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Global Maximum of a Convex Function: Necessary and Sufficient Conditions

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Dedicated to the memory of Simon Fitzpatrick.

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In this note we prove that an extended-real-valued lower semi-continuous convex function Φ defined on a reflexive Banach space X achieves its supremum on every nonempty bounded and closed convex set of its effective domain Dom Φ, if and only if the restriction of Φ to Dom Φ is sequentially continuous with respect to the weak topology on the underlying space X.

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1. Introduction and notation

"*The theory of the maximum of a convex function with respect to a closed and convex set*" as remarked by Rockafellar in [12, p. 342] "*has an entirely different character from the theory of the minimum*". A first significant difference between these two problems concerns the nature of the respective optimality condition.

In order to fix the ideas, we suppose that X is a real normed space with topological dual X^*. Given an extended-real-valued function $\Phi : X \to \mathbb{R} \cup \{+\infty\}$, let us write as usual Dom Φ for the set of all elements $x \in X$ for which $\Phi(x)$ is finite, and say that Φ is *proper* if $\mathrm{Dom}\,\Phi \neq \emptyset$. Let $\Gamma_0(X)$ denote the set of all the convex, proper and extended-real-valued functions defined on X and recall that the subdifferential of Φ at x is given by

$$\partial\Phi(x) = \{f \in X^* : \Phi(y) - \Phi(x) \geq \langle f, y - x\rangle \quad \forall y \in X\}$$

while the normal cone to C at x is defined by

$$N_C(x) = \{f \in X^* : \langle f, y - x\rangle \leq 0 \quad \forall y \in C\}.$$

Given a closed and convex set C, it is well-known that a function $\Phi \in \Gamma_0(X)$ achieves its infimum at $x \in C$ if and only if the celebrated Pshenichnyi-Rockafellar condition (see [10])

$$0 \in \partial\Phi(x) + N_C(x)$$

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holds. When considering the maximization problem of a convex function Φ over a closed and convex C, the specialization of the Pshenichnyi-Rockafellar condition provides a local necessary optimality condition. This condition says that the subdifferential of Φ at every point $x \in C$ where Φ attains its maximum lies within the normal cone to C at x, i.e.,

$$\partial\Phi(x) \subset N_C(x). \tag{1}$$

However, this simple condition is only necessary, and far from being sufficient, as shown (see [10]) by the maximization over the closed interval $C = [-1, 0]$ of the continuous and convex real function

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \max\{x^2, x\}.$$

The classical way of transforming this condition into a necessary and sufficient optimality criterion is to use more general definitions for the subdifferential and the normal cone.

Due to its importance in applications, the maximization of a convex function has recently received attention of many researchers. Without the slightest claim of being exhaustive (we refer the reader to [5] for a complete survey of the topic), we can mention here the work by Hiriart-Urruty ([8]), who gave a necessary and sufficient for $\Phi \in \Gamma_0(X)$ to achieve its maximum at $x \in C$:

$$\partial_\varepsilon\Phi(x) \subset N_{C,\varepsilon}(x) \quad \forall\varepsilon \geq 0,$$

where $\partial_\varepsilon\Phi(x)$ represents the ε-subdifferential of Φ at x and is given by

$$\partial_\varepsilon\Phi(x) = \{f \in X^* : \Phi(y) - \Phi(x) \geq \langle f, y - x\rangle - \varepsilon \quad \forall y \in X\}$$

and $N_{C,\varepsilon}(x)$ represents the set of ε-normals of C at x given by

$$N_{C,\varepsilon}(x) = \{f \in X^* \,:\, \langle f, y - x\rangle \leq \varepsilon \quad \forall y \in C\}.$$

The Hiriart-Urruty condition, initially stated in [8] has been extended to the class of tangential convex functions in [9, Theorem 2.1], and some related sufficient local optimality conditions were established in [4].

