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G-Majorization Inequalities and Canonical Forms of Matrices

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An Eaton system is connected with a decomposition statement for vectors of a linear space and with a scalar inequality related to the decomposition. The Singular Value Decomposition for the space of complex matrices associated with von Neumann's trace inequality is a typical example. In this paper we present a G-majorization inequality involving two orthoprojectors related to an Eaton system. The inequality generalizes a variety of majorization results on eigenvalues and singular values of matrices. A relationship between the inequality and canonical form theorems for certain spaces of matrices is shown. G-doubly stochastic operators are discussed.

Keywords: G-majorization, Eaton system, normal decomposition system, finite reflection group, G-doubly stochastic operator, eigenvalue, singular value

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1. Introduction

The theory of majorization has many applications in pure and applied mathematics. The best general reference here is [16]. Special attention in the literature is devoted to majorization inequalities (see e.g. [1, 2, 3, 17, 18, 20, 23]). Many of these inequalities concern eigenvalues and singular values of matrices. In the current paper we offer a unified approach to the task of obtaining such inequalities. In addition to this, we show a correspondence between the inequalities and decomposition theorems for matrices. Our method is based on the theory of group majorization, Eaton systems and normal decomposition systems (see the definitions below). We develop some ideas from [1, 13, 14, 18].

Section 1 is expository. We review here some of the standard facts on G-majorization orderings and related notions. The results are collected in Sections 2-4. In Theorem 2.1 we present a general G-majorization inequality involving two orthoprojectors associated with an Eaton system. The inequality embraces a variety of known results. Theorem 2.7 concerns Eaton systems connected with finite reflection groups. For such systems we give simple conditions implying a version of the inequality. In Section 3 we study the notion of subsystem introduced by Lewis [14]. Each subsystem of an Eaton system provides a decomposition statement inducing a canonical form of vectors in some subspace. In Theorem 3.1 we characterize a subsystem by, among others, the mentioned inequality. Thus there is a close connection between some decomposition results and G-majorization

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orderings. Finally, in Section 4, we show a role of G-doubly stochastic operators in the theory. Theorems 2.1-4.1 are illustrated in Examples 2.3-4.2 by concrete matrix spaces and groups.

Throughout the paper V is a finite-dimensional real linear space with inner product $\langle \cdot, \cdot \rangle$, and G is a closed subgroup of the orthogonal group $O(V)$ acting on V. Given points $x, y \in V$, we write $y \preceq_G x$, if y lies in the convex hull $C_G(x)$ of the orbit $Gx = \{gx : g \in G\}$, that is

$$y \preceq_G x \quad \text{iff } y \in \operatorname{conv} Gx$$

(see [4], [16, p. 422]). The relation $\preceq_G$ is called *group majorization* w.r.t. G (*G-majorization* for short). It is easy to see that $\preceq_G$ is a G-invariant ordering on V. For instance, if G is the group of permutation matrices acting on $V = \mathbb{R}^n$, then $\preceq_G$ is the (classical) majorization ordering on $\mathbb{R}^n$ [16, p. 23, p. 113] (see Example 1.1 for details).

It is well known that

$$y \preceq_G x \quad \text{iff } m(z, y) \le m(z, x) \quad \text{for } z \in V, \tag{1}$$

where $m(z, v) = \sup_{g \in G} \langle z, gv \rangle$, $z \in V$, is the *support function* of $C_G(v)$ for $v \in V$ [21, Section 13]. Assume that there exists a closed convex cone $D \subset V$ such that [4, 5]

(A1) $Gx \cap D$ is nonempty for each $x \in V$, or equivalently, $V = \bigcup_{g \in G} gD$,
(A2) $\langle x, gy \rangle \le \langle x, y \rangle$, or equivalently, $\|x - gy\| \ge \|x - y\|$, for $x, y \in D$ and $g \in G$,

where $\| \cdot \|$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle$. In this event, a vector x in V has its canonical (normal) form

$$x = g x_{\downarrow} \quad \text{for some } g \in G. \tag{2}$$

Here the *normal map* $(\cdot)_{\downarrow} : V \to D$ is defined by

$$\{x_{\downarrow}\} = D \cap Gx \quad \text{for } x \in V.$$

By virtue of (A1)-(A2), the intersection $D \cap Gx$ is a singleton set, and therefore the vector $x_{\downarrow}$ is uniquely determined by x [18, p. 14]. For notational convenience, we ignore the dependence of the map $(\cdot)_{\downarrow}$ on (V, G, D).

The map $(\cdot)_{\downarrow}$ is G-invariant and idempotent. Its range is the cone D. The restriction of $(\cdot)_{\downarrow}$ to D is the identity. For each $x \in V$, the vectors $x_{\downarrow}$ and x are equivalent in the sense that $x_{\downarrow} \preceq_G x$ and $x \preceq_G x_{\downarrow}$.

Under axioms (A1)-(A2), condition (1) takes the form

$$y \preceq_G x \quad \text{iff } y_{\downarrow} \preceq_G x_{\downarrow} \quad \text{iff } \langle z, y_{\downarrow} \rangle \le \langle z, x_{\downarrow} \rangle \quad \text{for } z \in D. \tag{3}$$

A useful version of (3) is the following

$$y \preceq_G x \quad \text{iff } \langle z, gy \rangle \le \langle z, x_{\downarrow} \rangle \quad \text{for } z \in D \text{ and } g \in G. \tag{4}$$

It is readily seen from (3) that the ordering $\preceq_G$ restricted to D is the cone ordering induced by the dual cone of D. For this reason, if (A1) and (A2) are met, the relation $\preceq_G$ is called a *group induced cone ordering* (*GIC ordering* for short), and the triple (V, G, D) is called an *Eaton system* [24]. It named after M. L. Eaton who introduced it in 1984 [4, 5]. Independently, A. S. Lewis [13, 14, 15] introduced the notion of a *normal decomposition* (*ND*) *system* $(V, G, (\cdot)_\downarrow)$. The above-mentioned notions play a unifying role in statistics, matrix theory and Lie theory. See [4]-[6], [13]-[15], [17]-[20] and [24] for examples, properties and applications of GIC orderings, Eaton systems and ND systems.

