_0^{h_n} x_n'(s)\,ds - y_\eta\right| \qquad (44)
\end{aligned}$$

by using (38) and (39). Letting $n \to \infty$ in (44), it follows from (42), that ω_n converges to $v - y_\eta$ and, therefore, $v - y_\eta \in T_K(x)$, for any $\eta > 0$. We now remember that $y_\eta \in \Phi(\hat{x}) + \eta\|\Phi\|_\Lambda B_X$. Hence, letting $\eta \to 0^+$, we obtain $\hat{y} \in \Phi(\hat{x})$ such that $v - \hat{y} \in T_K(x)$. Finally, since $|x - x_n(0)| = h_n|\omega_n| \to 0$, we get $x = \hat{x}$ and then $v \in \Phi(x) + T_K(x)$. Thus we have the inclusion

$$T_{\mathcal{V}}(x, K) \subset \{(v, \Phi) \in X \times \mathrm{LIP}(X, X) : v \in \Phi(x) + T_K(x)\}. \qquad (45)$$

The converse is also true. Indeed, if we take $y \in \Phi(x)$ with $v - y \in T_K(x)$, then there are $h_n \to 0^+$, $\omega_n \to v - y$ such that $x + h_n\omega_n \in K$. Let us consider the sequence

$$z_n := x + h_n\omega_n + h_n y \qquad (46)$$

and the family of compact sets

$$C_n := \overline{\bigcup_{z\in K} (z + h_n(\Phi(z) + \eta_n B_X))} \qquad (47)$$

where $\eta_n \to 0^+$. Clearly, $y \in \Phi(x) \subset \Phi(x + h_n\omega_n) + \|\Phi\|_\Lambda h_n|\omega_n|B_X$, which implies that $(z_n, C_n) \in \mathcal{V}$, provided that $\|\Phi\|_\Lambda h_n|\omega_n| \leq \eta_n$. Furthermore

$$|x + h_n v - z_n| = h_n|v - \omega_n - y| \qquad (48)$$

where $|v - y - \omega_n| \to 0^+$. Let us take $z \in K$. Now we use Lemma 4.2 to get

$$\vartheta_\Phi(h_n, z) \subset z + h_n\Phi(z) + \left(\|\Phi(z)\|\left(e^{h_n\|\Phi\|_\Lambda} - \|\Phi\|_\Lambda h_n - 1\right)/\|\Phi\|_\Lambda\right) B_X \subset C_n$$

if $\|\Phi(K)\|\left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1\right) < h_n\eta_n$. Therefore

$$\vartheta_\Phi(h_n, K) \subset C_n. \qquad (49)$$

On the other hand, given $z \in K$, $y \in \Phi(z)$ and $u \in B_X$, as a consequence of the Filippov theorem (see [2] or [11]), we can get $z(\cdot) \in \mathcal{S}_\Phi(z)$ such that $z'(0) = y$, also satisfying

$$|z(t) - z - ty| \leq e^{\|\Phi\|_\Lambda t}\int_0^t d_{\Phi(z+sy)}(y)\, e^{-\|\Phi\|_\Lambda s}\,ds. \qquad (50)$$

But, $d_{\Phi(z+sy)}(y) \le \mathbf{dl}(\Phi(z), \Phi(z+sy)) \le \|\Phi\|_\Lambda |y| s$, and combining this inequality with the integral on the right-hand side of (50) we conclude that

$$\int_0^t d_{\Phi(z+sy)}(y)\, e^{-\|\Phi\|_\Lambda s}\, ds \le |y| \left(1 - e^{-\|\Phi\|_\Lambda t} - \|\Phi\|_\Lambda t e^{-\|\Phi\|_\Lambda t}\right) / \|\Phi\|_\Lambda. \tag{51}$$

Thus, from (50)-(51), we can write

$$|z(h_n) - z - h_n y| \le |y| \left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1\right) / \|\Phi\|_\Lambda. \tag{52}$$

Finally (52) provides the estimate

$$|z(h_n) - (z + h_n y + h_n \eta_n u)| \le h_n \left(\eta_n + \frac{\|\Phi(K)\| \left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1\right)}{\|\Phi\|_\Lambda h_n}\right).$$

Hence

$$C_n \subset \vartheta_\Phi(h_n, K) + h_n \left(\eta_n + \frac{\|\Phi(K)\| \left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1\right)}{\|\Phi\|_\Lambda h_n}\right) B_X$$

and taking $\varepsilon_n := \max\left(|v - y - \omega_n|, \eta_n + \|\Phi(K)\| \left(\frac{e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1}{\|\Phi\|_\Lambda h_n}\right)\right)$, we can combine (48), (49) and the above-stated inclusion to obtain

$$|x + h_n v - z_n| \le h_n \varepsilon_n, \quad \mathbf{dl}(C_n, \vartheta_\Phi(h_n, K)) \le h_n \varepsilon_n \tag{53}$$

which clearly implies that $(v, \Phi) \in T_{\mathcal{V}}(x, K)$, and the proof is done. □

As a corollary of this result we can ensure that the contingent transition set and the sequential tangent set to $\mathcal{V}$ coincide at any point $(x, K) \in \mathcal{V}$.