In the case of continuous functions, a different necessary and sufficient optimality condition of the same type was achieved by using the notions of γ-subdifferential of Φ at x,

$$\partial_\gamma\Phi(x) = \{f : X \to \mathbb{R} :\ \Phi(y) - \Phi(x) \geq f(y) - f(x) \quad \forall y \in X\},$$

and of γ-normal cone of C at x,

$$N_{C,\gamma}(x) = \{f : X \to \mathbb{R} :\ f(y) \leq f(x) \quad \forall y \in C\}.$$

Thus, Flores-Bazan establishes (see [1]) that any continuous function $\Phi \in \Gamma_0(X)$ achieves its supremum on C at x if and only if

$$\partial_\gamma\Phi(x) \subset N_{C,\gamma}(x).$$

An extension of this condition to the maximization of vector-valued functions has been also provided in [2].

A different approach was given by Strekalovski (first published in [13]; a much more detailed account may be found in [14]). He proved that a function $\Phi \in \Gamma_0(X)$ achieves its supremum on C at x if and only if the condition (1) is valid at all the points y at which $\Phi(y) = \Phi(x)$. Let us remark that two more general variants of this condition have been obtained in [9, Theorem 1.1] and [15, Theorem 1], and a similar condition was proved in [15, Theorem 2] for piecewise-convex continuous functions.

Recently, several articles aimed to extend this analysis to nonconvex functions, by defining appropriate notions of subdifferentials. Let us particularly mention several works. Hiriart-Urruty and Ledyaev developed a necessary and sufficient condition for global maximization for a class of locally Lipschitz functions which are regular in the sense of Clarke. Let us also mention the use by Dutta ([6]) of a general nonsmooth tool introduced by Demyanov and called convexifactors to deduce optimality conditions for the maximization of locally Lipschitz functions, as well as Mordukhovich's sharp analysis ([11]) of lower and upper subdifferentials in order to obtain optimality conditions for a broad class of nonsmooth and nonconvex functions.

The purpose of this note is to point out another noticeable difference between minimizing and maximizing convex functions. Namely, criteria of existence of a minimum on convex sets apply to a much broader class of convex functions than the criteria for the existence of the maximum. Indeed, if the underlying space X is a reflexive Banach space, using standard convex analysis techniques we know that every lower semi-continuous (*lsc*, for short) function $\Phi \in \Gamma_0(X)$ attains its infimum on every closed bounded and convex set C. However, it is well-known that in every Hilbert spaces there are bounded convex sets which do not contain any element of maximal norm. We thus conclude that the norm of a Hilbert space, although being a convex and continuous mapping, does not achieve its maximum on every bounded closed convex set. Consequently, in order to ensure for a convex function the existence of a global maximum on every bounded closed and convex set, we should impose to the function a more restrictive condition than the mere lower semi-continuity. Namely, from the Eberlein-Smulian Theorem, it follows that in a every reflexive Banach space, every convex function which is sequentially weakly continuous on its domain reaches its maximum on every bounded closed and convex set. (A detailed account of conditions under which a convex functional achieves its supremum over every compact subset of a locally convex space is given in [7]).

The goal of this note is to prove that this restrictive condition of sequentially weak continuity on the domain of the function is in fact necessary and sufficient to ensure that a function $\Phi \in \Gamma_0(X)$ achieves its supremum on every bounded closed and convex set. Namely, we establish (Proposition 2.1, Section 2) that, if X is a reflexive Banach space, for every *lsc* function $\Phi \in \Gamma_0(X)$ that is not sequentially weakly continuous on its domain, it is possible to find a bounded closed and convex set over which Φ does not reach its supremum.

The remaining part of this note is essentially devoted to the proof of the main result. We begin with fixing some definitions and notation. We assume throughout that X is a reflexive Banach space with closed closed unit ball denoted by $\mathbb{B}_X$. The topological dual of X will be denoted by X^*, and for a set $S \subset X$ we use the notations $\mathrm{co}(S)$ and $\overline{\mathrm{co}}(S)$ for the convex hull and the closed convex hull of S.

2. The main result

The main result of this note is the following.