Recall that a finite group G acting on a linear space V is called a *finite reflection group* [11] if G is generated by some set of reflections. It is known that finite reflection groups induce GIC orderings and Eaton systems (see [6, Lemma 4.1, (35)], [22, Theorem 4.1]).

In our examples, we shall use the following matrix notation (for the field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$).

$$\begin{aligned}
\mathbb{R}^n_\downarrow &= \text{the convex cone of nonincreasing vectors in } \mathbb{R}^n,\\
\mathbb{R}^n_{+\downarrow} &= \text{the convex cone of nonnegative nonincreasing vectors in } \mathbb{R}^n,\\
\mathbb{M}_n(\mathbb{F}) &= \text{the vector space of } n \times n \text{ matrices over } \mathbb{F},\\
\mathbb{H}_n &= \text{the vector space of } n \times n \text{ Hermitian matrices},\\
\mathbb{S}_n(\mathbb{F}) &= \text{the vector space of } n \times n \text{ symmetric matrices over } \mathbb{F},\\
\mathbb{K}_n(\mathbb{R}) &= \text{the vector space of } n \times n \text{ real skew-symmetric matrices},\\
\mathbb{D}_n(\mathbb{F}) &= \text{the vector space of } n \times n \text{ diagonal matrices over } \mathbb{F},\\
\mathbb{U}_n &= \text{the group of } n \times n \text{ unitary matrices},\\
\mathbb{O}_n &= \text{the group of } n \times n \text{ real orthogonal matrices},\\
\mathbb{P}_n &= \text{the group of } n \times n \text{ permutation matrices},\\
\mathbb{GP}_n(\mathbb{F}) &= \text{the group of } n \times n \text{ generalized permutation matrices over } \mathbb{F}, \text{ i.e., matrices}\\
&\ \text{with exactly one nonzero entry with magnitude 1 in each row and column},\\
\mathbb{DO}_n &= \text{the group of } n \times n \text{ diagonal orthogonal matrices},\\
\mathbb{DU}_n &= \text{the group of } n \times n \text{ diagonal unitary matrices},\\
\lambda(X) &= \text{the vector of eigenvalues of Hermitian matrix } X \text{ stated in nonincreasing}\\
&\ \text{order},\\
s(X) &= \text{the vector of singular values of matrix } X \text{ stated in nonincreasing order},\\
d(X) &= \text{the vector of diagonal entries of matrix } X,\\
\operatorname{Re} X &= \text{the real part of matrix } X,\\
U_1(\cdot)U_2 &= \text{the matrix operator of the form } X \to U_1 X U_2, \text{ where } X,\ U_1 \text{ and } U_2\\
&\ \text{are matrices.}
\end{aligned}$$

Let $z_{[1]} \geq z_{[2]} \geq \ldots \geq z_{[n]}$ denote the entries of $z \in \mathbb{R}^n$ in nonincreasing order. For $x, y \in \mathbb{R}^n$, if $\sum_{j=1}^{i} y_{[j]} \leq \sum_{j=1}^{i} x_{[j]}$, $i = 1, \ldots, n$, then we write $y \prec_w x$. This is the *weak majorization* [16, p. 10]. If, in addition, $\sum_{j=1}^{n} y_{[j]} = \sum_{j=1}^{n} x_{[j]}$, we write $y \prec_m x$, the (*classical*) *majorization* [16, p. 7].

Example 1.1 ([4, p. 16]). It is known that if $V = \mathbb{R}^n$, $G = \mathbb{P}_n$ and $D = \mathbb{R}^n_\downarrow$, then (V, G, D) is an Eaton system, G is a finite reflection group, and $y \preceq_G x$ means $y \prec_m x$. Furthermore, $x_\downarrow = (x_{[1]}, \ldots, x_{[n]})^T$.

Likewise, replacing V, G, and D with $\mathbb{R}^n$, $\mathbb{GP}_n(\mathbb{R})$ and $\mathbb{R}^n_{+\downarrow}$, respectively, one obtains Eaton system (V, G, D) with finite reflection group G. Here $y \preceq_G x$ reduces to $|y| \prec_w |x|$, and, in addition, $x_\downarrow = (|x|_{[1]}, \ldots, |x|_{[n]})^T$, where $|x|$ is the vector of absolute values of the entries of $x \in \mathbb{R}^n$.

Example 1.2 ([4, p. 17]). Take $V = \mathbb{H}_n$ with the real inner product $\langle X, Y\rangle = \operatorname{Re}\operatorname{tr} XY$. Let G be the group of operators $X \to UXU^*$, $X \in \mathbb{H}_n$, with U running over $\mathbb{U}_n$. Then (V, G, D) is an Eaton system for $D = \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_{\downarrow}\}$. In fact, (A1) is the Spectral Theorem, and (A2) is Fan-Theobald's trace inequality [7, 25]. It is known that $X_{\downarrow} = \operatorname{diag}\lambda(X)$ for $X \in \mathbb{H}_n$, and $Y \preceq_G X$ iff $\lambda(Y) \preceq_m \lambda(X)$ for $X, Y \in \mathbb{H}_n$. So, the ordering $\preceq_G$ on $\mathbb{D}_n(\mathbb{R})$ may be identified with the classical majorization $\preceq_m$ on $\mathbb{R}^n$.

Example 1.3 ([4, p. 17-18]). Let V be $\mathbb{M}_n(\mathbb{C})$ with the real inner product $\langle X, Y\rangle = \operatorname{Re}\operatorname{tr} XY^*$. Let G be the group of all linear operators $X \to U_1XU_2$, $X \in \mathbb{M}_n(\mathbb{C})$, where U_1 and U_2 vary over $\mathbb{U}_n$. Put $D = \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_{+\downarrow}\}$. Then (V, G, D) is an Eaton system, where (A1) is the Singular Value Decomposition Theorem [16, p. 498], and (A2) is von Neumann's trace inequality [16, p. 514]. Moreover, $X_{\downarrow} = \operatorname{diag} s(X)$ for $X \in \mathbb{M}_n(\mathbb{C})$. Here $Y \preceq_G X$ iff $s(Y) \preceq_w s(X)$ for $X, Y \in \mathbb{M}_n(\mathbb{C})$. Thus $\preceq_G$ restricted to $\mathbb{D}_n(\mathbb{R})$ can be described as the weak majorization ordering $\preceq_w$ on $\mathbb{R}^n$.