Corollary 4.4. *Let X be a finite dimensional vector space and $\mathcal{K}(X)$ be the metric space of all its compact subsets equipped with the Hausdorff metric. Then for any $(x, K) \in \mathcal{V}$,*

$$T_{\mathcal{V}}(x, K) = T_{\mathcal{V}}^{\diamondsuit}(x, K). \tag{54}$$

Proof. It suffices to show that $T_{\mathcal{V}}(x, K) \subset T_{\mathcal{V}}^{\diamondsuit}(x, K)$. Thus, let us take (v, Φ) in $T_{\mathcal{V}}(x, K)$. By Theorem 4.3 we can find $y \in \Phi(x)$ such that $v - y \in T_K(x)$, which means that there exist sequences $h_n \to 0^+$ and $\omega_n \to v - y$ satisfying $x + h_n \omega_n \in K$. Given a map $x_n(\cdot) \in \mathcal{S}_\Phi(x + h_n \omega_n + h_n y)$, we define the absolutely continuous function

$$z_n(t) := \begin{cases} x + h_n \omega_n + ty, & 0 \le t < h_n \\ x_n(t - h_n), & t \ge h_n. \end{cases} \tag{55}$$

Clearly, for a.e. $0 < t < h_n$

$$\begin{aligned} z_n'(t) &= y \in \Phi(x) \subset \Phi(z_n(t)) + |h_n \omega_n + ty| \, \|\Phi\|_\Lambda B_X \\ &\subset \Phi(z_n(t)) + h_n (|\omega_n| + |y|) \, \|\Phi\|_\Lambda B_X \end{aligned}$$

and, since $x_n(\cdot) \in \mathcal{S}_\Phi$, we have that $z_n(\cdot)$ is a solution of

$$\left.\begin{aligned} &z_n'(t) \in \Phi_n(z_n(t)) \\ &z_n(0) = x + h_n \omega_n \in K \end{aligned}\right\} \tag{56}$$

where $\Phi_n(z) := \Phi(z) + h_n(|\omega_n| + |y|)\,\|\Phi\|_\Lambda B_X$. Therefore $z_n(\cdot) \in \mathcal{S}_{\Phi_n}(K)$ and setting $v_n := \omega_n + y$, $\vartheta_n := (v_n, \Phi_n)$, we have

$$x + h_n v_n = z_n(h_n) \in \vartheta_{\Phi_n}(h_n, K) \;\Leftrightarrow\; \vartheta_n(h_n, (x, K)) \in \mathcal{V}. \tag{57}$$

Finally, since $v_n \to v$ and $\mathbf{d\!l}_\infty(\Phi, \Phi_n) \le h_n(|\omega_n| + |y|)\,\|\Phi\|_\Lambda \to 0$, we conclude that the sequence ϑ_n of transitions converges to (v, Φ) in the topology defined by the metric d_Λ and, therefore, $(v, \Phi) \in T^\diamondsuit_\mathcal{V}(x, K)$. □

A relationship similar to (36) can be established between circatangent transitions to $\mathcal{V}$ at (x, K) and vectors in the Clarke tangent cone to K at x. Actually only slight changes in the proof of Theorem 4.3 are needed.

Theorem 4.5. *Let X be a finite dimensional vector space and $\mathcal{K}(X)$ be the metric space of all its compact subsets equipped with the Hausdorff metric. Then for any $(x, K) \in \mathcal{V}$,*

$$\{(v, \Phi) \in X \times \mathrm{LIP}(X, X) : v \in \Phi(x) + C_K(x)\} \subset C_\mathcal{V}(x, K). \tag{58}$$

Proof. Let $(x, K) \in \mathcal{V}$ and $v - y \in C_K(x)$, with $y \in \Phi(x)$. Given sequences $h_n \to 0^+$ and $(x_n, K_n) \to (x, K)$, with $(x_n, K_n) \in \mathcal{V}$, we can find $x'_n \in K$ such that $|x'_n - x_n| \le \mathbf{d\!l}(K, K_n)$. Thus $K \ni x'_n \to x$ and, since $v - y \in C_K(x)$, there exists a sequence $\omega_n \to v - y$ satisfying

$$x'_n + h_n\omega_n \in K. \tag{59}$$

Let us consider the family of compact sets

$$C_n := \overline{\bigcup_{z \in K_n} (z + h_n(\Phi(z) + \eta_n B_X))} \tag{60}$$

