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Minmax Convex Pairs

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This paper gives some general criteria for recognizing minmax convex pairs, i.e. *pairs* (X, Y) of *convex* subsets of a Hilbert space for which the bilinear *minmax* equality $\inf_{x\in X}\sup_{y\in Y}\langle x,y\rangle = \sup_{y\in Y}\inf_{x\in X}\langle x,y\rangle$ holds. Based on new notions of *normality*, *consistency*, *closure feasibility* and *boundary negligibility* of pairs of convex sets, such criteria yield new minmax equalities besides the old ones. Included are the celebrated Classical Minmax Theorem (von Neumann 1928, [8] and Kneser 1952, [7]) for bounded, closed convex sets, Fenchel's Minmax Theorem for polyhedral convex sets (Fenchel 1951, [2]), the Fenchel Minmax Theorem for strongly feasible pairs of convex sets (Borwein, Lewis [1]) and new minmax theorems (for locally compact sets, for polar sets,...). In the last section minmax convex pairs are used to characterize bounded, closed convex sets. Further investigation on minmax convex pairs relatively to closed hyperplanes and on attainment of extrema in their associated bilinear minmax equalities are left to subsequent papers, [3], [4] respectively.

1. Two criteria for minmax convex pairs

Let E be a (real) Hilbert space with inner product $\langle\cdot,\cdot\rangle$. A couple (X, Y) of nonempty convex subsets of E is called a **minmax (convex) pair**, whenever the following equality

(**bilinear minmax equality**)
$$\inf_{x\in X}\sup_{y\in Y}\langle x,y\rangle = \sup_{y\in Y}\inf_{x\in X}\langle x,y\rangle$$

holds. Let δ_Y^+ be the *convex (or upper) support function* of Y, and δ_X^- the *concave (or lower) support function* of X, from E to extended real line $\overline{\mathbb{R}} := \mathbb{R}\cup\{\pm\infty\}$, defined by

(**support functions**)
$$\delta_Y^+(x) := \sup_{y\in Y}\langle x,y\rangle \qquad \text{and} \qquad \delta_X^-(y) := \inf_{x\in X}\langle x,y\rangle.$$

A pair (X, Y) is declared **consistent** if[1]

$$(A)\qquad \begin{cases} \text{either} & \forall\,\varepsilon\in\mathbb{R}_{++} \quad \inf_{B_\varepsilon(X)}\delta_Y^+ < +\infty \quad (\textit{inf-consistency}) \\ \text{or} & \forall\,\varepsilon\in\mathbb{R}_{++} \quad \sup_{B_\varepsilon(Y)}\delta_X^- > -\infty \quad (\textit{sup-consistency}). \end{cases}$$

A pair (X, Y) is said to be **normal** if the following two properties hold:

(B) $\inf_X \delta_Y^+ = \sup_{\varepsilon>0}\inf_{B_\varepsilon(X)}\delta_Y^+$ (*inf-normality*),

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[1]$\mathbb{R}_{++} := \{t\in\mathbb{R} : t>0\}$. And $B_\varepsilon(X) := X + B_\varepsilon(0)$, $B_\varepsilon(Y) := Y + B_\varepsilon(0)$ are open neighborhoods of X, Y, generated by the ball $B_\varepsilon(0) := \{v\in E : \|v\| < \varepsilon\}$ with radius $\varepsilon > 0$.

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(C) $\sup_Y \delta_X^- = \inf_{\varepsilon>0} \sup_{B_\varepsilon(Y)} \delta_X^-$ (*sup-normality*).

Theorem 1.1 (A minmax criterion for consistent pairs of convex sets).
A couple of nonempty convex subsets of a Hilbert space is a minmax pair iff it is consistent and normal.

The properties (A) consistency, (B) inf-normality and (C) sup-normality which characterize minmax convex pairs, are independent (see the following examples A, B and C, respectively). And, the extrema in the bilinear minmax equality, associated to a minmax convex pair, are not necessary attained. This can be seen from example D.

(Ex A**)** Set $E := \mathbb{R}^2$, $X := \{(a,b) \in \mathbb{R}^2 : a = 1, b \geq 1\}$ and $Y := \{(a,b) \in \mathbb{R}^2 : a \geq 1, b = -1\}$. Since

$$\{\delta_Y^+ = +\infty\} \supset \{(a,b) \in \mathbb{R}^2 : a > 0\} \quad \text{and} \quad \{\delta_X^- = -\infty\} \supset \{(a,b) \in \mathbb{R}^2 : b < 0\},$$

one has $\inf_X \delta_Y^+ = \inf_{B_\varepsilon(X)} \delta_Y^+ = +\infty$ and $\sup_Y \delta_X^- = \sup_{B_\varepsilon(Y)} \delta_X^- = -\infty$ for every real number $0 < \varepsilon < 1$. Thus the pair (X, Y) is normal, but not consistent. □

(Ex B**)** Set $E := \mathbb{R}$, $X := \{x \in \mathbb{R} : x > 0\}$ and $Y := \mathbb{R}$. Then $\delta_Y^+ = \delta_{\{0\}}$ and $\delta_X^- = -\delta_{\{t \in \mathbb{R}: t \geq 0\}}$.[2] Hence

$$\inf_X \delta_Y^+ = +\infty > 0 = \inf_{B_\varepsilon(X)} \delta_Y^+ \quad \text{and} \quad \sup_{B_\varepsilon(Y)} \delta_X^- = \sup_Y \delta_X^- = 0$$

for every real number $\varepsilon > 0$. Thus (X, Y) is consistent and sup-normal, but not inf-normal. □