2. G-majorization and orthoprojectors

We begin with G-majorization inequality (7) which gives a more conceptual understanding of some results existing in the literature (see Examples 2.3-2.5, 2.9, 2.10, 3.3 and 4.2).

Theorem 2.1. *Let (V, G, D) be an Eaton system. Assume W is a linear subspace of V, H is a subset of G, and E is a subset of D. Let P and Q be the orthoprojectors, respectively, from V onto W and from* $\operatorname{span} D$ *onto* $\operatorname{span} E$. *If*

$$QD = E, \tag{5}$$

$$\textit{and} \quad W = \bigcup_{h\in H} hE, \tag{6}$$

then the following inequality holds:

$$Px \preceq_G Qx_{\downarrow} \quad \textit{for } x \in V. \tag{7}$$

Proof. Fix $x \in V$. By virtue of (3), it is sufficient to show

$$\langle z, (Px)_{\downarrow}\rangle \le \langle z, Qx_{\downarrow}\rangle \quad \text{for } z \in D, \tag{8}$$

since $Qx_{\downarrow} \in E \subset D$ by (5). From (6) it follows that $Px = he$ for some $h \in H$ and $e \in E$. But $E \subset D$ and $H \subset G$, so $(Px)_{\downarrow} = e$ and $Px = h(Px)_{\downarrow}$. Hence $(Px)_{\downarrow} = h^{-1}Px$ with $h^{-1} \in G$. We have $Qz \in E$ for $z \in D$ by (5). Therefore $hQz \in hE \subset W$ by (6). As a consequence, $PhQz = hQz$.

For $z \in D$ we now obtain

$$\begin{aligned} \langle z, (Px)_{\downarrow}\rangle &= \langle Qz, (Px)_{\downarrow}\rangle = \langle Qz, h^{-1}Px\rangle = \langle Qz, h^TPx\rangle \\ &= \langle PhQz, x\rangle = \langle hQz, x\rangle \le \langle Qz, x_{\downarrow}\rangle = \langle z, Qx_{\downarrow}\rangle = \langle Qz, Qx_{\downarrow}\rangle. \end{aligned} \tag{9}$$

The last inequality holds by (5) and (A2). Thus (8) and (7) are proved. □

Remark 2.2.

(a) If $E = D$ (or $\operatorname{span} E = \operatorname{span} D$) then Q reduces to the identity operator, and condition (5) can be deleted (see Example 2.3, cf. also Theorem 2.7, (12)).

(b) Condition (6) says that each vector $x \in W$ has its decomposition $x = hx_\downarrow$ for some $h \in H$ and $x_\downarrow \in E$ (cf. Examples 2.3-2.5).

(c) Note that Theorem 2.1 does not require that H is a subgroup of G and $E \subset W$ (see Example 2.5). However, if $E \subset W$ and (W, H, E) is an Eaton system then (9) yields $Px \preceq_H Qx_\downarrow$ for $x \in V$ (see Theorem 3.1).

Example 2.3. Assume V, G and D are defined as in Example 1.2. Letting

$$W := \mathbb{S}_n(\mathbb{R}), \quad H := \{U(\cdot)U^T : U \in \mathbb{O}_n\} \quad \text{and} \quad E := D$$

(cf. [14, Example 7.4]), we obtain

$$PX = \operatorname{Re} X \quad \text{for } X \in \mathbb{H}_n.$$

Condition (6) follows from the Spectral Decomposition for matrices in $\mathbb{S}_n(\mathbb{R})$. We now deduce from (7) that (cf. [1, p. 111])

$$\lambda(\operatorname{Re} X) \preceq_m \lambda(X) \quad \text{for } X \in \mathbb{H}_n.$$

Example 2.4. Let V, G and D be defined as in Example 1.3. For $1 \le k \le n$, set

$$\begin{aligned} W &:= \mathbb{M}_k(\mathbb{C}) \oplus 0_{n-k}, \\ H &:= \{U_1(\cdot)U_2 : U_1, U_2 \in \mathbb{U}_k \oplus I_{n-k}\}, \\ E &:= \{Z \in \mathbb{D}_k(\mathbb{R}) \oplus 0_{n-k} : z_{11} \ge \ldots \ge z_{kk} \ge 0\}. \end{aligned}$$

Condition (6) amounts to the Singular Value Decomposition for matrices in $\mathbb{M}_k(\mathbb{C})$. The orthoprojector P is given by

$$PX = X_{11} \oplus 0_{n-k} \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}),$$

where X_{11} is the $k \times k$ principal submatrix of X. Furthermore, Q is the restriction of P to $\mathbb{D}_n(\mathbb{R})$. Therefore inequality (7) becomes

$$s(X_{11}) \preceq_w (s_1, \ldots, s_k) \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}),$$

where $s(X) = (s_1, \ldots, s_n)$. This is a direct consequence of the interlacing inequalities for singular values (see [9, Corollary 3.1.3], cf. also [1, p. 101]).

