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The Mazur Intersection Problem

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Dedicated to the memory of Simon Fitzpatrick.

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Bounded closed convex sets in Euclidean space can be characterised by two distinct ball separation properties which in a general normed linear space are not equivalent. The study of these two separation properties has led to interesting developments in classifying those Banach spaces where these different characterisations of bounded closed convex sets hold.

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1. Introduction – The Mazur problem

A major theme in the study of Banach spaces is to determine classes of those spaces where particular Euclidean space properties hold. But an intriguing complication in such analysis is that different properties which we are accustomed to see as equivalent in Euclidean space can prove to be quite distinct in more general spaces.

Mazur [7] studied the Euclidean space property that every bounded closed convex set is an intersection of closed balls. Classes of spaces with this property were studied by Phelps [8] and a characterisation of such spaces was given by Giles *et al.* [3]. Surprisingly, Sevilla and Moreno [9] showed that spaces with this property are not necessarily Asplund. However, recently Granero *et al.* [4, 5] contributed an idea which sheds considerable light on the study by discerning two distinct separation properties involved.

In Euclidean space, a subset C is bounded closed and convex if and only if

(i) for every point $x \notin C$ there exists a closed ball B such that $x \notin B$ and $C \subseteq B$.
(ii) for every hyperplane H where $d(C, H) > 0$ there exists a closed ball B such that $B \cap H = \emptyset$ and $C \subseteq B$.

In the earlier study of the Mazur Intersection Property, concentration was on the first separation characterisation (i). But Granero *et al.* have drawn attention to the second separation characterisation (ii) which had been given considerable attention by Chen and Lin [2]. So then for a Banach space we can study three separate classes of subsets:

$\mathcal{H}$ the family of all bounded closed convex subsets,
$\mathcal{M}$ the family of all intersections of closed balls, and
$\mathcal{P}$ the family of all Mazur sets; (bounded closed convex subset C is a *Mazur set* if for every closed hyperplane H where $d(C, H) > 0$ there exists a closed ball B such that $B \cap H = \emptyset$ and $C \subseteq B$).

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Clearly in general $\mathcal{P} \subseteq \mathcal{M} \subseteq \mathcal{H}$. A Banach space where $\mathcal{M} = \mathcal{H}$ has the *Mazur Intersection Property*. Granero *et al.* call a Banach space where $\mathcal{P} = \mathcal{M}$ a *Mazur space*, and they give examples of spaces where $\mathcal{P} \neq \mathcal{M} = \mathcal{H}$ and $\mathcal{P} = \mathcal{M} \neq \mathcal{H}$, [4, p. 186].

2. Weak* denting and semi weak* denting points

We denote by $S(X)$ the unit sphere and by $B(X)$ the closed unit ball of a normed linear space X. The two separation properties are associated with two extreme point properties of $B(X^\star)$ in the dual space $X^\star$. To discuss them we need the following notation

Given $f \in S(X^\star)$ and $0 < \delta < 1$, the set

$$S\ell\,(B(X), f, \delta) \equiv \{x \in S(X) : f(x) > 1 - \delta\}$$

is called a *slice* of $B(X)$. A slice of $B(X^\star)$ generated by an element of $\widehat{X}$ is called a *weak* slice* of $B(X^\star)$

Given $\epsilon > 0$, an element $f \in S(X^\star)$ is said to be an *ϵ-weak* denting point* of $B(X^\star)$ if $B[f; \epsilon]$ contains a weak* slice of $B(X^\star)$ containing f; the element $f \in S(X^\star)$ is said to be a *weak* denting point* of $B(X^\star)$ if it is an ϵ-weak* denting point of $B(X^\star)$ for every $\epsilon > 0$.

We will see that the weak* denting points are associated with separation property (ii).

Chen and Lin [1] introduced semi weak* denting points which are associated with separation property (i). An element $f \in S(X^\star)$ is said to be a *semi weak* denting point* of $B(X^\star)$ if for every $\epsilon > 0$ there exists a weak* slice $S\ell$ of $B(X^\star)$ such that $\operatorname{diam}(\{f\} \cup S\ell) < \epsilon$.

It is clear that every weak* denting point of $B(X^\star)$ is a semi weak* denting point of $B(X^\star)$ and every element in the closure of the set of weak* denting points is a semi weak* denting point.

To develop characterisations of these two extreme point properties we need the following theory. The first is a consequence of the Parallel Hyperplane Lemma [8, p. 978].

Lemma 2.1. *In a normed linear space X, if for $f, g \in S(X^\star)$ and $0 < \epsilon < 2$*

$$S\ell\left(B(X), f, 1 - \frac{\epsilon}{2}\right) \subseteq \{x \in X : g(x) > 0\}$$

then $\|f - g\| \leq \epsilon$.

Proof. If $x \in B(X)$ and $g(x) = 0$ then $f(x) \leq \frac{\epsilon}{2}$ and $|f(x)| \leq \frac{\epsilon}{2}$. By the Parallel Hyperplane Lemma either

$$\|f + g\| \leq \epsilon \quad \text{or} \quad \|f - g\| \leq \epsilon.$$

For any $0 < \delta < 1 - \frac{\epsilon}{2}$ there exists $x \in S(X)$ such that

$$f(x) > 1 - \delta > \frac{\epsilon}{2} \quad \text{and so } g(x) > 0.$$

Then $\|f + g\| \geq |(f + g)(x)| > 1 - \delta$ and we conclude that $\|f - g\| \leq \epsilon$. □

We have the following conditions for weak* denting points and semi weak* denting points of the dual ball.