(Ex C**)** Set $E := \mathbb{R}$, $X := \mathbb{R}$ and $Y := \{y \in \mathbb{R} : y < 0\}$. Then $\delta_Y^+ = \delta_{\{t \in \mathbb{R}: t \geq 0\}}$ and $\delta_X^- = -\delta_{\{0\}}$. Hence

$$\inf_X \delta_Y^+ = \inf_{B_\varepsilon(X)} \delta_Y^+ = 0 \quad \text{and} \quad \sup_{B_\varepsilon(Y)} \delta_X^- = 0 > \sup_Y \delta_X^- = -\infty$$

for every real number $\varepsilon > 0$. Thus (X, Y) is consistent and inf-normal, but not sup-normal. □

(Ex D**)** Set $E := \mathbb{R}$, $X := \{x \in \mathbb{R} : 1 < x < 2\}$ and $Y := X$. Then for every $(x, y) \in X \times Y$ one has $\delta_Y^+(x) = 2x$ and $\delta_X^-(y) = y$; hence $\delta_Y^+(x) = 2x > \inf_X \delta_Y^+ = 2 = \sup_Y \delta_X^- > y = \delta_X^-(y)$. Thus (X, Y) is a minmax pair, but no extrema are attained. □

A pair (X, Y) is said to be **feasible**, if either $\inf_X \delta_Y^+ < +\infty$ or $\sup_Y \delta_X^- > -\infty$. Moreover, a pair (X, Y) is said to be **closure feasible**, if $(\overline{X}, \overline{Y})$ is feasible[3] or, equivalently, if

$$(A') \qquad \begin{cases} \text{either} & \inf_{\overline{X}} \delta_Y^+ < +\infty \quad (\textit{closure inf-feasibility}) \\ \text{or} & \sup_{\overline{Y}} \delta_X^- > -\infty \quad (\textit{closure sup-feasibility}). \end{cases}$$

Obviously, a minmax convex pair is (closure) feasible pair; on the other hand, a (closure) feasible pair is consistent. Therefore, Theorem 1.1 still holds if consistency is replaced by (closure) feasibility.

Call (X, Y) **boundary negligible**, if the following two properties hold:

[2]For an arbitrary subset A of E, the *indicator function* δ_A is a function from E to $\overline{\mathbb{R}}$, defined by $\delta_A(y) := 0$ (resp. $:= +\infty$), if $y \in A$ (resp. $y \notin A$).

[3]The topological closure of a subset A of E is denoted by $\overline{A}$.

(D) $\inf_X \delta_Y^+ \leq \delta_Y^+(x)$ for every $x \in \overline{X} \setminus X$ (*boundary inf-negligibility*),
(E) $\sup_Y \delta_X^- \geq \delta_X^-(y)$ for every $y \in \overline{Y} \setminus Y$ (*boundary sup-negligibility*).

Every minmax convex pair is boundary negligible. Clearly, a pair (X, Y) is boundary inf- (resp. sup-) negligible, if X (resp. Y) is closed or Y (resp. X) is bounded.

Theorem 1.2 (A minmax criterion for feasible pairs of convex sets). *Let X, Y be nonempty convex subsets of a Hilbert space. Then the following properties are equivalent:*

(1) (X, Y) *is minmax,*
(2) (X, Y) *is closure feasible, inf-normal and boundary sup-negligible,*
(3) (X, Y) *is closure feasible, sup-normal and boundary inf-negligible.*

Each one of the following corollaries is logically equivalent to Theorem 1.2.

Corollary 1.3 (A minmax criterion for closed sets). *Let X and Y be nonempty convex sets with Y closed (resp. X closed). Then (X, Y) is minmax iff it is both closure feasible and inf- (resp. sup-) normal.* □

Corollary 1.4 (A minmax criterion for closure sets). *Let X and Y be nonempty convex sets. Then*

(4) $(X, \overline{Y})$ *is minmax iff* (X, Y) *is closure feasible and inf-normal,*
(5) $(\overline{X}, Y)$ *is minmax iff* (X, Y) *is closure feasible and sup-normal.* □

It is worthwhile to notice that (*closure*) *feasibility* cannot be replaced by *consistency* in Theorem 1.2 and in its Corollary 1.3 and 1.4 (see following ex. E). Similarly, *closure feasibility* cannot be replaced by *feasibility* in Corollary 1.4, as we can see from ex. F.

(Ex E**)** Define $E := \mathbb{R}^3$ and let X, Y denote the nonempty closed convex subsets of E defined by $X := \{(a, b, c) \in \mathbb{R}^3 : b \leq -a^2, c = 1\}$ and $Y := \{(a, b, c) \in (\mathbb{R}_{++})^3 : a(b + c) \geq 1, b(c + a) \geq 1, c(a + b) \geq 1\}$. Since

$$X \subset \mathbb{R}^2 \times \mathbb{R}_{++} \subset \{\delta_Y^+ = +\infty\} \quad \text{and} \quad Y \subset \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R} \subset \{\delta_X^- = -\infty\},$$

(X, Y) is not a minmax pair. On the other hand, (X, Y) is "boundary sup-negligible" (because Y is closed). Moreover, (X, Y) is "inf-normal", since

$$B_\varepsilon(X) \subset \mathbb{R}^2 \times \mathbb{R}_{++} \subset \{\delta_Y^+ = +\infty\} \quad \text{for every real number} \quad 0 < \varepsilon \leq 1.$$