Proposition 2.1. *Let Φ be a proper extended-real-valued lsc convex function defined on a reflexive Banach space X. The map Φ achieves its supremum on every bounded, closed and convex subset of its effective domain Dom Φ, if and only if the restriction of Φ to its domain is sequentially continuous with respect to the weak topology on X.*

As already remarked in the introduction, the Eberlein-Smulian theorem allow us to deduce that a function $\Phi \in \Gamma_0(X)$ which is sequentially continuous on its domain with respect to the weak topology achieves its supremum on every bounded closed and convex set. It remains to prove that if a proper extended-real-valued *lsc* convex function Φ fails to be sequentially weakly continuous on its domain, then there exists a bounded closed and convex subset C of Dom Φ such that Φ does not attain its supremum on C.

The following proposition proves that this result is valid in an arbitrary real normed space.

Proposition 2.2. *Let X be a real normed space, and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a proper extended-real-valued lsc convex function which is not sequentially weakly continuous on* $\operatorname{Dom}\Phi$*. Then, there exists a bounded closed and convex subset is C of* $\operatorname{Dom}\Phi$ *over which* Φ *does not reach its supremum.*

Proof of Proposition 2.2. Pick a point $\overline{x} \in \operatorname{Dom}\Phi$ at which Φ is not sequentially weakly continuous. The lack of sequentially weak continuity of Φ at $\overline{x}$ ensures the existence of a sequence $(x_n)_{n\in\mathbb{N}^*} \subset \operatorname{Dom}\Phi$ weakly converging to $\overline{x}$, and of a positive value $\zeta > 0$ such that

$$|\Phi(x_n) - \Phi(\overline{x})| \geq \zeta \quad \forall n \in \mathbb{N}^*. \tag{2}$$

Remark that, if there is a subsequence $(x_k)_{k\in\mathbb{N}^*}$ of $(x_n)_{n\in\mathbb{N}^*}$ such that

$$\Phi(x_k) \leq \Phi(\overline{x}) \tag{3}$$

for every $k \in \mathbb{N}^*$, then, by virtue of relation (2), it follows that

$$\liminf_{k\to\infty} \Phi(x_k) \leq \Phi(\overline{x}) - \zeta,$$

a relation which contradicts the fact that Φ is *lsc*.

Thus, the set of those elements x_k fulfilling relation (3) is finite. Equivalently, there exists an integer $\overline{n} \in \mathbb{N}^*$ such that

$$\Phi(x_n) - \Phi(\overline{x}) \geq \zeta \quad \forall n \geq \overline{n}. \tag{4}$$

For every $n \geq \overline{n}$, let us consider the mapping $f_n : [0,1] \to \mathbb{R}$, defined by $f_n(\lambda) = \Phi\left((1-\lambda)\overline{x} + \lambda x_n\right)$. Let us remark that $f_n(0) = \Phi(\overline{x})$ and $f_n(1) = \Phi(x_n) \geq \Phi(\overline{x}) + \zeta$. Since f is continuous (as a finite convex function defined on a real interval), it follows by the Mean Value Theorem that there is $\lambda_n \in [0,1]$ such that

$$f_n(\lambda_n) = \Phi(\overline{x}) + \frac{n-1}{n}\,\zeta.$$

Set $p_n = (1-\lambda_n)\overline{x} + \lambda_n x_n$ and remark that p_n belongs to the line segment $[\overline{x}, x_n] \subset \operatorname{Dom}\Phi$, that

$$\Phi(p_n) = \Phi(\overline{x}) + \frac{n-1}{n}\,\zeta, \tag{5}$$

and that the sequence $(p_n)_{\in\mathbb{N}^*}$ converges weakly to $\overline{x}$.

Set $C = \overline{\mathrm{co}}\,((p_n)_{n\geq\overline{n}})$; obviously C is closed and convex. On one hand, C is a bounded set as it is contained in the closed convex hull of the weakly convergent sequence $(x_n)_{n\geq\overline{n}}$. On the other, as Φ is a *lsc* convex function, we deduce from relation (5) that C lies within the level set $\Phi^{-1}((-\infty, \Phi(\overline{x}) + \zeta])$, and thus within Dom Φ.