Lemma 2.2.

(*i*) *Given* $0 < \epsilon < 1$, *the element* $f \in S(X^\star)$ *is an* ϵ*-weak** *denting point of* $B(X^\star)$ *if there exists a closed ball* B_ϵ *such that*

$$S\ell\left(B(X), f, 1 - \frac{\epsilon}{2}\right) \subseteq B_\epsilon \subseteq \{x \in X : f(x) > 0\}$$

(*ii*) *An element* $f \in S(X^\star)$ *is a semi weak** *denting point of* $B(X^\star)$ *if given* $0 < \epsilon < 1$ *there exists a closed ball* B_ϵ *such that*

$$S\ell\left(B(X), f, 1 - \frac{\epsilon}{2}\right) \subseteq B_\epsilon \quad \text{and } 0 \notin B_\epsilon.$$

Proof. (*i*) If $B_\epsilon \equiv B[x_0; r]$ then $r \leq \|x_0\|$ and $f(x_0 + ry) > 0$ for all $y \in B(X)$, which implies that $f(x_0) > r$; that is,

$$f \in S\ell\left(B(X^\star), \frac{x_0}{\|x_0\|}, 1 - \frac{r}{\|x_0\|}\right).$$

But also if $g \in S\ell\left(B(X^\star), \frac{x_0}{\|x_0\|}, 1 - \frac{r}{\|x_0\|}\right)$ then $g(x_0) > r$, which implies that $g(z) > 0$ for all $z \in B[x_0; r]$. So $g(z) > 0$ for all $z \in S\ell\left(B(X), f, 1 - \frac{\epsilon}{2}\right)$. If also $g \in S(X^\star)$ then by Lemma 2.1 we deduce that $\|f - g\| \leq \epsilon$ and

$$S\ell\left(B(X^\star), \frac{x_0}{\|x_0\|}, 1 - \frac{r}{\|x_0\|}\right) \subseteq B[f; \epsilon]$$

and we conclude that f is an ϵ-weak* denting point of $B(X^\star)$.

(*ii*) Consider $0 < \eta < \frac{\epsilon}{7}$. There exists a closed ball B_η such that

$$S\ell\left(B(X), f, 1 - \frac{\eta}{2}\right) \subseteq B_\eta \quad \text{and } 0 \notin B_\eta.$$

By the Separation Theorem there exists $g \in S(X^\star)$ such that $B_\eta \subseteq \{x \in X : g(x) > 0\}$. So by Lemma 2.1 we have $\|f - g\| \leq \eta$. If $g(x) > \frac{3}{2}\,\eta$ then $f(x) \geq g(x) - \eta > \frac{\eta}{2}$, so

$$S\ell\left(B(X), g, 1 - \frac{3}{2}\,\eta\right) \subseteq B_\eta \subseteq \{x \in X : g(x) > 0\}.$$

By Lemma 2.2(*i*), g is a 3η-weak* denting point of $B(X^\star)$; that is, $B[g; 3\eta]$ contains a weak* slice $S\ell$ of $B(X^\star)$ containing g. So diam $\{\{f\} \cup S\ell\} < 7\eta < \epsilon$, and we conclude that f is a semi weak* denting point of $B(X^\star)$. □

An element $f \in S(X^\star)$ is said to be a *weak** *strongly exposed point* of $B(X^\star)$ if there exists $x \in S(X)$ such that $f \in \partial\|x\|$ and given $0 < \epsilon < 1$ there exists a weak* slice of $B(X^\star)$ containing f and generated by x of diameter less than ϵ. It is clear from Lemma 2.2(*i*) that if there exists $x_0 \in S(X)$ and for $0 < \epsilon < 1$, B_ϵ is a closed ball centred on λx_0 for $\lambda > 0$ then $f \in \partial\|x_0\|$ and is a weak* strongly exposed point of $B(X^\star)$.

3. Approximate Fréchet differentiability

It was Sullivan [10] who employed a form of approximate Fréchet differentiability which is useful for a discussion of our problem.

Given $\epsilon > 0$, we denote by $M_\epsilon(X)$ the set of points in $S(X)$ such that for some $\delta(\epsilon, x) > 0$

$$\sup_{\substack{0<\lambda<\delta \\ y\in S(X)}} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} < \epsilon.$$

The set of points in $S(X)$ where the norm is Fréchet differentiable is precisely $\bigcap_{\epsilon>0} M_\epsilon(X)$.

Given $x \in S(X)$, the *subdifferential* of the norm at x is the set

$$\partial\|x\| \equiv \{f \in S(X^\star) : f(x) = 1\}.$$

This is a nonempty convex weak* compact subset of $X^\star$. The subdifferential mapping $x \mapsto \partial\|x\|$ of $S(X)$ into subsets of $S(X^\star)$ is *weak* upper semicontinuous*; that is, given $x \in S(X)$ and weak* open subset W of $X^\star$ such that $\partial\|x\| \subseteq W$ then there exists $\delta > 0$ such that $\partial\|B(x;\delta) \cap S(X)\| \subseteq W$.