Example 2.5. Let V be $\mathbb{M}_n(\mathbb{R})$ with the trace inner product $\langle X, Y\rangle = \operatorname{Re}\operatorname{tr} XY^T$, and let

$$G := \{U_1(\cdot)U_2 : U_1, U_2 \in \mathbb{O}_n\} \quad \text{and} \quad D := \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_{+\downarrow}\}$$

(cf. Example 1.3). We put

$$\begin{aligned} W := \mathbb{K}_n(\mathbb{R}), \quad H &:= \{UU_0(\cdot)U^T : U \in \mathbb{O}_n\} \\ \text{and} \quad E &:= \{s_1 I_2 \oplus \ldots \oplus s_k I_2 : s_1 \ge \ldots \ge s_k \ge 0\}, \end{aligned}$$

where $n = 2k$ is even, $I_2 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and U_0 is the $n \times n$ block-diagonal matrix $B \oplus \ldots \oplus B$ with $B := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Here $E \not\subset W$. Condition (6) follows from the Autonne Decomposition (cf. [14, Example 7.5]).

It is evident that

$$PX = \frac{X - X^T}{2} \quad \text{for } X \in \mathbb{M}_n(\mathbb{R}),$$

and

$$QX = \frac{s_1 + s_2}{2} I_2 \oplus \ldots \oplus \frac{s_{n-1} + s_n}{2} I_2 \quad \text{for } X = \operatorname{diag}(s_1, \ldots, s_n) \in \mathbb{D}_n(\mathbb{R}).$$

An application of Theorem 2.1 results in

$$s\left(\frac{X - X^T}{2}\right)$$
$$\preceq_w \left(\frac{s_1(X) + s_2(X)}{2}, \frac{s_1(X) + s_2(X)}{2}, \ldots, \frac{s_{n-1}(X) + s_n(X)}{2}, \frac{s_{n-1}(X) + s_n(X)}{2}\right)^T$$

for $X \in \mathbb{M}_n(\mathbb{R})$ (cf. [1, p. 109]).

As we saw in Examples 2.3-2.5, the property (6) in Theorem 2.1 is closely related to canonical forms of some classes of matrices. On the other hand, (6) is a key assumption for (7) to hold. In Theorem 2.7 below we shall give some practical conditions for a pair of Eaton systems to have property (6). In consequence, we shall obtain inequality (7) in the form (12) (cf. Examples 2.9-2.10; see also [1, 18]).

For this end, we introduce the notion of a reduced system. Given an Eaton system (V, G, D), set

$$V_0 := \operatorname{span} D \quad \text{and} \quad G_0 := \{g \in G : gV_0 = V_0\}.$$

If $G_0|_{V_0}$ is a finite reflection group acting on V_0, that is, if (V_0, G_0, D) is an Eaton system (see [6, Lemma 4.1, (35)], [22, Theorem 4.1]), then

$$y \preceq_G x \quad \text{iff } y \preceq_{G_0} x \quad \text{for } x, y \in V_0 \tag{10}$$

[18, Theorem 3.2]. In this event, (V_0, G_0, D) is called the *reduced system* of (V, G, D) (cf. [24]). For simplicity of notation, we write here G_0 instead of $G_0|_{V_0}$.

Many examples of Eaton system are indeed associated with reduced systems.

Example 2.6. If V, G and D are as in Example 1.2, then the reduced system is (V_0, G_0, D) for $V_0 = \mathbb{D}_n(\mathbb{R})$, $G_0 = \{S(\cdot)S^T : S \in \mathbb{P}_n\}$ and $D = \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_\downarrow\}$, which can be identified with the Eaton system $(\mathbb{R}^n, \mathbb{P}_n, \mathbb{R}^n_\downarrow)$ via the linear map $\mathbb{D}_n(\mathbb{R}) \ni Z \to d(Z) \in \mathbb{R}^n$ (cf. Example 1.1).

In the case of Example 1.3, the reduced system (V_0, G_0, D) of (V, G, D) consists of $V_0 = \mathbb{D}_n(\mathbb{R})$, $G_0 = \{CS(\cdot)S^T : S \in \mathbb{P}_n, C \in \mathbb{DO}_n\}$ and $D = \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_{+\downarrow}\}$. Here the reduced system corresponds to the triple $(\mathbb{R}^n, \mathbb{GP}_n(\mathbb{R}), \mathbb{R}^n_{+\downarrow})$ (cf. Example 1.1).

The symbols cl and ri stand, respectively, for the closure operation and for the relative interior operation of sets [21, Section 6].

Theorem 2.7. *Let (V, G, D) and (W, H, F) be Eaton systems with reduced systems (V_0, G_0, D) and (W_0, H_0, F), respectively. Assume that $W \subset V$, $H \subset G$ and $D \subset F$. If*

$W_0 = V_0$ then there exists a subset $\widehat{G} \subset G$ such that

$$W = \bigcup_{g \in \widehat{G}} gD, \tag{11}$$

and the following inequality holds:

$$Px \preceq_G x_\downarrow \quad \text{for } x \in V, \tag{12}$$

where P is the orthoprojector from V onto W.

Proof. The triples (V_0, G_0, D) and (W_0, H_0, F) are Eaton systems with finite reflection groups $G_0|_{V_0}$ and $H_0|_{W_0}$ such that $W_0 = V_0$ and $H_0 \subset G_0$. Set

$$\widetilde{G}_0 := \{g \in G_0 : F \cap \operatorname{ri}(gD) \text{ is nonempty}\}.$$

Since $\operatorname{ri} D \subset D \subset F$ and $\operatorname{ri} D$ is nonempty [21, Theorem 6.2], the identity operator id belongs to $\widetilde{G}_0$, so $\widetilde{G}_0$ is nonempty.