We claim that the function Φ does not achieve its supremum on C which is a bounded closed and convex subset of its domain.

By contradiction, let us suppose that there is a point $\tilde{x} \in C$ such that

$$\Phi(x) \leq \Phi(\tilde{x}) \quad \forall x \in C. \tag{6}$$

Substituting p_n, $n \geq \overline{n}$ to x in relation (6) we obtain that

$$\Phi(\tilde{x}) \geq \Phi(p_n) = \Phi(\overline{x}) + \frac{n-1}{n}\,\zeta \quad \forall n \geq \overline{n}. \tag{7}$$

Taking the supremum over n in (7) yields

$$\Phi(\tilde{x}) \geq \sup_{n\geq\overline{n}} \Phi(\overline{x}) + \frac{n-1}{n}\,\zeta = \Phi(\overline{x}) + \zeta. \tag{8}$$

As $\tilde{x} \in \overline{\mathrm{co}}\,((p_n)_{n\geq\overline{n}})$, $\tilde{x}$ is the norm-limit of a sequence $(u_m)_{m\in\mathbb{N}^*}, u_m \in C$. For every $m \in \mathbb{N}^*$, u_m is thus a convex combination of a finite number of $\{p_n : n \geq \overline{n}\}$:

$$u_m = \sum_{n\geq\overline{n}} \alpha_{m,n} p_n,$$

for some sequence $(\alpha_{m,n})_{n\geq\overline{n}} \subseteq [0,1]$ such that $\sum_{n\geq\overline{n}} \alpha_{m,n} = 1$ (remark that for every fixed m only a finite number among the elements of the sequence $(\alpha_{m,n})_{n\geq\overline{n}}$ are non null).

Let us prove that, for every $n \in \mathbb{N}^*$, the sequence $(\alpha_{m,n})_{m\in\mathbb{N}^*}$ converges to zero.

Indeed, by the convexity of Φ we have

$$\begin{aligned} \Phi(u_m) &= \Phi\left(\sum_{n\geq\overline{n}} \alpha_{m,n} p_n\right) \\ &\leq \sum_{n\geq\overline{n}} \alpha_{m,n}\Phi(p_n) = \sum_{n\geq\overline{n}} \alpha_{m,n}\left(\Phi(\overline{x}) + \frac{n-1}{n}\,\zeta\right). \end{aligned} \tag{9}$$

For every $q \in \mathbb{N}^*$, $q \geq \overline{n}$ we may write

$$\begin{aligned} &\sum_{n\geq\overline{n}} \alpha_{m,n}\left(\Phi(\overline{x}) + \frac{n-1}{n}\,\zeta\right) \\ \leq\; & \alpha_{m,q}\left(\Phi(\overline{x}) + \frac{q-1}{q}\,\zeta\right) + \sum_{n\geq\overline{n},\, n\neq q} \alpha_{m,n}(\Phi(\overline{x}) + \zeta) \\ =\; & \Phi(\overline{x}) + \left(\sum_{n\geq\overline{n}} \alpha_{m,n} - \frac{\alpha_{m,q}}{q}\right)\zeta = \Phi(\overline{x}) + \left(1 - \frac{\alpha_{m,q}}{q}\right)\zeta. \end{aligned} \tag{10}$$

Combining relations (9) and (10) we obtain

$$\Phi(u_m) \le \Phi(\overline{x}) + \left(1 - \frac{\alpha_{m,q}}{q}\right)\zeta \qquad \forall m \in \mathbb{N}^*,\ q \ge \overline{n}.$$