The set of points of approximate Fréchet differentiability has a useful characterisation.

Lemma 3.1 [3, Lemma 2.1, p. 109]**.** *For a normed linear space X, given $0 < \epsilon < 1$ and $x \in S(X)$, the following are equivalent*

(i) $x \in M_\epsilon(X)$,
(ii) *x determines a slice of $B(X^\star)$ of diameter less than ϵ,*
(iii) *there exists $\delta(\epsilon, x) > 0$ such that* diam $\partial\|B(x;\delta) \cap S(X)\| < \epsilon$.

Proof. *(i)* $\implies$ *(ii)* Suppose that for all $n \in \mathbb{N}$, diam $S\ell\left(B(X^\star), x, \frac{1}{n^2}\right) \geq \epsilon$. Then there exist $f_n, g_n \in S\ell\left(B(x^\star), x, \frac{1}{n^2}\right)$ such that $\|f_n - g_n\| > \epsilon - \frac{1}{n^2}$. For each $n \in \mathbb{N}$ there exists $y_n \in S(X)$ such that $(f_n - g_n)(y_n) > \epsilon - \frac{1}{n}$. Then

$$\begin{aligned} \|x + \tfrac{1}{n} y_n\| + \|x - \tfrac{1}{n} y_n\| &> f_n(x + \tfrac{1}{n} y_n) + g_n(x - \tfrac{1}{n} y_n) \\ &> 2 - \tfrac{2}{n^2} + \tfrac{1}{n}(f_n - g_n)(y_n) > 2 - \tfrac{3}{n^2} + \tfrac{\epsilon}{n}, \end{aligned}$$

so

$$\frac{\|x + \frac{1}{n} y_n\| + \|x - \frac{1}{n} y_n\| - 2}{\frac{1}{n}} > \epsilon - \frac{3}{n} \quad \text{for all } n \in \mathbb{N}.$$

(ii) $\implies$ *(iii)* Consider $S\ell(B(X^\star), x, \delta)$ such that diam $S\ell(B(X^\star), x, \delta) < \epsilon$. Then for all $y \in S(X)$ such that $\|x - y\| < \delta$ we have

$$|f_y(x) - 1| \leq |f_y(x - y)| \leq \|x - y\| < \delta \quad \text{for all } f_y \in \partial\|y\|.$$

Then $f_y(x) > 1 - \delta$ which implies that

$$\partial\|B(x;\delta) \cap S(X)\| \subseteq S\ell(B(X^\star), x, \delta), \qquad \text{so diam } \partial\|B(x;\delta \cap S(X)\| < \epsilon.$$

$(iii) \Longrightarrow (i)$ Now $\dfrac{x+\lambda y}{\|x+\lambda y\|} \in B(x;\delta) \cap S(X)$ for all $0 < |\lambda| < \dfrac{\delta}{2}$ and $y \in S(X)$. So for

$f \in \partial \left\| \dfrac{x+\lambda y}{\|x+\lambda y\|} \right\|$ and $g \in \partial \left\| \dfrac{x-\lambda y}{\|x-\lambda y\|} \right\|$ we have $\|f-g\| < \epsilon$.

Then $\dfrac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} \leq (f-g)(y) \leq \|f-g\| < \epsilon$ for $0 < \lambda < \dfrac{\delta}{2}$. □

It follows that, given $0 < \epsilon < 1$, the set $M_\epsilon(X)$ is always an open subset of $S(X)$. Clearly, for a Banach space X if $M_\epsilon(X)$ is dense in $S(X)$ for all $0 < \epsilon < 1$ then by the Baire Category Theorem, the norm of X is Fréchet differentiable at the points of a dense G_δ subset of $S(X)$.

4. Linkage to the separation properties

We have the following characterisation of semi weak* denting points.

Theorem 4.1. *In any Banach space X, the following are equivalent:*

(i) *$f \in S(X^\star)$ is a semi weak* denting point of $B(X^\star)$.*
(ii) *for every bounded subset C of X with $\inf f(C) > 0$ there exists a closed ball B such that $C \subseteq B$ and $0 \notin B$,*
(iii) *given $0 < \epsilon < 1$ there exists an $x \in S(X)$ and $\delta(\epsilon) > 0$ such that $\partial \|B(x;\delta) \cap S(X)\| \subseteq B(f;\epsilon)$.*
(iv) *$f \in \bigcap_{\epsilon>0} \overline{\partial \|M_\epsilon(X)\|}$.*

Proof. $(ii) \Longrightarrow (i)$ Follows immediately from Lemma 2.2(ii).