where $\eta_n \to 0^+$. Setting $\xi_n := 2\mathbf{d\!l}(K, K_n)$ we can write

$$x_n + h_n\omega_n \in x'_n + h_n\omega_n + (\xi_n/2)B_X \subset K + (\xi_n/2)B_X \subset K_n + \xi_n B_X \tag{61}$$

which yields the existence of $u_n \in B_X$ satisfying

$$\hat{z}_n := x_n + h_n\omega_n + \xi_n u_n \in K_n. \tag{62}$$

Hence, if we define $z_n := \hat{z}_n + h_n y$, the following inclusion is satisfied

$$z_n \in \hat{z}_n + h_n\Phi(x) \subset \hat{z}_n + h_n(\Phi(\hat{z}_n) + \|\Phi\|_\Lambda|\hat{z}_n - x|B_X) \tag{63}$$

and, therefore, choosing η_n such that $\|\Phi\|_\Lambda|\hat{z}_n - x| < \eta_n$, we have $z_n \in C_n$ or, equivalently, $(z_n, C_n) \in \mathcal{V}$. Furthermore,

$$|x_n + h_n v - z_n| = h_n|v - \omega_n - y - (\xi_n/h_n)\,u_n| \tag{64}$$

and taking $\xi_n/h_n \to 0^+$ (assuming, for instance, $\mathbf{d\!l}(K, K_n) < h_n^2$, which involves no loss of generality) we have $|v - \omega_n - y - (\xi_n/h_n)\,u_n| \to 0^+$. In addition, for any $z \in K_n$, Lemma 4.2 gives

$$\begin{aligned} \vartheta_\Phi(h_n, z) \;&\subset\; z + h_n\Phi(z) + \frac{\|\Phi(z)\|\left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1\right)}{\|\Phi\|_\Lambda}B_X \\ &\subset\; C_n \end{aligned} \tag{65}$$

whenever $\|\Phi(K_n)\| \left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1\right) / \|\Phi\|_\Lambda h_n < \eta_n$ and, therefore,

$$\vartheta_\Phi(h_n, K_n) \subset C_n. \tag{66}$$

Conversely, following the same procedure as in the proof of Theorem 4.3 we obtain

$$C_n \subset \vartheta_\Phi(h_n, K_n) + h_n \rho_n B_X \tag{67}$$

$\rho_n := \left(\eta_n + \|\Phi(K_n)\| \left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1\right) / \|\Phi\|_\Lambda h_n\right)$ being. Finally, setting $\varepsilon_n := \max\left(|v - \omega_n - (\xi_n/h_n)\, u_n - y|, \rho_n\right)$, (64), (66) and (67) yield

$$|x_n + h_n v - z_n| \le h_n \varepsilon_n, \quad \mathbf{dl}(C_n, \vartheta_\Phi(h_n, K_n)) \le h_n \varepsilon_n \tag{68}$$

with $(z_n, C_n) \in \mathcal{V}$. This clearly implies that

$$\frac{d_{\mathcal{V}}(\vartheta(h_n, (x_n, K_n)))}{h_n} \le \varepsilon_n$$

where $\vartheta := (v, \Phi)$. Hence $(v, \Phi) \in C_{\mathcal{V}}(x, K)$, and the proof is done. □

4.2. Tangential regularity of $\mathcal{V}$

We shall finally show that the inclusion established by Theorem 3.1 becomes an equality for the subset $\mathcal{V} \subset X \times \mathcal{K}(X)$ defined by (33). A characterization of points where $\mathcal{V}$ is sleek will be obtained as a corollary.

Theorem 4.6. *Let X be a finite dimensional vector space and $\mathcal{K}(X)$ be the metric space of all its compact subsets equipped with the Hausdorff metric. Then for any $(x, K) \in \mathcal{V}$,*

$$\operatorname*{Liminf}_{\mathcal{V} \ni (z,M) \to (x,K)} T_{\mathcal{V}}(z, M) = C_{\mathcal{V}}(x, K). \tag{69}$$

Proof. From Theorem 3.1 it suffices to show that any transition in $C_{\mathcal{V}}(x, K)$ belongs to the lower limit of the contingent transition sets $T_{\mathcal{V}}(z, M)$. Given $\vartheta = (v, \Phi) \in C_{\mathcal{V}}(x, K)$ and $\varepsilon > 0$, from the very definition of Clarke tangent transition set, we have $\delta > 0$ such that for any $0 < h < \delta$ and any $(x', K') \in \mathcal{V}$ with $|x' - x| < \delta$ and $\mathbf{dl}(K', K) < \delta$, then

$$d_{\mathcal{V}}(\vartheta(h, (x', K'))) < h\varepsilon,$$

or, equivalently, there exists $(z(h), M(h)) \in \mathcal{V}$ satisfying

$$|x' + hv - z(h)| < h\varepsilon, \quad \mathbf{dl}(M(h), \vartheta_\Phi(h, K')) < h\varepsilon. \tag{70}$$