Finally, (X, Y) is "sup-consistent", since for every natural number $n \geq 1$

$$(n, \frac{1}{2n}, n) \in Y \quad \text{and} \quad \delta_X^-(n, -\frac{1}{2n}, n) > -\infty.$$

□

(Ex F**)** Define $E := \mathbb{R}^3$ and let X, Y denote the nonempty convex subsets of E defined by $X := \{(a, b, c) \in \mathbb{R}^3 : a \leq 0, b = 0, c = 1\}$ and $Y := \{(a, b, c) \in \mathbb{R}^3 : a > 0, b = 0\}$. Since

$$X \subset \mathbb{R}^2 \times \{1\} \subset \{\delta_Y^+ = +\infty\} \quad \text{and} \quad Y \subset \mathbb{R}_{++} \times \mathbb{R}^2 \subset \{\delta_X^- = -\infty\},$$

(X, Y) is not a feasible pair; but $(X, \overline{Y})$ is minmax, because $\inf_X \delta_Y^+ = +\infty$ and $\sup_{\overline{Y}} \delta_X^- = \sup_{\overline{Y} \setminus Y} \delta_X^- = +\infty$. □

In the sequel "**convex set**" always stand for "**convex subset of a Hilbert space E**", if not otherwise explicitly specified. For later reference, we recall the following equalities relating well known notions and support functions:

(6) $Y^\circ := \{\delta_Y^+ \le 1\}$ (*polar set* of Y)
(7) $Y^- := \{\delta_Y^+ \le 0\}$ (*polar cone* of Y)
(8) $\operatorname{bar}^+ Y := \{v \in E : \delta_Y^+(v) < +\infty\}$ (*upper barrier cone* of Y)
(9) $\overline{\operatorname{conv}}\, A = \{y \in E : \delta_A^-(v) \le \langle v, y\rangle \le \delta_A^+(v)$ for every $v \in E\}$ (*Peano's convex hull formula*)[4]
(10) $Y^\infty := \{v \in E : Y + v \subset Y\} = (\operatorname{bar}^+(Y))^-$ (*asymptotic cone of closed convex nonempty sets*).

2. Proof of Theorems 1.1 and 1.2

Obviously, if (X, Y) is a minmax convex pair, then so are $(Y, -X)$, $(-Y, X)$ and $(-X, -Y)$. Moreover, since

$$-\delta_X^- = \delta_{-X}^+, \; -\delta_Y^+ = \delta_{-Y}^- \text{ and } -\inf_X \delta_Y^+ = \sup_{-X} \delta_Y^- = \sup_X \delta_{-Y}^-, \; -\sup_Y \delta_X^- = \inf_{-Y} \delta_X^+ = \inf_Y \delta_{-X}^+,$$

if a pair (X, Y) is inf- (resp. sup-) $\bullet$, then $(Y, -X)$, $(-Y, X)$ are sup- (resp. inf-) $\bullet$; where the bullet stands for consistent, closure feasible, feasible, boundary negligible or normal.

We will demonstrate Theorem 1.1 that normality and consistency characterize minmax convex pairs. For that purpose and for proving Theorem 1.2, the following lemma is basic.

Lemma 2.1. *Let X and Y be nonempty convex sets and let $\varepsilon \in \mathbb{R}_{++}$. Then*

$$\sup_{\overline{Y}} \delta_X^- \ge \inf_{B_\varepsilon(X)} \delta_Y^+ \tag{1}$$

if (and only if) either $\inf_{B_\varepsilon(X)} \delta_Y^+ < +\infty$ or $\sup_{\overline{Y}} \delta_X^- > -\infty$.

Proof. For proving the inequality (1), we will show that, for every real number a such that $\inf_{B_\varepsilon(X)} \delta_Y^+ > a$, we have $\sup_{\overline{Y}} \delta_X^- \ge a$. *First case:* "$\inf_{B_\varepsilon(X)} \delta_Y^+ < +\infty$". Fix $a \in \mathbb{R}$ such that $\inf_{B_\varepsilon(X)} \delta_Y^+ > a$; then

$$\delta_Y^+ > a - \delta_{B_\varepsilon(X)}. \tag{$*1$}$$

Obviously, the epigraph $\operatorname{epi} \delta_Y^+ := \{(x, t) \in E \times \mathbb{R} : \delta_Y^+(x) \le t\}$ and the strict hypograph $\operatorname{hypo_s}(a - \delta_{B_\varepsilon(X)}) := \{(x, t) \in E \times \mathbb{R} : x \in B_\varepsilon(X) \text{ and } t < a\}$ are nonempty convex subsets of the Hilbert space $E \times \mathbb{R}$; they are disjoint and, in addition, $\operatorname{epi} \delta_Y^+$ is a cone and $\operatorname{hypo_s}(a - \delta_{B_\varepsilon(X)})$ is open. Therefore, there is a closed hyperplane H of $E \times \mathbb{R}$ with $0 \in H$ such that

$$\operatorname{epi} \delta_Y^+ \text{ and } \operatorname{hypo_s}(a - \delta_{B_\varepsilon(X)}) \text{ are separated by } H. \tag{$*2$}$$

By $\inf_{B_\varepsilon(X)} \delta_Y^+ < +\infty$, the support function δ_Y^+ is finite at some point of the open set $B_\varepsilon(X)$. Therefore the separating hyperplane H is not "vertical"; that is, there is an $\bar{y} \in E$ such that $H = \{(x, t) \in E \times \mathbb{R} : t = \langle x, \bar{y}\rangle\}$ and

$$\delta_Y^+(x) \ge \langle x, \bar{y}\rangle \ge a - \delta_{B_\varepsilon(X)}(x) \quad \text{for every } x \in E. \tag{$*3$}$$

[4] $\overline{\operatorname{conv}}\, A$ denotes the smallest closed convex subset of E containing A. See Peano (1888 [9], p. 132) for a first appearance of statement (9). An useful reformulation of (7) is "$y \in \overline{\operatorname{conv}}\, A \iff \langle v, y\rangle \le \delta_A^+(v)$ for every $v \in E$".