$(i) \Longrightarrow (iii)$ Given $0 < \epsilon < 1$, consider $B(f;\epsilon)$. Since f is a semi weak* denting point of $B(X^\star)$ there exists an $x \in S(X)$ and $0 < \gamma < 1$ and weak* slice $S\ell\,(B(X^\star), x, \gamma) \subseteq B(f;\epsilon)$. Then $\partial\|x\| \subseteq S\ell\,(B(X^\star), x, \gamma)$ and since the subdifferential mapping $x \mapsto \partial\|x\|$ is weak* upper semicontinuous there exists a $\delta(\epsilon) > 0$ such that

$$\partial \|B(x;\delta) \cap S(X)\| \subseteq S\ell\,(B(X^\star), x, \gamma) \subseteq B(f;\epsilon).$$

$(iii) \Longrightarrow (iv)$ Given $0 < \eta < \dfrac{\epsilon}{2}$ there exists $\delta(\eta) > 0$ such that $\partial \|B(x;\delta) \cap S(X)\| \subseteq B(f;\eta)$. But by Lemma 3.1 we have that $x \in M_\epsilon(X)$ and so $d\,(f, \partial \|M_\epsilon(X)\|) < \eta$.

$(iv) \Longrightarrow (ii)$ We may assume that $C \subseteq B(X)$. For if $C \subseteq B[0;m]$ then $\dfrac{1}{m} C \subseteq B(X)$. If there exists a closed ball $B[z;r]$ such that $\dfrac{1}{m} C \subseteq B[z;r]$ and if $\inf f\,(B[z;r]) > 0$ then $C \subseteq B[mz;mr]$ and $\inf f\,(B[mz;mr]) > 0$. So for $C \subseteq B(X)$ write $\epsilon \equiv \dfrac{1}{3}\inf f(C)$. Since $f \in \overline{\partial \|M_\epsilon(X)\|}$ there exists an $x \in M_\epsilon(X)$ and $f_x \in \partial\|x\|$ such that $\|f - f_x\| < \epsilon$. That is, there exists $0 < \delta < 1$ such that

$$\sup_{\substack{y \in S(X) \\ 0<\lambda<\delta}} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} < \epsilon$$

which implies that

$$\sup_{y\in B(X)} \frac{\|x+\delta y\| + \|x-\delta y\| - 2}{\delta} \leq \epsilon.$$

Now $C \subseteq B(\frac{x}{\delta}; \frac{1}{\delta} - \epsilon)$ for if not there exists $y \in C$ such that $\left\|\frac{x}{\delta} - y\right\| > \frac{1}{\delta} - \epsilon$. Then

$$\begin{aligned}\frac{\|x+\delta y\| + \|x-\delta y\| - 2}{\delta} &= \frac{\|x+\delta y\| - \|x\|}{\delta} + \|\frac{x}{\delta} - y\| - \frac{1}{\delta} \\ &\geq f_x(y) + \frac{1}{\delta} - \epsilon - \frac{1}{\delta} \\ &\geq f(y) - \|f - f_x\| - \epsilon > \epsilon\end{aligned}$$

a contradiction. □

We should note that the continuity property (iii) was established by Phelps [8, Lemma 4.1, p. 979]. The characterisation $(i) \iff (ii)$ has been given by Chen and Lin [1, Proposition 3, p. 194]. The equivalences $(ii) \iff (iii) \iff (iv)$ were given in Giles *et al.* [3, Lemma 2.2, p. 112]. However, this current presentation is much more concise.

We notice that the characterisation $(i) \iff (iv)$ tells us that the set of semi weak* denting points is always a closed set.

It is clear from the proof of Theorem $(iv) \iff (i)$ and our previous observations from Lemma 2.2(i) that $f \in \bigcap_{\epsilon>0} \partial\|M_\epsilon(X)\|$ if and only if f is a weak* strongly exposed point of $B(X^\star)$.

Corollary 4.2. *An element $f \in S(X^\star)$ is a semi weak* denting point of $B(X^\star)$ if and only if for every $0 < \epsilon < 1$, $\overline{S\ell\left(B(X), f, 1-\frac{\epsilon}{2}\right)}$ is an intersection of closed balls.*

Proof. Suppose that f is a semi weak* denting point of $B(X^\star)$ and given $0 < \epsilon < 1$ consider $\overline{S\ell\left(B(X), f, 1-\frac{\epsilon}{2}\right)}$ and $x_0 \notin \overline{S\ell\left(B(X), f, 1-\frac{\epsilon}{2}\right)}$.
If f does not separate x_0 from $\overline{S\ell\left(B(X), f, 1-\frac{\epsilon}{2}\right)}$ then $x_0 \notin B(X)$ so $B(X)$ is a closed ball containing $S\ell\left(B(X), f, 1-\frac{\epsilon}{2}\right)$ and not containing x_0.

If f separates x_0 from $\overline{S\ell\left(B(X), f, 1-\frac{\epsilon}{2}\right)}$ then since f is a semi weak* denting point of $B(X^\star)$ it follows from Theorem 4.1$(i) \Longrightarrow (ii)$ that there exists a closed ball B such that $B \supseteq \overline{S\ell\left(B(X), f, 1-\frac{\epsilon}{2}\right)}$ and $x_0 \notin B$.

The converse follows immediately from Lemma 2.2(ii). □

A similar result was given by Chen and Lin [1, Prop. 5, p. 197] but our proof is more direct.

A property of Mazur spaces now follows.