Let us take arbitrary sequences $\mathcal{V} \ni (x_r, K_r) \to (x, K)$ and $h_n \to 0^+$, and let $(z_{rn}, M_{rn}) \in \mathcal{V}$ denote the element associated by (70). Clearly

$$|x_r + h_n v - z_{rn}| < h_n \varepsilon \;\Leftrightarrow\; |v - (z_{rn} - x_r)/h_n| < \varepsilon$$

and since $(z_{rn} - x_r)/h_n$ is bounded (and X is a finite dimensional vector space), a classical argument (Cantor's diagonalization) shows that for any r we can assume that there exists the limit

$$v_r := \lim_{n \to \infty} (z_{rn} - x_r)/h_n \tag{71}$$

satisfying $|v_r - v| \le \varepsilon$ and also

$$|x_r + h_n v_r - z_{rn}| = h_n \left| v_r - (z_{rn} - x_r)/h_n \right|. \tag{72}$$

On the other hand, since $z_{rn} \in M_{rn} \subset \vartheta_\Phi(h_n, K_r) + h_n \varepsilon B_X$, we can find $x_{rn}(\cdot) \in \mathcal{S}_\Phi(K_r)$ satisfying

$$z_{rn} = x_{rn}(h_n) + h_n \varepsilon u_{rn} \tag{73}$$

where $u_{rn} \in B_X$. Hence,

$$\frac{z_{rn} - x_r}{h_n} = \frac{x_{rn}(h_n) + \varepsilon h_n u_{rn} - x_r}{h_n}. \tag{74}$$

Let us consider the map $z_{rn}(t) := x_{rn}(t) + t\varepsilon u_{rn}$. Clearly,

$$\begin{aligned} z'_{rn}(t) &= x'_{rn}(t) + \varepsilon u_{rn} \in \Phi(x_{rn}(t)) + \varepsilon u_{rn} \\ &\subset \Phi(z_{rn}(t)) + \|\Phi\|_\Lambda |t\varepsilon u_{rn}| B_X + \varepsilon u_{rn} \\ &\subset \Phi_\varepsilon(z_{rn}(t)) + \|\Phi\|_\Lambda t\varepsilon B_X \end{aligned} \tag{75}$$

$\Phi_\varepsilon(z) := \Phi(z) + \varepsilon B_X$, being. This map belongs to LIP(X, X), also satisfying $\mathbf{dl}_\infty(\Phi, \Phi_\varepsilon) \le \varepsilon$. Moreover, $d_{\Phi_\varepsilon(z_{rn}(t))}(z'_{rn}(t)) \le \|\Phi\|_\Lambda \varepsilon t$, and Filippov theorem provides a function $y_{rn}(\cdot) \in \mathcal{S}_{\Phi_\varepsilon}(K_r)$, such that $y_{rn}(0) = x_{rn}(0) \in K_r$, and also

$$|z_{rn}(t) - y_{rn}(t)| \le \frac{\varepsilon}{\|\Phi\|_\Lambda} \left(e^{\|\Phi\|_\Lambda t} - 1 - \|\Phi\|_\Lambda t \right), \quad t > 0. \tag{76}$$

Combining (74) with (76) we can write

$$\frac{z_{rn} - x_r}{h_n} = \frac{y_{rn}(h_n) + \frac{\varepsilon}{\|\Phi\|_\Lambda} \left(e^{\|\Phi\|_\Lambda h_n} - 1 - \|\Phi\|_\Lambda h_n \right) \tilde{u}_{rn} - x_r}{h_n} \tag{77}$$

with $\tilde{u}_{rn} \in B_X$. From Lemma 4.2

$$\frac{y_{rn}(h_n) - y_{rn}(0)}{h_n} \in \Phi_\varepsilon(y_{rn}(0)) + \frac{\rho \left(e^{\|\Phi\|_\Lambda h_n} - 1 - \|\Phi\|_\Lambda h_n \right)}{\|\Phi\|_\Lambda h_n} B_X$$

where $\rho > 0$ is a constant with $\Phi_\varepsilon(x) \subset \rho B_X$ for all $x \in X$, $\varepsilon > 0$. Moreover

$$|y_{rn}(0) - x_r| \le |y_{rn}(0) - z_{rm}| + |z_{rn} - x_r| \le h_n (\rho + 2\varepsilon + |v|) \underset{n \to \infty}{\longrightarrow} 0$$

by using (73) and the very definition of z_{rn}. Since Φ_ε is continuous, the above claims imply that $(y_{rn}(h_n) - y_{rn}(0))/h_n \to \tilde{y}_r \in \Phi_\varepsilon(x_r)$ as $n \to \infty$ for any r, and

$$\tilde{v}_{rn} := \frac{y_{rn}(0) + \frac{\varepsilon}{\|\Phi\|_\Lambda} \left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1 \right) \tilde{u}_{rn} - x_r}{h_n} \tag{78}$$