By equality (9§1), (∗3) implies

$$\bar{y} \in \overline{Y} \quad \text{and} \quad \langle x, \bar{y}\rangle \geq a \quad \text{for every } x \in X. \tag{$*4$}$$

Thus $\sup_{\overline{Y}} \delta^-_X \geq a$. *Second case:* "$\inf_{B_\varepsilon(X)} \delta^+_Y = +\infty$ and $\sup_Y \delta^-_X > -\infty$". Then

$$\text{bar}\,^+Y \cap B_\varepsilon(X) = \emptyset \quad \text{and} \quad \exists\, \bar{y} \in Y \text{ such that } \delta^-_X(\bar{y}) > -\infty. \tag{$*5$}$$

Hence, the barrier cone $\text{bar}\,^+Y$ is strongly separated from X by a closed hyperplane of E. Therefore, there are a $\bar{w} \in E$ and a real number b such that

$$\langle v, \bar{w}\rangle \leq 0 < b \leq \langle x, \bar{w}\rangle \quad \text{for every} \quad v \in \text{bar}\,^+Y,\ x \in X. \tag{$*6$}$$

By equality (10§1), this amounts to

$$\bar{w} \in (\overline{Y})^\infty \quad \text{and} \quad \delta^-_X(\bar{w}) > 0. \tag{$*7$}$$

By the definition of asymptotic cone, from $\bar{y} \in Y$ it follows that $\bar{y} + t\bar{w} \subset \overline{Y}$ for $t \in \mathbb{R}_{++}$. Therefore, sublinearity of δ^-_X, (∗5) and (∗7) combined give:

$$\sup_{\overline{Y}} \delta^-_X \geq \sup_{t>0} \delta^-_X(\bar{y} + t\bar{w}) \geq \sup_{t>0}(\delta^-_X(\bar{y}) + t\delta^-_X(\bar{w})) = +\infty.$$

Thus $\sup_{\overline{Y}} \delta^-_X \geq \inf_{B_\varepsilon(X)} \delta^+_Y$. □

Lemma 2.2. *Let X, Y be nonempty sets. Then the following properties hold:*

(2) $\sup\limits_{Y} \delta^-_X = \sup\limits_{y\in Y} \sup\limits_{\varepsilon>0} \inf\limits_{x\in B_\varepsilon(\overline{X})} \langle x, y\rangle \leq \sup\limits_{\varepsilon>0} \inf\limits_{B_\varepsilon(\overline{X})} \delta^+_Y = \sup\limits_{\varepsilon>0} \inf\limits_{B_\varepsilon(X)} \delta^+_Y \leq \inf\limits_{\overline{X}} \delta^+_Y \leq \inf\limits_{X} \delta^+_Y$,

(3) $\inf\limits_{X} \delta^+_Y = \inf\limits_{x\in X} \inf\limits_{\varepsilon>0} \sup\limits_{y\in B_\varepsilon(\overline{Y})} \langle x, y\rangle \geq \inf\limits_{\varepsilon>0} \sup\limits_{B_\varepsilon(\overline{Y})} \delta^-_X = \inf\limits_{\varepsilon>0} \sup\limits_{B_\varepsilon(Y)} \delta^-_X \geq \sup\limits_{\overline{Y}} \delta^-_X \geq \sup\limits_{Y} \delta^-_X$,

(4) $\sup\limits_{Y} \delta^-_X = \sup\limits_{Y} \delta^-_{\overline{X}} \leq \sup\limits_{\overline{Y}} \delta^-_X = \sup\limits_{\overline{Y}} \delta^-_{\overline{X}} \leq \inf\limits_{\overline{X}} \delta^+_{\overline{Y}} = \inf\limits_{\overline{X}} \delta^+_Y \leq \inf\limits_{X} \delta^+_{\overline{Y}} = \inf\limits_{X} \delta^+_Y$.

Proof. It follows immediately from elementary equalities

$$\delta^+_{\overline{Y}} = \delta^+_Y = \inf_{\varepsilon\in\mathbb{R}_{++}} \delta^+_{B_\varepsilon(Y)} \quad , \quad \delta^-_{\overline{X}} = \delta^-_X = \sup_{\varepsilon\in\mathbb{R}_{++}} \delta^-_{B_\varepsilon(X)}$$

and from the general minmax inequality "$\sup_A \inf_B f \leq \inf_B \sup_A f$". □

Proposition 2.3 (Necessary conditions). *A minmax pair of nonempty sets is feasible, closure feasible, consistent, normal and boundary negligible.*

Proof. Normality and boundary negligibility of a minmax pair (X, Y) follow from (2)-(4) of Lemma 2.2. Clearly, every minmax pair is feasible; hence, closure feasible and, consequently, consistent. □

Proposition 2.4 (Sufficient conditions). *A pair of nonempty convex sets is minmax, if it is inf-normal, boundary sup-negligible and either inf-consistent or closure sup-feasible.*