Corollary 4.3. *On the dual sphere $S(X^\star)$ of a Mazur space X, every semi weak* denting point is a weak* denting point.*

Proof. If $f \in S(X^\star)$ is a semi weak* denting point of $B(X^\star)$ then from Corollary 4.2, for every $0 < \epsilon < 1$, $\overline{S\ell\left(B(X), f, 1 - \frac{\epsilon}{2}\right)} \in \mathcal{M}$. But for a Mazur space $\mathcal{P} = \mathcal{M}$. So by Lemma 2.2($i$) we see that f is a weak* denting point of $B(X^\star)$. □

This result was first given by Granero *et al.* [5, Cor. 2.4, p. 413] and they raised the problem of determining whether this property characterises Mazur spaces.

The characterisation of Banach spaces with Mazur Intersection Property given in Giles *et al.* [3, Lemma 2.1, p. 114] follows readily.

Theorem 4.4. *A Banach space X has the Mazur Intersection Property if and only if the set of weak* denting points of $B(X^\star)$ is dense in $S(X^\star)$.*

Proof. If the set of weak* denting points of $B(X^\star)$ is dense in $S(X^\star)$ then every point $f \in S(X^\star)$ is a semi weak* denting point of $B(X^\star)$ so from Theorem 4.1(i) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) we have that X has the Mazur Intersection Property.

Conversely, if X has the Mazur Intersection Property we have from Theorem 4.1(ii) $\Longleftrightarrow$ (i) that every point $f \in S(X^\star)$ is a semi weak* denting point of $B(X^\star)$. Given $0 < \epsilon < 1$, consider D_ϵ the union of points in $S(X^\star)$ which are in the interior of weak* slices of $B(X^\star)$ of diameter less than ϵ. Now D_ϵ is open and dense in $S(X^\star)$. By the Baire Category Theorem $\bigcap\limits_{\epsilon>0} D_\epsilon$ is dense in $S(X^\star)$, but these are precisely the weak* denting points of $B(X^\star)$. □

The following denting point property is due to Chen and Lin and the proof is contained in part of [2, Theorem 1.3, p. 841].

Lemma 4.5. *In any normed linear space X, if $f_0 \in S(X^\star)$ is a weak* denting point of $B(X^\star)$ then given $0 < \epsilon < 1$ there exists $x \in S(X)$, $\gamma > 0$ and $k \in \mathbb{N}$ where $0 < \frac{1}{2k} < \epsilon < 1$ such that $f_0 \in S\ell\left(B(X^\star), x, \gamma\right) \subseteq S\ell\left(B(X^\star), x, 2k\gamma\right)$ and diam $S\ell\left(B(X^\star), x, 2k\gamma\right) < \epsilon$.*

Proof. Since f_0 is a weak* denting point of $B(X^\star)$ there exists $x_1 \in S(X)$ and $0 < \alpha < 1$ such that $f_0 \in S\ell\left(B(X^\star), x_1, \alpha\right)$ and diam $S\ell\left(B(X^\star), x_1, \alpha\right) < \epsilon$. Write $\beta \equiv 1 - f_0(x_1)$ and choose $\beta_1 > 0$ such that $\beta < \beta_1 < \alpha$. Again since f_0 is a weak* denting point of $B(X^\star)$ we can choose $x_2 \in S(X)$, $\gamma > 0$ and $k \in \mathbb{N}$ where $2\gamma < \frac{1}{k} < \epsilon < 1$ such that $f_0 \in S\ell\left(B(X^\star), x_2, \gamma\right)$ and

$$\text{diam } S\ell\left(B(X^\star), x_2, \gamma\right) < \min\left(\frac{\alpha - \beta_1}{2k}, \beta_1 - \beta\right).$$

Now for $f \notin S\ell\left(B(X^\star), x_1, \beta_1\right)$,

$$\|f_0 - f\| \geq f_0(x_1) - f(x_1) \geq 1 - \beta - (1 - \beta_1) = \beta_1 - \beta > \text{ diam } S\ell\left(B(X^\star), x_2, \gamma\right),$$

so $S\ell\left(B(X^\star), x_2, \gamma\right) \subseteq S\ell\left(B(X^\star), x_1, \beta_1\right)$.
Choose $f_{x_2} \in \partial\|x_2\|$. For $f_1 \in B(X^\star)$ where $f_1(x_1) \geq 1 - \alpha$ there exists $0 < \lambda < 1$ such that

$$g = \lambda f_{x_2} + (1 - \lambda) f_1 \quad \text{and} \quad g(x_2) = 1 - \gamma.$$

Then $\|f_{x_2} - g\| = (1-\lambda)\|f_{x_2} - f_1\|$ and so

$$1-\lambda = \frac{\|f_{x_2} - g\|}{\|f_{x_2} - f_1\|} \le \frac{\alpha - \beta_1}{2k\|f_{x_2} - f_1\|}.$$

But $\|f_{x_2} - f_1\| \ge f_{x_2}(x_1) - f_1(x_1) > 1 - \beta_1 - (1-\alpha) = \alpha - \beta_1$, so $1-\lambda < \frac{1}{2k}$.