converges to a vector, denoted by $\tilde{v}_r$, as $n \to \infty$. Furthermore,

$$\begin{aligned} \frac{d_{K_r}(x_r + h_n \tilde{v}_r)}{h_n} &\le \frac{d_{K_r}(x_r + h_n \tilde{v}_{rn})}{h_n} + |\tilde{v}_{rn} - \tilde{v}_r| \\ &\le \frac{\varepsilon \left(e^{\|\Phi\|_\Lambda h_n} - \|\Phi\|_\Lambda h_n - 1 \right)}{\|\Phi\|_\Lambda h_n} + |\tilde{v}_{rn} - \tilde{v}_r| \underset{n \to \infty}{\longrightarrow} 0 \end{aligned}$$

which implies $\tilde{v}_r \in T_{K_r}(x_r)$. Hence, $v_r = \tilde{y}_r + \tilde{v}_r \in \Phi_\varepsilon(x_r) + T_{K_r}(x_r)$, or, equivalently, (see Theorem 4.3) $(v_r, \Phi_\varepsilon) \in T_{\mathcal{V}}(x_r, K_r)$. We have, therefore, obtained a transition $\vartheta_{r,\varepsilon} := (v_r, \Phi_\varepsilon)$ in $T_{\mathcal{V}}(x_r, K_r)$ satisfying

$$d_\Lambda(\vartheta, \vartheta_{r,\varepsilon}) \leq \max\left(|v - v_r|, \, \mathbf{dl}_\infty(\Phi, \Phi_\varepsilon)\right) \leq \varepsilon,$$

which implies $d_{T_{\mathcal{V}}(x_r,K_r)}(\vartheta) \to 0$ as $r \to \infty$ for every sequence $(x_r, K_r) \to (x, K)$ in $\mathcal{V}$. Hence

$$\vartheta = (v, \Phi) \in \operatorname*{Liminf}_{\mathcal{V} \ni (z,M) \to (x,K)} T_{\mathcal{V}}(z, M)$$

and the proof is done. □

Theorem 4.7. *Let X be a finite dimensional vector space and $\mathcal{K}(X)$ be the metric space of all its compact subsets equipped with the Hausdorff metric. If K is sleek at x, $(x, K) \in \mathcal{V}$, then $\mathcal{V}$ is sleek at (x, K).*

Proof. Since K is assumed to be sleek at x, Theorems 4.3 and 4.5 give

$$T_{\mathcal{V}}(x, K) = \{(v, \Phi) \in X \times \mathrm{LIP}(X, X) : v \in \Phi(x) + C_K(x)\} \subset C_{\mathcal{V}}(x, K)$$

and, by Theorem 4.6, we can conclude that

$$T_{\mathcal{V}}(x, K) = \operatorname*{Liminf}_{\mathcal{V} \ni (z,M) \to (x,K)} T_{\mathcal{V}}(z, M)$$

and the proof is complete. □

5. An application to the control of joint systems

Let us consider a control problem where the evolution of states is governed by the joint evolutionary-morphological system

$$\left.\begin{array}{l} x'(t) = f(t, x(t), K(t), u(t)) \\ \overset{\circ}{K}(t) \ni \Phi(t, x(t), K(t), u(t)) \\ u(t) \in U(x(t), K(t)) \end{array}\right\} \tag{79}$$

We assume that $f : \mathbb{R}_+ \times \mathcal{V} \times Y \longrightarrow X$ is a continuous function (Y a finite dimensional vector space), $\Phi : \mathbb{R}_+ \times \mathcal{V} \times Y \longrightarrow \mathrm{LIP}(X, X)$ is continuous and, finally, $U : \mathcal{V} \longrightarrow 2^Y$ is a lower semicontinuous set-valued map with closed convex values. Trajectories (states) to this problem on an interval $I \subset \mathbb{R}$, will be pairs $(x(\cdot), K(\cdot))$, where $x(\cdot)$ is an absolutely continuous function (we write $x(\cdot) \in \mathcal{AC}(I; X)$), and $K(\cdot)$ is a compact-valued Lipschitz tube, denoted $K(\cdot) \in \mathcal{C}^{0,1}(I; \mathcal{K}(X))$, such that:

(1) For a.e. $t \in I$, $x'(t) = f(t, x(t), K(t), u(t))$

(2) For every $t \in I$

$$\lim_{h \to 0^+} \frac{\mathbf{dl}(K(t+h), \vartheta_{\Phi(t,x(t),K(t),u(t))}(h, K(t)))}{h} = 0$$

where $\vartheta_{\Psi(t)}(h, K) = \{z(h) : z'(s) \in \Psi(t)(z(s)), \; z(0) \in K\}$ is the morphological transition associated with $\Psi(t) := \Phi(t, x(t), K(t), u(t))$. In this case, it is said that $K(\cdot)$ is a morphological primitive of $\Phi(\cdot, x(\cdot), K(\cdot), u(\cdot))$ (we refer to [2], [5] for more information about morphological equations).