We first prove that

$$F = \bigcup_{g \in \widetilde{G}_0} gD. \tag{13}$$

To do this, take any $g \in \widetilde{G}_0$. We intend to show that $gD \subset F$. On the contrary, suppose that gD is not included in F. Then there exists $y \in gD \subset V_0$ such that $y \notin F$. According to the definition of $\widetilde{G}_0$, there exists $x \in F \cap \operatorname{ri}(gD)$. Clearly, $x \in F$, $x \in \operatorname{ri}(gD)$ and $x = gx_\downarrow$ with $x_\downarrow \in \operatorname{ri} D$, where the normal map $(\cdot)_\downarrow$ is taken with respect to (V_0, G_0, D). Because (W_0, H_0, F) is an Eaton system with $W_0 = V_0$, and $H_0|_{V_0}$ is a finite reflection group, we have $F = \{z \in V_0 : \langle z, r_i \rangle \geq 0,\ i = 1, \ldots, m\}$ for the simple roots $r_i \in V_0$ of $H_0|_{V_0}$ (see [11, Section 1.12], [22, Theorem 4.1]). In consequence, $\langle y, r_0 \rangle < 0$ and $\langle x, r_0 \rangle \geq 0$ for some $r_0 = r_{i_0}$, because $y \notin F$ and $x \in F$. Additionally, the real function $t \to \langle (1-t)x + ty, r_0 \rangle$ is continuous on the interval $[0, 1]$. For this reason there exists a point $z_0 = (1 - t_0)x + t_0 y$ for some $t_0 \in [0, 1)$ such that $\langle z_0, r_0 \rangle = 0$. Therefore $z_0 \neq y$, and, consequently, $z_0 \in \operatorname{ri}(gD)$, since $x \in \operatorname{ri}(gD)$ and $y \in gD$ (see [21, Theorem 6.1]).

Let $S_{r_0} \in H_0|_{V_0}$ be the reflection on V_0 corresponding to r_0. Then $S_{r_0} \in G_0|_{V_0}$, since $H_0 \subset G_0$. Moreover, it is not hard to verify that (V_0, G_0, gD) is an Eaton system. We therefore get $S_{r_0} z = z$ for all $z \in \operatorname{span} gD = V_0$, because $S_{r_0} z_0 = z_0$ and $z_0 \in \operatorname{ri}(gD)$ (see [18, Lemma 2.1, Theorem 3.1]). Thus S_{r_0} is the identity on V_0, a contradiction.

We have shown that $gD \subset F$ for each $g \in \widetilde{G}_0$. From this $\bigcup_{g \in \widetilde{G}_0} gD \subset F$. It remains to prove the opposite inclusion. Assume that $x \in F$. Since $F = \operatorname{cl} \operatorname{ri} F$ [21, Theorem 6.3], the set $B_k(x) \cap \operatorname{ri} F$ is nonempty for each positive integer k, where $B_k(x)$ is the open ball in V_0 with the centre x and the radius $\frac{1}{k}$. But the group $G_0|_{V_0}$ is finite, and axiom (A1) holds for (V_0, G_0, D), so the open set $\bigcup_{g \in G_0} \operatorname{ri}(gD)$ is dense in V_0. This implies that the intersection $B_k(x) \cap \operatorname{ri} F \cap \bigcup\limits_{g \in G_0} \operatorname{ri}(gD)$ is nonempty. So there exists a point $x_k \in B_k(x) \cap \operatorname{ri} F \cap \operatorname{ri}(g_k D)$ for some $g_k \in G_0$. Hence $x_k \in F \cap \operatorname{ri}(g_k D)$, which forces $g_k \in \widetilde{G}_0$ for each k. Therefore

$$x_k \in \operatorname{ri}(g_k D) \subset \bigcup_{g \in \widetilde{G}_0} \operatorname{ri}(gD)$$

for each k. Letting $k \to \infty$, we have $x_k \to x$ as $x_k \in B_k(x)$. So, finally

$$x \in \mathrm{cl}\,(\bigcup_{g\in\widetilde{G}_0} \mathrm{ri}\,(gD)) = \bigcup_{g\in\widetilde{G}_0} \mathrm{cl}\,\mathrm{ri}\,(gD) = \bigcup_{g\in\widetilde{G}_0} gD.$$

Thus the inclusion $F \subset \bigcup_{g\in\widetilde{G}_0} gD$ is established. This proves (13).

To see (11), apply (13) and (A1) for (W, H, F) to obtain $W = \bigcup_{g\in\widehat{G}} gD$ with $\widehat{G} := H\widetilde{G}_0$. Now, the proof of (12) is completed by using Theorem 2.1. □

Remark 2.8. The first part of above proof is similar to the proofs of [8, Lemma 4] and of [22, Theorem 4.1, parts (ii)-(iii)].

In Examples 2.9 and 2.10, we apply Theorem 2.7 for the reduced systems described in Example 2.6.

Example 2.9. We take V, G and D to be as in Example 1.3. Setting

$$W := \mathbb{H}_n, \quad H := \{U(\cdot)U^* : U \in \mathbb{U}_n\} \quad \text{and} \quad F := \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_\downarrow\}$$

yields

$$PX = \frac{X + X^*}{2} \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}).$$

According to Theorem 2.7 and (12), we obtain

$$s\,(\frac{X + X^*}{2}) \preceq_w s(X) \quad \text{for all } X \in \mathbb{M}_n(\mathbb{C}),$$

Fan-Hoffman's inequality (see [1, p.109], [16, p. 240]).

Example 2.10. Suppose that V, G and D are as in Example 1.2. For $1 \le k \le n$, we put

$$\begin{aligned} W &:= \mathbb{H}_k \oplus \mathbb{H}_{n-k}, \\ H &:= \{U(\cdot)U^* : U \in \mathbb{U}_k \oplus \mathbb{U}_{n-k}\}, \\ F &:= \{Z \in \mathbb{D}_n(\mathbb{R}) : z_{11} \ge \ldots \ge z_{kk},\ z_{k+1\,k+1} \ge \ldots \ge z_{nn}\}. \end{aligned}$$

One can now check that

$$PX = X_{11} \oplus\ X_{22} \quad \text{for } X \in \mathbb{H}_n,$$

where X_{11} and X_{22} are the $k \times k$ and $(n-k) \times (n-k)$ diagonal blocks of X, respectively. Now, Theorem 2.7 implies that

$$(\lambda(X_{11}), \lambda(X_{22})) \preceq_m \lambda(X) \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}).$$

This is a result of Fan (see [16, p. 225], cf. also [1, p. 97]).

3. Subsystems of Eaton systems

Remind that the triple (V, G, D) is an Eaton system if axioms (A1)-(A2) hold (see Section 1). In this event, the (nonlinear) operator $(\cdot)_\downarrow : V \to D$ given by $\{x_\downarrow\} = D \cap Gx$ for $x \in V$ is called a normal map, and the triple $(V, G, (\cdot)_\downarrow)$ is called a normal decomposition (ND) system.