Proof. By definition, the closure sup-feasibility (A') and the inf-consistency (A) of a pair (X, Y) of nonempty convex set imply that conditions of Lemma 2.1 are satisfied for every $\varepsilon \in \mathbb{R}_{++}$. Hence, property (1) of Lemma 2.1 gives: $\inf_X \delta_Y^+ \geq \sup_{\overline{Y}} \delta_X^- \geq \sup_{\varepsilon>0} \inf_{B_\varepsilon(X)} \delta_Y^+$; therefore, by the inf-normality (B) of (X, Y) we infer:

$$\inf_X \delta_Y^+ = \sup_{\overline{Y}} \delta_X^- \quad (\textit{the pair } (X, \overline{Y}) \textit{ is minmax}). \tag{$*2$}$$

On the other hand, boundary sup-negligibility amounts to

$$\sup_{\overline{Y}} \delta_X^- = \sup_Y \delta_X^-. \tag{$*3$}$$

Hence, from ($*2$) and ($*3$) it follows that (X, Y) is a minmax pair. □

Proof of Theorem 1.1. "*if part*". *First case*: "(X, Y) is normal and inf-consistent". From (2) and (3) of Lemma 2.2, it follows that normality of (X, Y) implies boundary negligibility of (X, Y). Hence, Proposition 2.4 shows that (X, Y) is minmax. *Second case*: "(X, Y) is normal and sup-consistent". Then $(-Y, X)$ is normal and inf-consistent. Hence, *first case* gives that $(-Y, X)$ is minmax; consequently, (X, Y) is minmax. "*only if part*": It follows from Proposition 2.3. □

Proof of Theorem 1.2. "(1§1) $\Longrightarrow$ (2§1) & (3§1)": it follows from Proposition 2.3. "(2§1) $\Longrightarrow$ (1§1)": observe that closure feasibility of (X, Y) implies that (X, Y) is either inf-consistent or closure sup-feasible. Hence, by Proposition 2.4 the pair (X, Y) is minmax. "(3§1) $\Longrightarrow$ (1§1)": by (3§1) the pair $(-Y, X)$ is closure feasible, inf-normal and boundary sup-negligible. Applying "(2§1) $\Longrightarrow$ (1§1)" to pair $(-Y, X)$ entails $(-Y, X)$ is minmax; thus (X, Y) is minmax. □

Remark 2.5 (Normality is preserved by closure). (2) and (3) of Lemma 2.2 imply:

(5) (X, Y) is inf-normal $\Leftrightarrow$ so is $(X, \overline{Y}) \Leftrightarrow (\overline{X}, Y)$ is inf-normal and (X, Y) is boundary inf-negligible

(6) (X, Y) is sup-normal $\Leftrightarrow$ so is $(\overline{X}, Y) \Leftrightarrow (X, \overline{Y})$ is sup-normal and (X, Y) is boundary sup-negligible. □

Remark 2.6 (Minmax pairs are preserved by closure). If the pair (X, Y) is minmax, (4) of Lemma 2.2 implies that the couples $(X, \overline{Y})$, $(\overline{X}, Y)$ and $(\overline{X}, \overline{Y})$ are minmax pairs as well. More clearly:

(7) (X, Y) is minmax iff $(\overline{X}, Y)$ and $(X, \overline{Y})$ are minmax

(8) (X, Y) is minmax iff $(\overline{X}, \overline{Y})$ is minmax and (X, Y) boundary negligible. □

Remark 2.7 (Generally, minmax pairs are not preserved by translations). Suitable translations transform arbitrary pairs of convex sets into minmax pairs; in fact, $(X+v, Y+w)$ is a minmax pair for every $(-v, -w) \in X \times Y$, because $0 \in (X+v) \cap (Y+w)$. Therefore, infinitely many minmax convex pairs are not preserved by translations, since there are infinitely many pairs of convex sets which are not minmax. □

3. Some classes of minmax convex pairs

Recall the upper and lower **barrier cones**:

$$\mathrm{bar}^+(Y) := \{x \in E : \delta^+_Y(x) < +\infty\} \quad \text{and} \quad \mathrm{bar}^-(X) := \{y \in E : \delta^-_X(y) > -\infty\}.$$

Then feasibility of a pair (X, Y) can alternatively be expressed by the following condition:

$$(AA) \qquad \text{either} \quad X \cap \mathrm{bar}^+(Y) \neq \emptyset \quad \text{or} \quad Y \cap \mathrm{bar}^-(X) \neq \emptyset.$$

Call a couple (X, Y) **strongly feasible**[5], if

$$\begin{cases} \text{either} & (X + v\mathbb{R}_{++}) \cap \mathrm{bar}^+ Y \neq \emptyset \quad \text{for every } v \in E \quad \textit{(strongly inf-feasible)} \\ \text{or} & (Y + w\mathbb{R}_{++}) \cap \mathrm{bar}^- X \neq \emptyset \quad \text{for every } w \in E \quad \textit{(strongly sup-feasible).} \end{cases}$$

Clearly, strong feasibility implies feasibility. Inf-normality (B) (and similarly for sup-normality) of a couple (X, Y) can be restated in the equivalent form:

$$(BB) \qquad \forall(\alpha, \epsilon) \in \mathbb{R} \times \mathbb{R}_{++} \text{ either } X \cap \{\delta^+_Y \le \alpha + \epsilon\} \neq \emptyset \text{ or } \mathrm{dist}\,(X, \{\delta^+_Y \le \alpha\}) \neq 0.$$[6]

Lemma 3.1 (A basic lemma for separation properties). *Let A, B be closed convex sets such that* $\mathrm{bar}^+ A + \mathrm{bar}^+ B = E$. *Then either* $A \cap B \neq \emptyset$ *or* $\mathrm{dist}\,(A, B) \neq 0$.