In this section we shall investigate the existence of trajectories of (79) viable in the sense that

$$x(t) \in K(t) \quad \Leftrightarrow \quad (x(t), K(t)) \in \mathcal{V}. \tag{80}$$

This problem has been studied in [2] for evolutionary-morphological systems without controls and also in [5], when the control variable only appears in f as $f(t,x,K,u) = g(t,x,K) - u$, and $U(x,K) = \varphi(x,K)B_X$, as a way of correcting the dynamics of a joint evolutionary-morphological system to get viable solutions.

Lemma 5.1. *Let X be a finite dimensional vector space and $\mathcal{K}(X)$ be the metric space of all its compact subsets equipped with the Hausdorff metric. Given a set-valued map $\Phi \in \mathrm{LIP}(X,X)$ such that* Graph(Φ) *is a convex set, then*

$$\{(v,\Phi) : v \in \Phi(x) + T_K(x)\} \subset T_{\mathcal{V}_\mathcal{C}}(x,K) \tag{81}$$

where $\mathcal{K}_\mathcal{C}(X)$ is the family of all compact convex subsets of X and $\mathcal{V}_\mathcal{C}$ is the closed subset

$$\mathcal{V}_\mathcal{C} := \{(x,K) \in X \times \mathcal{K}_\mathcal{C}(X) : x \in K\}. \tag{82}$$

Proof. Since the set C_n defined by (47) is also convex when the graph of Φ is assumed to be convex, it suffices to reproduce the proof of Theorem 4.3. □

Let us denote by $\mathrm{LIP}_\mathcal{C}(X,X)$ the family of all the maps in $\mathrm{LIP}(X,X)$ having convex graph.

Theorem 5.2. *Let $f : \mathbb{R}_+ \times \mathcal{V} \times Y \longrightarrow X$ be a continuous map affine with respect to the variable u and having linear growth*

$$|f(t,x,K,u)| \leq c(t)\left(1 + |x| + \sup_{z\in K}|z| + |u|\right), \quad t \geq 0,\ (x,K) \in \mathcal{V},\ u \in Y$$

where $c(\cdot) \in L^1_{loc}(0,\infty)$, $\Phi : \mathbb{R}_+ \times \mathcal{V} \times Y \longrightarrow \mathrm{LIP}_\mathcal{C}(X,X)$, be continuous, affine with respect to u and also skirted

$$\|\Phi\|_\Lambda := \sup_{t\geq 0,\ (x,K)\in\mathcal{V},\ u\in Y} \|\Phi(t,x,K,u)\|_\Lambda < \infty, \tag{83}$$

and bounded, $\|\Phi\|_\infty := \sup_{t\geq 0,\ (x,K)\in\mathcal{V},\ u\in Y}\left(\sup_{y\in\Phi(t,x,K,u)}|y|\right) < \infty$. Let $U : \mathcal{V} \longrightarrow 2^Y$ be lower semicontinuous with closed convex values and linear growth, $U(x,K) \subset \beta\left(1 + |x| + \sup_{z\in K}|z|\right)$, $(x,K) \in \mathcal{V}$, also satisfying that for any $t \geq 0$, $(x,K) \in \mathcal{V}$, there are $\gamma, \delta, \varepsilon > 0$ such that $\gamma B_{X\times LIP(X,X)}$ is contained in

$$(f(s,z,M,U(z,M)\cap\varepsilon B_Y), \Phi(s,z,M,U(z,M)\cap\varepsilon B_Y)) - T_\mathcal{V}(z,M) \tag{84}$$

whenever $|s-t| < \delta$ and $(z,M) \in \mathcal{V}$ with $|z-x| < \delta$, $\mathbf{dl}(M,K) < \delta$. Then, for any initial state $(x_0,K_0) \in \mathcal{V}_\mathcal{C}$, there exists a joint trajectory $(x(\cdot),K(\cdot))$ in $\mathcal{AC}(0,\infty;X) \times \mathcal{C}^{0,1}(0,\infty;\mathcal{K}(X))$ to (79) starting from (x_0,K_0) and such that (80) holds for all $t \geq 0$.

Proof. Let us consider the regulation map R taking any $(t,x,K) \in \mathbb{R}_+ \times \mathcal{V}_\mathcal{C}$ into the nonempty closed convex set

$$\{u \in U(x,K) : (f(t,x,K,u), \Phi(t,x,K,u)) \in T_\mathcal{V}(x,K)\}. \tag{85}$$

By virtue of Theorem 4.7 we have that the set-valued map $(x,K) \rightsquigarrow T_{\mathcal{V}}(x,K)$ is lower semicontinuous at every $(x,K) \in \mathcal{V}_{\mathcal{C}}$. Hence, Proposition 1.5.1 in [4] states that the regulation map R is lower semicontinuous at every $(t,x,K) \in \mathbb{R}_+ \times \mathcal{V}_{\mathcal{C}}$, since (84) is satisfied.