Lewis [13, 14, 15] showed the usefulness of ND systems in convex matrix analysis and Lie theory. Following his ideas, we now introduce the notion of a subsystem. Let (V, G, D) be an Eaton system. Assume H is a closed subgroup of G, W is an H-invariant subspace of V (with the inherited inner product), and E is a closed convex subcone of D. The triple (W, H, E) is called a *subsystem* of (V, G, D), if (W, H, E) is an Eaton system (see Examples 2.3-2.4). For notation simplicity, we write (W, H, E) in place of $(W, H|_W, E)$. A special class of subsystems is formed by the reduced systems (see Example 2.6).

In Theorem 3.1 we give a characterization of a subsystem of an Eaton system (cf. [18, Theorem 3.2]).

Theorem 3.1. *Suppose (V, G, D) is an Eaton system. Let H be a closed subgroup of G, let W be an H-invariant subspace of V, and let $E \subset W$ be a closed convex subcone of D. Let P and Q be the orthoprojectors, respectively, from V onto W and from* $\operatorname{span} D$ *onto* $\operatorname{span} E$. *Assume $QD = E$. Then the following statements are mutually equivalent:*

(a) *(W, H, E) is a subsystem of (V, G, D).*

(b) *The following inequality holds*

$$Px \preceq_H Qx_\downarrow \quad \text{for } x \in V. \tag{14}$$

(c) *The following inclusion holds*

$$W \subset \bigcup_{g \in G} gE, \tag{15}$$

and, in addition, the orderings $\preceq_H$ and $\preceq_G$ coincide on W, i.e.

$$y \preceq_H x \quad \text{iff } y \preceq_G x \quad \text{for } x, y \in W. \tag{16}$$

(d) *The following decomposition holds*

$$W = \bigcup_{h \in H} hE. \tag{17}$$

Proof. $(a) \Rightarrow (b)$: Analysis similar to that in the proof of Theorem 2.1 shows that (see (9)), $\langle y, (Px)_\downarrow \rangle \le \langle y, Qx_\downarrow \rangle$ for $x \in V$ and $y \in E$. Consequently, using (3) applied for the Eaton system (W, H, E), we conclude that $Px \preceq_H Qx_\downarrow$ for $x \in V$.

$(b) \Rightarrow (c)$: To see (15), fix arbitrarily $x \in W$. It follows from (2) that $x = gx_\downarrow$ for some $g \in G$. It suffices to show that $x_\downarrow \in E$. Since $Px = x = gx_\downarrow$, (14) gives $gx_\downarrow \preceq_H Qx_\downarrow$. Hence $\|x_\downarrow\| = \|gx_\downarrow\| \le \|Qx_\downarrow\| \le \|x_\downarrow\|$. The first inequality follows from the convexity and H-invariance of the norm map $\|\cdot\|$. The second is due to Pythagorean Theorem. Therefore $\|Qx_\downarrow\| = \|x_\downarrow\|$, which implies $Qx_\downarrow = x_\downarrow$. From this $x_\downarrow \in E$, because $QD = E$. This proves (15).

In order to prove (16), let $x \in W$. The implication $y \preceq_H x \Rightarrow y \preceq_G x$ for $y \in W$ is obvious, because $H \subset G$ and $C_H(x) \subset C_G(x)$.

For the reverse implication, let $y \in W$ be such that $y \preceq_G x$. Then $y \preceq_G x_\downarrow$ with $x_\downarrow \in E \subset W$ by (15). We thus get $y = \sum_i \lambda_i g_i x_\downarrow$, a finite convex combination for some $g_i \in G$. Hence $y = Py = \sum_i \lambda_i P g_i x_\downarrow$. However, $P g_i x_\downarrow \preceq_H Q x_\downarrow = x_\downarrow$ by (14). Therefore $\sum_i \lambda_i P g_i x_\downarrow \in \operatorname{conv} H x_\downarrow = C_H(x_\downarrow)$. In this way $y \in C_H(x_\downarrow)$, that is $y \preceq_H x_\downarrow$.

In particular, for $y = x$ we find that $x \preceq_H x_\downarrow$. But $\|x\| = \|x_\downarrow\|$. So we may conclude that $x \in H x_\downarrow$ by the strict convexity of the norm $\|\cdot\|$. This means that there exists an $h \in H$ satisfying $x = h x_\downarrow$. Hence $C_H(x_\downarrow) = C_H(x)$.

Consequently, $y \in C_H(x)$, that is $y \preceq_H x$. Thus we have obtained the wanted inequality.

$(c) \Rightarrow (d)$: Let $x \in W$ be arbitrarily chosen. By (2) and (15), $x = g x_\downarrow$ for some $g \in G$ and $x_\downarrow \in E \subset W$. For this reason $x \preceq_G x_\downarrow$ and $x_\downarrow \preceq_G x$. It now follows from (16) that $x \preceq_H x_\downarrow$ and $x_\downarrow \preceq_H x$. Hence $x = h x_\downarrow$ for some $h \in H$. Therefore $x \in \bigcup_{h\in H} hE$.

By using the arbitrariness in choice of $x \in W$, we can obtain $W \subset \bigcup_{h\in H} hE$. The opposite inclusion is clear by the H-invariance of W. Thus (17) is proved.

$(d) \Rightarrow (a)$: Axioms (A1) and (A2) for (W, H, E) are implied by, respectively, (17) and (A2) used for (V, G, D). So (W, H, E) is an Eaton system, and therefore it is a subsystem of (V, G, D). □

Remark 3.2.

(a) In view of Theorem 3.1, given an Eaton system (V, G, D) and its subsystem (W, H, E), the normal map $(\cdot)_\downarrow$ for (W, H, E) is the restriction to W of the one for (V, G, D) (cf. [18, Lemma 2.1]). Therefore we use the same symbol for them.