Proof. Let $\mathrm{dist}\,(A, B) = 0$. Then there are sequences $\{a_n\}_n \subset A$ and $\{b_n\}_n \subset B$ such that $\lim_n \|a_n - b_n\| = 0$ and $\|a_n - b_n\| \le 1$. The sequence $\{a_n\}_n$ is weakly bounded in the Hilbert space E. In fact, choose an arbitrary $a \in E$. Since $\mathrm{bar}^+ A + \mathrm{bar}^+ B = E$, there is $(\bar{a}, \bar{b}) \in \mathrm{bar}^+ A \times \mathrm{bar}^+ B$ such that $a = \bar{a} + \bar{b}$. Then we have

$$\langle a_n, a\rangle = \langle a_n, \bar{a}\rangle + \langle a_n, \bar{b}\rangle = \langle a_n, \bar{a}\rangle + \langle a_n - b_n, \bar{b}\rangle + \langle b_n, \bar{b}\rangle \le \delta^+_A(\bar{a}) + \|\bar{b}\| + \delta^+_B(\bar{b})$$

for every natural number n. Hence, $\{a_n\}_n$ is weakly bounded; consequently, it has a weakly convergent subsequence $\{a_{n_k}\}_k$. Assume $\{a_{n_k}\}_k$ weakly convergent to $\hat{a}$. Then, by $\lim_k \|a_{n_k} - b_{n_k}\| = 0$, we infer $\{b_{n_k}\}_k \rightharpoonup \hat{a}$. Therefore, the closedness of convex sets A and B implies $\hat{a} \in A \cap B$. Thus $A \cap B \neq 0$. □

Lemma 3.2. *A pair (X, Y) of nonempty convex sets is inf-normal in any of the following cases:*

(1) *X and Y are polyhedral convex subsets of a finite-dimensional space,*
(2) *X is closed, locally compact and $X^\infty \cap Y^- = \{0\}$,*
(3) *X is closed and (X, Y) strongly feasible.*

Proof. *1st case.* Without loss of generality, assume E finite dimensional. Since Y is polyhedral, the support function δ^+_Y is polyhedral (see [1], p. 98); hence all its level sets $\{\delta^+_Y \le \alpha\}$ are polyhedral. Therefore the polyhedral sets X and $\{\delta^+_Y \le \alpha\}$ either are disjoint or have positive distance, because "two nonempty disjoint finite dimensional polyhedral

[5]Strong feasibility holds, if at least one of the following conditions is satisfied: "either X or Y is bounded", "$\mathrm{int}\, Y \cap \mathrm{bar}^- X \neq \emptyset$", "$Y \cap \mathrm{int}\,(\mathrm{bar}^- X) \neq \emptyset$", "$\mathrm{int}\, X \cap \mathrm{bar}^+ Y \neq \emptyset$", "$X \cap \mathrm{int}\,(\mathrm{bar}^+ Y) \neq \emptyset$". In terms of *core*, the strong feasibility can be expressed by "either $0 \in \mathrm{core}(X - \mathrm{bar}^+ Y)$ or $0 \in \mathrm{core}(Y - \mathrm{bar}^- X)$".
[6]Recall $\mathrm{dist}\,(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}$; if either A or B is empty, their distance is defined as equal to $+\infty$.

sets are strongly separated by some closed hyperplane" (see [10], p. 175). Thus (BB) hold.

2nd case. For every non empty sublevel of δ_Y^+, one has $\{\delta_Y^+ \leq \alpha\}^\infty = \{\delta_Y^+ \leq 0\} = Y^-$. Hence for proving (BB) under (2) it is enough to use Dieudonné's separation theorem: "two closed nonempty convex subsets of a locally convex vector space having no common non-null asymptotic vector are strongly separated by some closed hyperplane, whenever they are disjoint and at least one of them is locally compact" (see [5], p. 103).

3rd case. First subcase: "Let (X, Y) be strongly sup-feasible". Strong sup-feasibility means "$\operatorname{cone} Y - \operatorname{bar}^- X = E$". Since

$$\operatorname{bar}^+(\{\delta_Y^+ \leq \alpha\}) \supset \operatorname{cone} Y \quad \text{and} \quad \operatorname{bar}^+ X = -\operatorname{bar}^- X,$$

we have that $\operatorname{bar}^+(\{\delta_Y^+ \leq \alpha\}) + \operatorname{bar}^+ X = E$. Thus, Lemma 3.1 entails (BB). *Second subcase*: "Let (X, Y) be strongly inf-feasible". Since (X, Y) is strongly inf-feasible, also the pair $(X, \overline{Y})$ is strongly inf-feasible. Hence $(-\overline{Y}, X)$ is strongly sup-feasible. Applying the *first subcase* to $(-\overline{Y}, X)$, we have that $(-\overline{Y}, X)$ is inf-normal; therefore, since X is closed, Corollary 1.3 entails that it is a minmax pair. Now, from Theorem 1.1 it follows that $(-\overline{Y}, X)$ is sup-normal; consequently $(X, \overline{Y})$ is inf-normal, that is: (X, Y) is inf-normal. □

From Lemma 3.2 we get the classical minmax theorem - proved by von Neumann (1928) for finite dimensions and by Kneser (1952) in the infinite dimensional case - and Fenchel's minmax theorem for polyhedral sets (Fenchel 1951, [2], Chap. III, §6), crucial for linear programming. Besides we get a minmax theorem (for strongly feasible pairs) - one of the by-products of the Fenchel duality (see [1], Ex. 16, p. 81) - and minmax theorems for locally compact convex sets and, finally, for polar sets, polar cones and closed vector subspaces.