Fixing q and taking the limit inferior over $m \in \mathbb{N}^*$ in the previous inequality yields

$$\begin{aligned} \liminf_{m\to\infty} \Phi(u_m) &\le \liminf_{m\to\infty}\left(\Phi(\overline{x}) + \left(1 - \frac{\alpha_{m,q}}{q}\right)\zeta\right) \\ &= \Phi(\overline{x}) + \left(1 - \frac{\limsup_{m\to\infty}\alpha_{m,q}}{q}\right)\zeta \qquad \forall q \ge \overline{n}. \end{aligned} \tag{11}$$

Now we make use of the norm-convergence of the sequence $(u_n)_{n\in\mathbb{N}^*}$ to $\tilde{x}$ and of the lower semi-continuity of the function Φ to deduce that

$$\Phi(\tilde{x}) \le \liminf_{m\to\infty} \Phi(u_m).$$

According to relation (8), the previous inequality implies that

$$\Phi(\overline{x}) + \zeta \le \liminf_{m\to\infty} \Phi(u_m). \tag{12}$$

Combining relations (11) and (12) gives

$$\limsup_{m\to\infty} \alpha_{m,q} \le 0 \qquad \forall q \ge \overline{n},$$

and as $\alpha_{m,n} \ge 0$ for every $m \in \mathbb{N}^*$ and $n \ge \overline{n}$, we obtain

$$\lim_{m\to\infty} \alpha_{m,q} = 0 \qquad \forall q \ge \overline{n}. \tag{13}$$

In order to obtain the desired contradiction, we prove now that the sequence $(u_n)_{n\in\mathbb{N}^*}$ converges weakly to $\overline{x}$. To this end, fix $f \in X^*$ and set M for the supremum of f on C. Hence

$$|\langle f, \overline{x}\rangle| \le M \quad \text{and} \quad |\langle f, p_n\rangle| \le M \quad \forall n \ge \overline{n}. \tag{14}$$

As the sequence $(p_n)_{n\in\mathbb{N}^*}$ converges weakly to $\overline{x}$, it follows that for every $\varepsilon > 0$ there is an integer $\overline{n}(\varepsilon) \ge \overline{n}$ such that

$$|\langle f, p_n\rangle - \langle f, \overline{x}\rangle| \le \varepsilon \qquad \forall n > \overline{n}(\varepsilon).$$

As a consequence we have

$$\begin{aligned} &\left|\sum_{n>\overline{n}(\varepsilon)} \alpha_{m,n}\left(\langle f, p_n\rangle - \langle f, \overline{x}\rangle\right)\right| \\ &\le \sum_{n>\overline{n}(\varepsilon)} \alpha_{m,n}\,|\langle f, p_n\rangle - \langle f, \overline{x}\rangle| \le \left(\sum_{n>\overline{n}(\varepsilon)} \alpha_{m,n}\right)\varepsilon \le \varepsilon \qquad \forall m \in \mathbb{N}^*. \end{aligned} \tag{15}$$

Recall (relation (13)) that for every fixed $q \in \mathbb{N}^*$, $\overline{n} \leq q \leq \overline{n}(\varepsilon)$, the sequence $(\alpha_{m,q})_{m\in\mathbb{N}^*}$ of positive real numbers converges to zero. Consequently, there is $\overline{m}(\varepsilon)$ such that

$$0 \leq \alpha_{m,q} \leq \frac{\varepsilon}{2M\overline{n}(\varepsilon)} \quad \forall m \geq \overline{m}(\varepsilon),\ \overline{n} \leq q \leq \overline{n}(\varepsilon).$$

Taking into account relation (14) we have

$$\begin{aligned} &\left| \sum_{n=\overline{n}}^{n=\overline{n}(\varepsilon)} \alpha_{m,n} \left(\langle f, p_n \rangle - \langle f, \overline{x} \rangle \right) \right| \qquad (16) \\ \leq &\sum_{n=\overline{n}}^{n=\overline{n}(\varepsilon)} \alpha_{m,n}(|\langle f, p_n \rangle| + |\langle f, \overline{x} \rangle|) \leq \sum_{n=\overline{n}}^{n=\overline{n}(\varepsilon)} \frac{\varepsilon}{\overline{n}(\varepsilon)} \leq \varepsilon \quad \forall m \geq \overline{m}(\varepsilon). \end{aligned}$$