(b) If Q is the restriction to $\operatorname{span} D$ of P then (14) can be written as

$$Px \preceq_H Px_\downarrow \quad \text{for } x \in V. \tag{18}$$

On the other hand, in the case $E = D$ the orthoprojector Q is the identity, so the assumption $QD = E$ and inclusion (15) in Theorem 3.1 can be dropped, and (14) leads to

$$Px \preceq_H x_\downarrow \quad \text{for } x \in V \tag{19}$$

(see Example 3.3). Furthermore, if the system (W, H, E) has its reduced system (W_0, H_0, E), then $\preceq_H$ can be replaced on W_0 by $\preceq_{H_0}$ according to (10).

Example 3.3. Put

$$W := \mathbb{D}_n(\mathbb{C}), \quad H := \{CS(\cdot)S^T : S \in \mathbb{P}_n,\ C \in \mathbb{DU}_n\} \quad \text{and} \quad E := D$$

with V, G and D as in Example 1.3. Condition (17) is met by the Polar Decomposition for diagonal matrices. The orthoprojector P is given by

$$PX = \operatorname{diag}(x_{11}, \dots, x_{nn}) \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}).$$

Using (19), one has

$$(|x_{11}|, \dots, |x_{nn}|) \preceq_w (s_1, \dots, s_n) \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}),$$

where $s(X) = (s_1, \dots, s_n)$. This is Fan's result [16, p. 228] (cf. also [23, Corollary 1.1]).

4. G-doubly stochastic operators

In this section we show that some G-doubly stochastic operators induce subsystems of Eaton systems. Let (V, G, D) be an Eaton system. A linear operator $L : V \to V$ is called *G-doubly stochastic* if

$$Lx \preceq_G x \quad \text{for } x \in V.$$

Using (3) and axioms (A1)-(A2), it is not difficult to check that L is G-doubly stochastic if and only if so is its adjoint L^* defined by $\langle Lx, y\rangle = \langle x, L^* y\rangle$ for all $x, y \in V$.

For instance, if $V = \mathbb{R}^n$, $G = \mathbb{P}_n$ and $D = \mathbb{R}^n_\downarrow$ as in Example 1.1, and if L is an $n \times n$ matrix, then the last inequality amounts to $Lx \preceq_m x$ for $x \in \mathbb{R}^n$, which is equivalent to the conditions $L \geq 0$, $Le = e$ and $L^T e = e$, where $e = (1, \ldots, 1)^T \in \mathbb{R}^n$ (see [2, p. 169, Theorem 3.1]). In other words, L is a matrix with nonnegative entries and with row and column sums equal to 1. Such matrices L are said to be *doubly stochastic*. Clearly, L is doubly stochastic if and only if so is L^T.

Recall that the symbols $\operatorname{ri}(\cdot)$ and $\operatorname{cl}(\cdot)$ mean "the relative interior of" and "the closure of", respectively. It is known that if $A \subset V$ is convex then $\operatorname{cl}\operatorname{ri} A = \operatorname{cl} A$ [21, Theorem 6.3]. Given an Eaton system (V, G, D), denote $V_r := \bigcup_{g\in G} g \operatorname{ri} D$. Since G is compact and (A1) holds, the set V_r is dense in V, that is $\operatorname{cl} V_r = V$. A point $x \in V$ is said to be *regular* if $x \in V_r$.

Theorem 4.1. *Suppose (V, G, D) is an Eaton system. Let W be a subspace of V such that $D \subset W$, and let $H := \{h \in G : hD \subset W\}$. Then the following statements are equivalent:*

(a) $W = \bigcup_{h\in H} hD$.

(b) *There exists a G-doubly stochastic operator $L : V \to V$ such that $W = \{x \in V : Lx = x\}$, and, in addition, the set $V_r \cap W$ is dense in W.*

Under condition (a), the set H is a group if and only if $H = \{h \in G : hW = W\}$. In this event, the triple (W, H, D) is a subsystem of (V, G, D).

Proof. $(a) \Rightarrow (b)$: To see the first part of (b), it is sufficient to define $L := P$, where P denotes the orthoprojector from V onto W. Indeed, by using Theorem 2.1 for $E := D$ and Remark 2.2, part (a), one has $Px \preceq_G x_\downarrow \preceq_G x$ for $x \in V$. This means that P is G-doubly stochastic. Clearly, the fixed points of P are exactly those in W. So, P has the required properties.

To prove that the set $V_r \cap W$ is dense in W, note that D is convex and closed, and that each $h \in H$ is a continuous operator. Hence $hD = h \operatorname{cl}\operatorname{ri} D = \operatorname{cl} h \operatorname{ri} D$. From (a) we find that

$$W = \bigcup_{h\in H} h\, D = \bigcup_{h\in H} \operatorname{cl} h \operatorname{ri} D \subset \operatorname{cl}\big(\bigcup_{h\in H} h \operatorname{ri} D\big) \subset \operatorname{cl}(V_r \cap W) \subset W.$$

Thus $W = \operatorname{cl}(V_r \cap W)$ is established as desired.

$(b) \Rightarrow (a)$: We first prove $V_r \cap W = \bigcup_{h\in H} h \operatorname{ri} D$. It suffices to show $V_r \cap W \subset \bigcup_{h\in H} h \operatorname{ri} D$, because the opposite inclusion is straightforward. Let $z \in V_r \cap W$. Then $z = gw \in W$ for some $g \in G$ and $w \in \operatorname{ri} D$. We only need to prove that $g \in H$. Clearly, $Lgw = gw$ and $w - g^{-1}Lgw = 0$. Therefore $\langle y, w - g^{-1}Lgw\rangle = 0$ and $\langle y - g^{-1}L^* gy, w\rangle = 0$ for all $y \in D$. On the other hand, $Lgv \preceq_G v$ for all $v \in D$, since L is G-doubly stochastic, and $\preceq_G$ is