Proposition 3.3 (Classical minmax theorem). *A couple of nonempty convex sets is minmax, whenever at least one of them is both bounded and closed.*

Proof. Let (X, Y) be a pair of nonempty convex sets. Without loss of generality, let X be closed and bounded. Boundedness of X amounts to have a full barrier cone $\operatorname{bar}^- X = E$; hence strong feasibility of (X, Y) obviously holds, because Y is not empty. Hence, by (3) from Lemma 3.2 it follows that (X, Y) is inf-normal. On the other hand, boundedness of X also ensures the continuity of its support functions. Thereby the boundary sup-negligibility (E) of (X, Y) holds. Finally, (2) of Theorem 1.2 says that (X, Y) is a minmax pair. □

Proposition 3.4 (Fenchel minmax theorem for polyhedral sets). *Every feasible couple of nonempty polyhedral convex sets is a minmax pair.*

For example, every couple (X, Y) of nonempty polyhedral sets such that $0 \in X \cup Y$, is feasible; consequently, it is minmax.

Proof. *1st case*: "E has finite dimension". Let (X, Y) be a feasible couple of nonempty polyhedral convex sets[7]. Applying Lemma 3.2(1) to (X, Y) and $(-Y, X)$ we get that

[7] For pairs of polyhedral sets, consistency=feasibility.

(X,Y) is normal. Hence, Theorem 1.1 entails that (X,Y) is a minmax convex pair. *2nd case*: "E has arbitrary dimension". By definition, the polyhedral sets X and Y are intersections of finitely many closed half-spaces: there are $\{a_i\}_{i=1}^{n+m} \subset E$ and $\{\alpha_i\}_{i=1}^{n+m} \subset \mathbb{R}$ such that $X = \cap_{i=1}^{n}\{x \in E : \langle x, a_i\rangle \le \alpha_i\}$ and $Y = \cap_{i=n+1}^{m}\{y \in E : \langle y, a_i\rangle \le \alpha_i\}$. Let E_0 be the finite dimensional subspace of E which is spanned by vectors $\{a_i\}_{i=1}^{n+m}$, and let $E_0^\perp$ be its orthogonal complement in E. Clearly,

(∗1) $X_0 := X \cap E_0$ and $Y_0 := X \cap E_0$ are polyhedral convex sets of the finite dimensional space E_0,
(∗2) $X = X_0 + E_0^\perp$ and $Y = Y_0 + E_0^\perp$,
(∗3) $\inf_X \delta_Y^+ = \inf_X(\delta_{Y_0}^+ + \delta_{E_0}) = \inf_{X\cap E_0} \delta_{Y_0}^+ = \inf_{X_0} \delta_{Y_0}^+$
(∗4) $\sup_Y \delta_X^- = \sup_Y(\delta_{X_0}^- - \delta_{E_0}) = \sup_{Y\cap E_0} \delta_{X_0}^- = \sup_{Y_0} \delta_{X_0}^-$.

Now, since (X,Y) is a feasible pair, from (∗3) and (∗4) the feasibility of (X_0,Y_0) follows. Therefore, applying the first case to the couple (X_0,Y_0) of polyhedral sets, we obtain that (X_0,Y_0) is a minmax pair in E_0. Hence (∗3) and (∗4) imply that (X,Y) is a minmax pair. □

Proposition 3.5 (Fenchel minmax theorem for strongly feasible pairs of convex sets). *Every strongly feasible couple of nonempty closed convex sets is a minmax pair.*

For example, every couple (X,Y) of nonempty closed convex sets such that $0 \in \operatorname{int} X \cup \operatorname{int} Y$, is strongly feasible; consequently, it is minmax.

Proof. Let X and Y be nonempty closed convex sets with (X,Y) strongly feasible. From (3) of Lemma 3.2 it follows that (X,Y) is inf-normal. Hence, Corollary 1.3 yields (X,Y) is minmax. □

Proposition 3.6 (Minmax pairs related to locally compact convex sets). *A feasible pair (X,Y) of nonempty closed convex sets is minmax, if at least one of the following properties holds:*

(4) *X is locally compact (in particular, finite dimensional) and $X^\infty \cap Y^- = \{0\}$*
(5) *Y is locally compact (in particular, finite dimensional) and $Y^\infty \cap (-X)^- = \{0\}$.*

Proof. *1st case*: "(4) holds". From (2) of Lemma 3.2 it follows that (X,Y) is inf-normal. Hence, by Corollary 1.3, (X,Y) is a minmax pair. *2nd case*: "(5) holds". Applying the *1st case* to pair $(-Y,X)$ entails $(-Y,X)$ is a minmax pair. Therefore, (X,Y) is minmax. □

Proposition 3.7 (Minmax pairs related to polar sets, polar cones and vector spaces). *Let X, Y be nonempty convex sets with Y closed and $0 \in Y$. Then (X,Y) is minmax iff*

$$\forall\,(\alpha,\varepsilon) \in \mathbb{R}_{++}\times\mathbb{R}_{++} \qquad \textit{either} \quad X \cap (\alpha+\varepsilon)Y^\circ \neq \emptyset \qquad \textit{or} \quad \operatorname{dist}(X, \alpha Y^\circ) \neq 0. \tag{6}$$

In particular, if Y is a closed cone, the pair (X,Y) is minmax iff

$$\textit{either } X \cap Y^- \neq \emptyset \textit{ or } \operatorname{dist}(X, Y^-) \neq 0. \tag{7}$$