Given $(x(\cdot), K(\cdot))$ a trajectory of (79) starting from (x_0, K_0), it is known (see [2]) that, under assumptions made on Φ, the tube $K(\cdot)$ is $\|\Phi\|_\Lambda$-Lipschitz and, therefore,

$$K(t) \subset K_0 + \|\Phi\|_\Lambda t B_X. \tag{86}$$

Setting $r(t) := \sup_{z\in K_0} |z| + \|\Phi\|_\Lambda t$, if $u(\cdot)$ is a control associated with this trajectory, then

$$|u(t)| \leq \beta\left(1 + r(t) + |x(t)|\right). \tag{87}$$

Thus, by using that f has a linear growth, we get

$$|x(t)| \leq |x_0| + \int_0^t c(s)(1+\beta)(1+r(s))\,ds + \int_0^t c(s)(1+\beta)|x(s)|\,ds$$

and Gronwall's inequality gives the estimate

$$|x(t)| \leq \underbrace{|x_0|e^{\mu(t)} + \int_0^t c(s)(1+\beta)(1+r(s))e^{\mu(t)-\mu(s)}\,ds}_{\varphi(t)} \tag{88}$$

where $\mu(\tau) := (1+\beta)\int_0^\tau c(s)ds$. This yields that, for any fixed $T > 0$, the regulation map R is lower semicontinuous with closed convex values on the compact

$$\mathcal{M}_T := \{(t,x,K) \in [0,T] \times \mathcal{V}_{\mathcal{C}} : |x| \leq \varphi(t),\ K \subset r(t)B_X\} \tag{89}$$

and the well-known Michael's Theorem provides a continuous selection, that is, there is a continuous map $\widehat{u} : \mathcal{M}_T \longrightarrow X$ such that $\widehat{u}(t,x,K) \in R(t,x,K)$. This allows to consider the joint system

$$\left.\begin{array}{l} x'(t) = f(t, x(t), K(t), \widehat{u}(t,x(t),K(t))) \\ \overset{\circ}{K}(t) \ni \Phi(t, x(t), K(t), \widehat{u}(t,x(t),K(t))) \end{array}\right\} \tag{90}$$

Furthermore, since $\widehat{u}$ is a selection of R, by Theorem 4.3,

$$f(t,x,K,\widehat{u}(t,x,K)) \in \Phi(t,x,K,\widehat{u}(t,x,K))(x) + T_K(x)$$

for any $(t,x,K) \in \mathcal{M}_T$ and, since Φ has been assumed taking values with convex graph, $\Phi(t,x,K,\widehat{u}(t,x,K)) \in \mathrm{LIP}_{\mathcal{C}}(X,X)$, Lemma 5.1 gives

$$(f(t,x,K,\widehat{u}(t,x,K)), \Phi(t,x,K,\widehat{u}(t,x,K))) \in T_{\mathcal{V}_{\mathcal{C}}}(x,K). \tag{91}$$

We are, therefore, under assumptions of the usual viability theorem for joint evolutionary-morphological systems (see Theorem 4.3.1 in [2]) and there will exist a solution $(x(\cdot), K(\cdot))$ to (90) such that $x(0) = x_0$, $K(0) = K_0$, with $(x(t), K(t)) \in \mathcal{V}_{\mathcal{C}}$, for all $t \in [0,T]$. This pair of maps is the desired viable trajectory to (79), with

$$u(t) := \widehat{u}(t, x(t), K(t)) \tag{92}$$

the associated control. Finally, since we have proved the existence of solution to (79)-(80) on $[0,T]$, for every $T > 0$, and assumptions made on f and Φ prevent from blow-up, a standard procedure allows to extend the solution to the whole interval $[0,\infty[$. □

Remark 5.3. Notice that we have proved more than it is claimed in the theorem. Actually we have shown that the tube $K(\cdot)$ remains convex and $(x(t), K(t)) \in \mathcal{V}_\mathcal{C}$ for every $t \geq 0$.

Remark 5.4. It is clear that, for every $T > 0$, the regulation map R is lower semicontinuous on the bounded subset

$$\mathcal{O}_T := \{(t, x, K) \in [0, T] \times \mathcal{V}_\mathcal{S} : |x| \leq \varphi(t),\ K \subset r(t)B_X\} \tag{93}$$

where $\mathcal{K}_\mathcal{S}(X)$ denotes the family of all the compact regular (sleek) subsets of X and $\mathcal{V}_\mathcal{S} := \{(x, K) \in X \times \mathcal{K}_\mathcal{S}(X) : x \in K\}$. However, this set is not closed and, therefore, Michael's Theorem does not work.