G-invariant. Making use of (4) we deduce that $\langle y, g^{-1}Lgv\rangle \le \langle y, v\rangle$ for all $v \in D$, which means $y - g^{-1}L^*gy \in \operatorname{dual} D$. It now follows that $y - g^{-1}L^*gy$ is orthogonal to $\operatorname{span} D$, because $y - g^{-1}L^*gy$ is orthogonal to $w \in \operatorname{ri} D$. In particular, $\langle y - g^{-1}L^*gy, y\rangle = 0$, and further $\langle y, y - g^{-1}Lgy\rangle = 0$. Hence, by Cauchy-Schwarz inequality,

$$\|y\|^2 = \langle g^{-1}Lgy, y\rangle \le |\langle g^{-1}Lgy, y\rangle| \le \|g^{-1}Lgy\|\,\|y\|. \tag{20}$$

However the relation $g^{-1}Lgy \preceq_G y$ forces $\|g^{-1}Lgy\| \le \|y\|$, since $\|\cdot\|$ is a G-invariant convex function on V. As a consequence, $|\langle g^{-1}Lgy, y\rangle| = \|g^{-1}Lgy\|\,\|y\|$ by (20). Therefore there exists a real number λ such that $g^{-1}Lgy = \lambda y$. But $\langle y, y - g^{-1}Lgy\rangle = 0$, so $\lambda = 1$. This gives $Lgy = gy$. For this reason $gy \in W$, because $W = \{x \in V : Lx = x\}$. Now, the arbitrariness of $y \in D$ implies $gD \subset W$. This yields $g \in H$. Remind that $z = gw$ for some $w \in \operatorname{ri} D$. Consequently we conclude that $z \in \bigcup_{h\in H} h \operatorname{ri} D$, as was to be proved.

Summarizing, we have $V_r \cap W = \bigcup_{h\in H} h \operatorname{ri} D$. Hence

$$W = \operatorname{cl}(V_r \cap W) = \operatorname{cl} \bigcup_{h\in H} h \operatorname{ri} D. \tag{21}$$

Next we claim that

$$\operatorname{cl} \bigcup_{h\in H} h \operatorname{ri} D = \bigcup_{h\in H} \operatorname{cl} h \operatorname{ri} D. \tag{22}$$

In fact, the inclusion $\bigcup_{h\in H} \operatorname{cl} h \operatorname{ri} D \subset \operatorname{cl} \bigcup_{h\in H} h \operatorname{ri} D$ is evident. To see the opposite inclusion, fix arbitrarily $x \in \operatorname{cl} \bigcup_{h\in H} h \operatorname{ri} D$. Then $x = \lim\limits_{n\to\infty} h_n z_n$ for some sequences $h_n \in H$ and $z_n \in \operatorname{ri} D$, $n \ge 1$. Because H is a closed subset of the compact group G, H is also compact. So, there exists a subsequence h_{n_k}, $k \ge 1$, converging to some $h_0 \in H$. Moreover, $x = \lim\limits_{k\to\infty} h_{n_k} z_{n_k}$, whence

$$\|x\| = \lim_{k\to\infty} \|h_{n_k} z_{n_k}\| = \lim_{k\to\infty} \|z_{n_k}\|.$$

We now have

$$\|h_0 z_{n_k} - x\| \le \|h_0 - h_{n_k}\| \cdot \|z_{n_k}\| + \|h_{n_k} z_{n_k} - x\|, \quad k \ge 1,$$

which implies $x = \lim\limits_{k\to\infty} h_0 z_{n_k}$. Therefore

$$x \in \operatorname{cl} h_0 \operatorname{ri} D \subset \bigcup_{h\in H} \operatorname{cl} h \operatorname{ri} D,$$

yielding the desired inclusion.

Combining (21) and (22), the required equality $W = \bigcup_{h\in H} hD$ follows.

The proof of the last part of the theorem is immediate and is therefore omitted. □

In some cases, the role of the set H in Theorem 4.1 can be played by a smaller set H_0.

Example 4.2. Define V, G and D as in Example 1.3. Set $W := \mathbb{S}_n(\mathbb{C})$. Let H be defined as in Theorem 4.1, and let

$$H_0 := \{h_0 = U(\cdot)U^T : U \in \mathbb{U}_n\}.$$

Then

$$W = \bigcup_{h_0 \in H_0} h_0 D$$

by the Takagi Decomposition for complex symmetric matrices (see [9, Cor. 4.4.4], [12, Theorem 2], cf. also [13, Example 3.5]). It can be derived by a straightforward calculation that

$$H = \{h = U(\cdot)AU^T : U \in \mathbb{U}_n,\ A = \operatorname{diag}(a_1, \ldots, a_n), |a_i| = 1\}.$$

So, $H_0 \subset H$, and therefore $\bigcup_{h_0 \in H_0} h_0 D \subset \bigcup_{h \in H} hD$. Additionally, for each $h = U(\cdot)AU^T$ in H there exists $h_0 = U_1(\cdot)U_1^T$ with $U_1 = UB$, $B = \operatorname{diag}(b_1, \ldots, b_n)$, $b_i^2 := a_i$ such that the restrictions to D of h and h_0 are the same. Finally, $W = \bigcup_{h_0 \in H_0} h_0 D = \bigcup_{h \in H} hD$. Thus condition (a) in Theorem 4.1 is fulfilled. Consequently, part (b) of this theorem follows. For instance, one can consider the operator $L : V \to V$ given by $LX = X^T$. Then $W = \{X \in V : LX = X\}$ and L is G-doubly stochastic, since the equality $s(X^T) = s(X)$ [10, p. 154] gives $LX \preceq_G X$ for $X \in V$.

On the other hand, an easy computation shows that the orthoprojector from V onto W is given by

$$PX = \frac{X + X^T}{2} \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}).$$

It follows from Theorem 4.1 that the Takagi's Decomposition and Theorems 2.1 and 3.1 imply

$$s\left(\frac{X + X^T}{2}\right) \preceq_w s(X) \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}).$$

This is an analogue of Fan-Hoffman's inequality (see Example 2.9). Conversely, the last inequality can be used to derive the decomposition.

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