Summing up relations (15) and (16) we deduce that

$$\begin{aligned} &|\langle f, u_m \rangle - \langle f, \overline{x} \rangle| \qquad (17) \\ = &\left| \left\langle f, \sum_{n\geq\overline{n}} \alpha_{m,n} p_n \right\rangle - \langle f, \overline{x} \rangle \right| = \left| \sum_{n\geq\overline{n}} \alpha_{m,n}(\langle f, p_n \rangle - \langle f, \overline{x} \rangle) \right| \\ \leq &\left| \sum_{n=\overline{n}}^{n=\overline{n}(\varepsilon)} \alpha_{m,n}(\langle f, p_n \rangle - \langle f, \overline{x} \rangle) \right| + \left| \sum_{n>\overline{n}(\varepsilon)} \alpha_{m,n}(\langle f, p_n \rangle - \langle f, \overline{x} \rangle) \right| \leq 2\varepsilon \quad \forall m \geq \overline{m}(\varepsilon). \end{aligned}$$

We have thus proved that the sequence $(u_m)_{m\in\mathbb{N}^*}$ converges weakly to $\overline{x}$; as $\tilde{x}$ is the norm-limit of the same sequence, it follows that $\tilde{x} = \overline{x}$. Making use of relation (8) we deduce that

$$\Phi(\overline{x}) < \Phi(\overline{x}) + \zeta \leq \Phi(\tilde{x}) = \Phi(\overline{x}),$$

a contradiction. Thus, the convex function Φ does not reach its supremum on C. □

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References

[1] F. Flores-Bazan: On minima of the difference functions, J. Optim. Theory Appl. 93 (1997) 525–531.

[2] F. Flores-Bazan, W. Oettli: Simplified optimality conditions for minimizing the difference of vector-valued functions, J. Optim. Theory Appl. 108 (2001) 571–586.

[3] V. F. Demyanov: Convexification and concavification of positively homogeneous functions by the same family of linear functions, Universita di Pisa, Report 3, 208 (1994).

[4] M. Dür: A parametric characterization of local optimality, Math. Methods Oper. Res. 57 (2003) 101–109.

[5] M. Dür, R. Horst, M. Locatelli: Necessary and sufficient global optimality conditions for convex maximization revisited, J. Math. Anal. Appl. 217 (1998) 637–649.

[6] J. Dutta: Optimality conditions for maximizing a locally Lipschitz function, Optimization 54 (2005) 377–389.

[7] J. Haberl: Maximization of generalized convex functionals in locally convex spaces, J. Optim. Theory Appl. 121 (2004) 327–359.

[8] J.-B. Hiriart-Urruty: From convex optimization to nonconvex optimization. Necessary and sufficient conditions for global optimality, in: Nonsmooth Optimization and Related Topics, F. H. Clarke, V. F. Demyanov, F. Gianessi (eds.), Plenum Press, New York (1989) 219–239.

[9] J.-B. Hiriart-Urruty, Y. S. Ledyaev: A note on the characterization of the global maxima of a (tangentially) convex function over a convex set, J. Convex Analysis 3 (1996) 55–61.

[10] V. Jeyakumar, Z. Y. Wu: A qualification free sequential Pshenichnyi-Rockafellar lemma and convex semidefinite programming, J. Convex Analysis 13 (2006) 773–784.

[11] B. Mordukhovich: Necessary conditions in nonsmooth minimization via lower and upper subgradients, Set-Valued Anal. 12 (2004) 163–193.

[12] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[13] A. S. Strekalovsky: On search of global maximum of convex functions on a constraint set, J. of Comp. Mathematics and Math. Physics 33 (1993) 349–363.

[14] A. S. Strekalovsky: Global optimality conditions for nonconvex optimization, J. Glob. Optim. 12 (1998) 415–434.

[15] I. Tsevendorj: Piecewise-convex maximization problems, J. Glob. Optim. 21 (2001) 1–14.