To close the paper we will show that $\mathcal{K}_\mathcal{S}(X)$ is not closed in $\mathcal{K}(X)$, when this space is endowed with the Hausdorff metric. With this aim let us consider the compact set

$$K := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ y \leq |x|\right\} \tag{94}$$

that is, the intersection of the unit closed ball in $\mathbb{R}^2$ and the hypograph of the absolute value function $|\cdot|$. It is a trivial matter to see that the Bouligand tangent cone to K at $(0, 0)$ coincides with the hypograph of $|\cdot|$, i.e.

$$T_K(0, 0) = \left\{(x, y) \in \mathbb{R}^2 : y \leq |x|\right\}. \tag{95}$$

Therefore K is not sleek at the origin because Clarke tangent cone is always convex. In order to construct a sequence of sleek sets converging to K, let us first consider the $\mathcal{C}^\infty$ function (standard mollifier)

$$\rho(x) := \begin{cases} e^{-1/(1-x^2)}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases} \tag{96}$$

then the sequence, $\rho_n(x) := n\rho(nx)/\|\rho\|_{L^1}$, and, finally, the sequence of mollifications

$$\phi_n(x) = |\cdot| * \rho_n(x) = \int_{\mathbb{R}} |x - y|\, \rho_n(y)\, dy. \tag{97}$$

This sequence provides smooth approximations to $|\cdot|$, in fact $\phi_n(\cdot) \to |\cdot|$ uniformly on compact subsets of $\mathbb{R}$. Let us consider the family of nonempty compact sets (see Figure 5.1):

$$K_n := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ y \leq \phi_n(x)\right\} \tag{98}$$

By the very definition of the Pompeiu-Hausdorff distance it follows that

$$\mathbf{d\!l}(K, K_n) \leq \sup_{x \in [-1,1]} |\phi_n(x) - |x|| \to 0 \tag{99}$$

as $n \to \infty$. Therefore, to have the announced claim only remains to show that sets K_n are sleek.

Proposition 5.5. *The set K_n defined by* (98) *is sleek.*

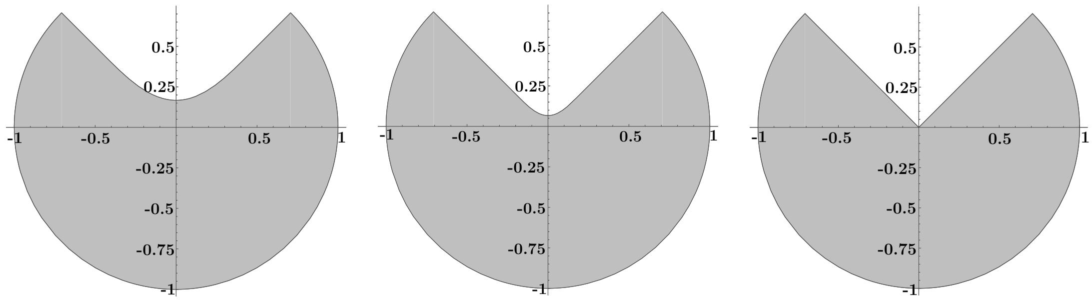

Figure 5.1: K_n converges to K

Proof. By using elementary properties of convolutions we have that $\phi_n(\cdot)$ is increasing (decreasing) on $]0,+\infty[$ ($]-\infty,0[$) with $\phi_n(1)=1$ and

$$0<\phi_n(0)=\frac{1}{n}-2\int_0^{1/n}\int_0^y \rho_n(s)\,ds\,dy<\frac{1}{n} \tag{100}$$

Hence there exits $\hat{x}_n \in\ ]0,1[$ such that $\phi_n(\hat{x}_n)^2+\hat{x}_n^2=1$ and, since $\phi_n(\cdot)$ is an even function, we have that the boundary of the unit ball in $\mathbb{R}^2$ cuts the graph of $\phi_n(\cdot)$ at points

$$(\hat{x}_n,\phi_n(\hat{x}_n)),\quad (-\hat{x}_n,\phi_n(\hat{x}_n)). \tag{101}$$

The boundary of K_n is smooth (and consequently K_n is sleek) at every point different from $(\pm\hat{x}_n,\phi_n(\hat{x}_n))$. Therefore to complete the proof it suffices to show that K_n is sleek at the nonsmooth points (101). This fact follows by a direct computation providing the desired equality

$$\begin{aligned} T_{K_n}(\hat{x}_n,\phi_n(\hat{x}_n)) &= \left\{(v,w)\in\mathbb{R}^2 : \hat{x}_n v+\phi_n(\hat{x}_n)w\le 0,\ w\le\phi_n'(\hat{x}_n)v\right\}\\ &= C_{K_n}(\hat{x}_n,\phi_n(\hat{x}_n)), \end{aligned}$$

and in an analogous way for the point $(-\hat{x}_n,\phi_n(\hat{x}_n))$. □

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