Then $f_1(x_2) = \dfrac{(g - \lambda f_{x_2})(x_2)}{1-\lambda} = \dfrac{g(x_2) - \lambda}{1-\lambda} = \dfrac{1-\gamma-\lambda}{1-\lambda} = 1 - \dfrac{\gamma}{1-\lambda} < 1 - 2k\gamma.$

We conclude that

$$f_0 \in S\ell\,(B(X^\star), x_2, \gamma) \subseteq S\ell\,(B(X^\star), x_2, 2k\gamma) \subseteq S\ell\,(B(X^\star), x_1, \alpha)$$

so diam $S\ell\,(B(X^\star), x_2, 2k\gamma) < \epsilon$. □

We have the following characterisation of weak* denting points.

Theorem 4.6. *In any Banach space X, the following are equivalent.*

(*i*) $f \in S(X^\star)$ *is a weak* denting point of* $B(X^\star)$,

(*ii*) *for every bounded subset C of X with inf $f(C) > 0$ there exists a closed ball B of X such that $C \subseteq B$ and inf $f(B) > 0$.*

Proof. $(ii) \implies (i)$ Follows immediately from Lemma 2.2(i).

$(i) \implies (ii)$ As in Theorem 4.1$(iv) \implies (ii)$ we may assume that $C \subseteq B(X)$. Again for $C \subseteq B(X)$ write $\epsilon \equiv \dfrac{1}{3}\mathrm{inf} f(C)$. From Lemma 4.5 there exists an $x \in S(X)$, $\gamma > 0$, and $k \in \mathbb{N}$ where $2\gamma < \frac{1}{k} < \epsilon < 1$ such that $f \in S\ell\,(B(X^\star), x, \gamma)$ and diam $S\ell\,(B(X^\star), x, 2k\gamma) < \epsilon$. Then from Lemma 3.1 we deduce that

$$\sup_{y \in B(X)} \frac{\|x + k\gamma y\| + \|x - k\gamma y\| - 2}{k\gamma} \le \epsilon$$

As in Theorem 4.1$(iv) \implies (ii)$ we have that $C \subseteq B\left(\frac{1}{k\gamma}x, \frac{1}{k\gamma} - \epsilon\right)$. But also

$$\inf f\left(B\left(\frac{x}{k\gamma}, \frac{1}{k\gamma} - \epsilon\right)\right) \ge f\left(\frac{x}{k\gamma}\right) - \left(\frac{1}{k\gamma} - \epsilon\right) > \frac{1-\gamma}{k\gamma} - \frac{1}{k\gamma} - \epsilon = \epsilon - \frac{1}{k} > 0.$$

□

It is now clear that a Banach space which is a Mazur space with the Mazur Intersection Property has very desirable geometric properties.

Theorem 4.7 [1, Proposition 5.3, p. 196]**.** *A Banach space X has $\mathcal{P} = \mathcal{H}$ if and only if every point of $S(X^\star)$ is a weak* denting point of $B(X^\star)$.*

Proof. From Theorem 4.4 we have that if X has $\mathcal{M} = \mathcal{H}$ then the set of weak* denting points of $B(X^\star)$ is dense in $B(X^\star)$. From Corollary 4.3 we have that if X has $\mathcal{P} = \mathcal{M}$ then every semi weak* denting point of $B(X^\star)$ is a weak* denting point of $B(X^\star)$.

But from Theorem 4.1$(i) \iff (iv)$ we have that the set of semi weak* denting points of $B(X^\star)$ is always a closed subset of $S(X^\star)$.

So we conclude that X with $\mathcal{P} = \mathcal{H}$ has every point of $S(X^\star)$ a weak* denting point of $B(X^\star)$.

Conversely, if every point of $S(X^\star)$ is a weak* denting point of $B(X^\star)$, it follows from Theorem 4.4 that $\mathcal{M} = \mathcal{H}$. But also from Theorem 4.6 it follows that $\mathcal{P} = \mathcal{M}$. □

Now a Banach space X with $\mathcal{P} = \mathcal{H}$ has dual $X^\star$ rotund and so X smooth. So every $f \in S(X^\star)$ which attains its norm is not only a a weak* denting point but a weak* strongly exposed point of $B(X^\star)$. This implies that the norm of X is Fréchet differentiable on $S(X)$ and so also is an Asplund space. Now it is a consequence of the Bishop-Phelps Theorem that a Banach space X with norm Fréchet differentiable on $S(X)$ is by Theorem 4.4 a space with $\mathcal{M} = \mathcal{H}$. But since Fréchet differentiability of the norm on $S(X)$ does not in general imply that $X^\star$ is rotund such a condition does not necessarily imply that the space X has $\mathcal{P} = \mathcal{H}$. However, if X is a reflexive space and has norm Fréchet differentiable on $S(X)$ then X has $\mathcal{P} = \mathcal{H}$.

Further, a Banach space X with locally uniformly rotund dual $X^\star$ has every point of $S(X^\star)$ a weak* denting point and so has $\mathcal{P} = \mathcal{H}$. Such a space is not necessarily reflexive because every Banach space with separable dual can be equivalently renormed to have locally uniformly rotund dual.

5. The dual problem

Giles *el al* [3, §3, p. 116] considered an appropriate intersection property for dual spaces called the weak* Mazur Intersection Property. We can pursue a similar investigation on the same lines as has been done above.