Consequently, if Y is a closed vector space, the pair (X,Y) is minmax iff

$$\textit{either } X \cap Y^\perp \neq \emptyset \textit{ or } \operatorname{dist}(X, Y^\perp) \neq 0. \tag{8}$$

Proof. *First case*: "Y is a polar set" (i.e. Y is closed convex and $0 \in Y$). The pair (X, Y) is feasible, because $0 \in Y \cap \mathrm{bar}^{-}X$. Moreover, since $0 \in Y$, one has $\delta_Y^+ \geq 0$. Hence, inf-normality (BB) of (X, Y) is expressed by (6), because $\mu Y^\circ = \{\delta_Y^+ \leq \mu\}$ for $\mu > 0$. Now, since Y is closed, Corollary 1.3 entails the required equivalence. *Second case*: "Y is a polar cone" (i.e. it is a closed nonempty convex cone). In this case we have $Y^- = Y^\circ$ and $\mu Y^\circ = Y^-$ for $\mu > 0$. Therefore, from the *first case* follows the required equivalence. *Third case*: "Y is a closed vector space". We have $Y^- = Y^\perp$. Therefore, the *second case* gives the desired equivalence. □

Remark 3.8 (Minmax pairs with respect to whole space). By Proposition 3.7 we have that (X, E) is a minmax pair iff either $0 \in X$ or $0 \notin \overline{X}$. In particular, $(\overline{X}, E)$ and $(E, \overline{X})$ are minmax pairs.

This "elementary" fact provides a proof of the well known formula on Fenchel biconjugation of indicator functions: "$(\delta_Y^+)^* = \delta_{\overline{Y}}$ for every convex set Y". To wit, for every $\hat{y} \in E$ one has: $(\delta_Y^+)^*(\hat{y}) := \sup_{x\in E}\big(\langle x, \hat{y}\rangle - \delta_Y^+(x)\big) = \sup_{x\in E}\inf_{y\in Y}\langle x, \hat{y} - y\rangle = \sup_{x\in E}\inf_{y\in \hat{y}-\overline{Y}}\langle x, y\rangle = \inf_{y\in\hat{y}-\overline{Y}}\sup_{x\in E}\langle x, y\rangle = \delta_{\overline{Y}}(\hat{y})$. □

4. A characterization of bounded sets by minmax convex pairs

In this section both boundedness and closedness of convex subsets of a Hilbert space are characterized by minmax pairs.

Proposition 4.1 (Boundedness characterization). *A nonempty convex set X of a Hilbert space E is bounded if and only if (X, Y) is minmax for every nonempty bounded convex set Y of E.*

Proof. To verify the "only if part", let Y and X be bounded, nonempty convex set and observe two facts. First, by Classical Minmax Theorem (see Prop. 3.3) the pair $(X, \overline{Y})$ is minmax; hence

$$\inf_{x\in X}\sup_{y\in Y}\langle x, y\rangle = \inf_{x\in X}\sup_{y\in\overline{Y}}\langle x, y\rangle = \sup_{y\in\overline{Y}}\inf_{x\in X}\langle x, y\rangle. \tag{$*1$}$$

Second, by Hormander's Theorem [6] the support function δ_X^- of the bounded set X is continuous, hence (X, Y) is boundary sup-negligible, i.e.

$$\sup_{y\in\overline{Y}}\inf_{x\in X}\langle x, y\rangle = \sup_{y\in Y}\inf_{x\in X}\langle x, y\rangle. \tag{$*2$}$$

In conclusion, combining $(*1)$ and $(*2)$ we have that (X, Y) is a minmax pair.

To verify the "if part", let (X, Y) be minmax for every nonempty, bounded convex set Y. We will show that X is weakly bounded. Let v be an arbitrary non-null element of E. The segment $Y_v := \{tv : 0 < t < 1\}$ is nonempty convex and bounded; hence, by hypothesis, (X, Y_v) is minmax, that is:

$$\inf_X \delta_{Y_v}^+ = \sup_{Y_v}\delta_X^-. \tag{$*3$}$$

Since $\delta_{Y_v}^+ \geq 0$, from $(*3)$ it follows that $\sup_{Y_v}\delta_X^- > -1$; consequently, $\delta_X^-(\bar{t}v) \geq -1$ for some real number $\bar{t} > 0$. Therefore, $\inf_{x\in X}\langle x, v\rangle > -\frac{1}{\bar{t}}$. Now, v being an arbitrary element of E, one has that X is weakly bounded and, consequently, norm bounded, in virtue of Banach-Steinhaus theorem. □

Proposition 4.2 (Closedness characterization). *A nonempty convex set X of a Hilbert space E is closed, if and only if $(X - v, E)$ is a minmax pair for every $v \in E$.*

Proof. Applying (8) of Proposition 3.7 with the couple $(X - v, E)$ we infer: "$\forall_{v \in E}$ $(X - v, E)$ is minmax" $\iff$ "$\forall_{v \in E}$ either $v \in X$ or $v \notin \overline{X}$" $\iff$ "X is closed". □

Theorem 4.3 (Boundedness and closedness characterization). *A nonempty convex set X of a Hilbert space E is both bounded and closed, if and only if $(X - v, Y)$ is a minmax pair for every nonempty convex set Y of E and for every $v \in E$.*

Proof. The "only if part" follows from Proposition 3.3, the Classical Minmax Theorem. The "if part" follows from Proposition 4.1 and 4.2. □

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