So then for a Banach space we denote by

1. $\mathcal{H}^*$ the family of all bounded weak* closed convex subsets in the dual space,
2. $\mathcal{M}^*$ the family of all intersections of closed dual balls in the dual space, which of course is $\mathcal{M}$ for this space.
3. $\mathcal{P}^*$ the family of all weak* Mazur sets; (a bounded weak* closed convex subset $C^\star$ in the dual space is a *weak* Mazur set* if for every weak* closed hyperplane $H^\star$ where $d(C^\star, H^\star) > 0$ there exists a closed dual ball $B^\star$ such that $B^\star \cap H^\star = \emptyset$ and $C^\star \subseteq B^\star$.

Again clearly, $\mathcal{P}^* \subseteq \mathcal{M}^* \subseteq \mathcal{H}^*$. A Banach space where $\mathcal{M}^* = \mathcal{H}^*$ has the *weak* Mazur Intersection Property.* Granero *et al.* call a Banach space where $\mathcal{P}^* = \mathcal{M}^*$ a *weak* Mazur space.*

These intersection properties for a Banach space X are closely related to extreme point properties of the ball $B(X)$. An element $x \in S(X)$ is said to be a *denting point* of $B(X)$ if for every $\epsilon > 0$ there exists a slice $B(X)$ containing x of diameter less than ϵ. Similarly, an element of $x \in S(X)$ is said to be a *semi denting point* of $B(X)$ if for every $\epsilon > 0$ there exists a slice $S\ell$ of $B(X)$ such that $\operatorname{diam}\{\{x\} \cup S\ell\} < \epsilon$.

Many of the results for denting and semi denting points can be deduced from those already given for weak* denting and semi weak* denting points.

Lemma 5.1. *For a Banach space X*

(*i*) $x \in S(X)$ *is a denting point (semi denting point) of* $B(X)$ *if and only if* $\hat{x} \in S(\widehat{X})$ *is a weak** *denting point (semi weak** *denting point) of* $B(X^{\star\star})$,

(*ii*) *if* $F \in S(X^{\star\star})$ *is a weak** *denting point (semi weak** *denting point) of* $(B(X^{\star\star})$ *then* $F \in S(\widehat{X})$ *and is the image under the natural embedding of a denting point (semi denting point) of* $B(X)$.

Proof. We will consider proofs for semi denting points

(*i*) Consider $x \in S(X)$ a semi denting point of $B(X)$ and $0 < \epsilon < 1$. Then there exists a $f \in S(X^\star)$ and $0 < \delta < 1$ such that

$$\text{diam } (\{x\} \cup S\ell\,(B(X), f, \delta)) < \epsilon.$$

For any $F \in S\ell\left(B(X^{\star\star}), \hat{f}, \delta\right)$ consider any weak* neighbourhood N of F. Then

$$W \equiv N \cap \{F \in X^{\star\star} : F(f) > 1 - \delta\}$$

is a weak* neighbourhood of F. Since $B(\widehat{X})$ is weak* dense in $B(X^{\star\star})$ then W contains an element of $B(\widehat{X})$ necessarily of $S\ell\left(B(\widehat{X}), \hat{f}, \delta\right)$. So

$$S\ell\left(B(X^{\star\star}), \hat{f}, \delta\right) \subseteq \overline{S\ell\left(B(\widehat{X}), \hat{f}, \delta\right)}^{w^\star}.$$

Now a closed ball in $X^{\star\star}$ of diameter less that 2ϵ contains $\{\hat{x}\} \cup S\ell\left(B(\widehat{X}), \hat{f}, \delta\right)$ and this ball is weak* closed and so contains $\{\hat{x}\} \cup \overline{S\ell\left(B(\widehat{X}), \hat{f}, \delta\right)}^{w^\star}$. Therefore

$$\text{diam } \left(\{\hat{x}\} \cup S\ell\left(B(X^{\star\star}), \hat{f}, \delta\right)\right) < 2\epsilon$$

and we conclude that $\hat{x}$ is a semi weak* denting point of $B(X^{\star\star})$. The converse is obvious.

(*ii*) Consider $F \in S(X^{\star\star})\backslash\widehat{X}$. Then $d(F, \widehat{X}) \equiv d > 0$ and $B(F, \frac{d}{2})$ contains no points of $\widehat{X}$. However since $B(\widehat{X})$ is weak* dense in $B(X^{\star\star})$ every weak* slice of $B(X^{\star\star})$ contains points of $B(\widehat{X})$. Therefore $B(F, \frac{d}{2})$ cannot contain any weak* slice of diameter less than $\frac{d}{2}$ and so F is not a semi weak* denting point of $B(X^{\star\star})$. So the semi weak* denting points of $B(X^{\star\star})$ are contained in $\widehat{X}$ and clearly each one is the image of a semi denting point of $B(X)$ under the natural embedding. □

It follows from Theorem 4.1(*i*) $\iff$ (*iv*) and Lemma 5.1 that the set of semi weak* denting points of $B(X^{\star\star})$ is a closed subset of $S(\widehat{X})$. So as the natural embedding is a homeomorphism we deduce that this set of semi denting points of $B(X)$ is a closed subset of $S(X)$.

We can immediately characterise semi denting and denting points.

Theorem 5.2. *For a Banach space* X *the following are equivalent.*

(*i*) $x \in S(X)$ *is a semi denting point of* $B(X)$, (*ii*) *for every bounded subset* $C^\star$ *of* $X^\star$ *with inf* $\hat{x}(C^\star) > 0$ *there exists a closed dual ball* $B^\star$ *of* $X^\star$ *such that* $C^\star \subseteq B^\star$ *and* $0 \notin B^\star$.

Theorem 5.3. *For a Banach space* X *the following are equivalent.*

(*i*) $x \in S(X)$ *is a denting point of* $B(X)$,

(*ii*) *for every bounded subset* $C^\star$ *of* $X^\star$ *with inf* $\hat{x}(C^\star) > 0$ *there exists a closed dual ball* $B^\star$ *of* $X^\star$ *such that* $C^\star \subseteq B^\star$ *and inf* $\hat{x}(B^\star) > 0$.

The proofs are direct from Lemma 5.1 and Theorem 4.1$(i) \iff (ii)$ and Theorem 4.6$(i) \iff (ii)$.

Theorem 5.2 enables us to establish the characterisation of Banach spaces with the weak* Mazur Intersection Property [3, Theorem 3.1(i) $\iff$ (v), p. 118].

Theorem 5.4. *A Banach space* X *has the weak** *Mazur Intersection Property if and only if the set of denting points of* $B(X)$ *is dense in* $S(X)$.

Proof. If the set of denting points of $B(X)$ is dense in $S(X)$ then every point $x \in S(X)$ is a semi denting point of $B(X)$ so by Theorem 5.2$(i) \iff (ii)$ X has the weak* Mazur Intersection Property.

Conversely, if X has the weak* Mazur Intersection Property then by Theorem 5.2$(ii) \iff (i)$ every point of $S(X)$ is a semi denting point of $B(X)$. Then given $0 < \epsilon < 1$, consider D_ϵ the union of points of $S(X)$ which are in the interior of slices of $B(X)$ of diameter less then ϵ. Now D_ϵ is the open and dense in $S(X)$. Since X is complete, by the Baire Category Theorem $\bigcap_{\epsilon>0} D_\epsilon$ is dense in $S(X)$ but these are precisely the denting points of $B(X)$. □

Theorem 5.3 enables us to establish a property for weak* Mazur spaces which corresponds to that for Mazur spaces given in Corollary 4.3.

Theorem 5.5. *On the sphere* $S(X)$ *of a weak** *Mazur space* X*, every semi denting point is a denting point.*

Proof. If $x \in S(X)$ is a semi denting point of $B(X)$ then by Lemma 5.1 $\hat{x} \in S(\widehat{X})$ is a semi weak* denting point of $B(X^{\star\star})$. Then from Corollary 4.2, for every $0 < \epsilon < 1$

$$S\ell\left(B(X^{\star\star}), \hat{f}, 1 - \frac{\epsilon}{2}\right) \subseteq \mathcal{M} = \mathcal{M}^\star.$$

But for a weak* Mazur space $\mathcal{P}^\star = \mathcal{M}^\star$, so by Lemma 2.2$(i)$ $\hat{x}$ is a weak* denting point of $B(X^{\star\star})$ and by Lemma 5.1, x is a denting point of $B(X)$. □

We then have a result corresponding to Theorem 4.7.

Theorem 5.6. *A Banach space* X *has* $\mathcal{P}^\star = \mathcal{H}^\star$ *if and only if every point of* $S(X)$ *is a denting point of* $B(X)$.

Proof. From Theorem 5.4 we have that if X has $\mathcal{M}^\star = \mathcal{H}^\star$ then the set of denting points of $B(X)$ is dense in $S(X)$. From Theorem 5.5 we have that if X has $\mathcal{P}^\star = \mathcal{M}^\star$ then every

semi denting point of $B(X)$ is a denting point of $B(X)$.
But we have noticed that the set of semi denting points of $B(X)$ is a closed subset of $S(X)$. So we conclude that X with $\mathcal{P}^\star = \mathcal{H}^\star$ has every point of $S(X)$ a denting point of $B(X)$.

Conversely, if every point of $S(X)$ is a denting point of $B(X)$ it follows from Theorem 5.2 that $\mathcal{M}^\star = \mathcal{H}^\star$ and from Theorem 5.3 that $\mathcal{P}^\star = \mathcal{M}^\star$. □

A Banach space X where every point of $S(X)$ is a denting point of $B(X)$, $\mathcal{P}^\star = \mathcal{H}^\star$ has been studied by Kenderov and Giles [6, Theorem 3.5, p. 472] and has been shown to have the differentiability property that every continuous convex function on an open convex subset of the dual possessing a weak* continuous subgradient at the points of a dense G_δ subset of its domain, is Fréchet differentiable at the points of a dense G_δ subset of its domain. Such a space is called a *dual differentiability space* and the class of such spaces include those with the Radon Nikodym Property. Troyanski [11, p. 306] has shown that a Banach space X where every point of $S(X)$ is a denting point can be equivalently renormed to be locally uniformly rotund.

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