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Positive Sets and Monotone Sets

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In this paper, we show how convex analysis can be applied to the theory of sets that are "positive" with respect to a continuous quadratic form on a Banach space. Monotone sets can be considered as a special case of positive sets, and we show how our results lead to very efficient proofs of a number of results on monotone sets. One of the key techniques that we use is a generalization of the Fitzpatrick function from monotone set theory to an analogous function for positive sets.

Introduction

In this paper, we discuss the theory of subsets of Banach space that are "positive" with respect to a certain quadratic form. In a sense to be made precise in Section 8, this theory subsumes the theory of monotone subsets of Banach spaces. We will see that this theory not only produces new results on positive and monotone sets, but also simpler proofs of some known results on monotone sets.

In Sections 1–2, we discuss the bilinear and quadratic forms on a Banach spaces that will be the object of this paper, and introduce the other notation that we will need. Bilinear forms on a Banach space have been studied extensively under the name "indefinite inner product". (See, for example, [1].) However, the results presented in this paper are of a totally different character.

In Section 3, we introduce q–positive sets, maximally q–positive sets, and the operation ${}^{\pi}$ of q–positive relationship. We also show how every q–positive subset, A, gives rise to a proper, convex, lower semicontinuous function, Φ_A. Conversely, we show in Lemma 3.5 how certain convex functions give rise to q–positive sets in a very natural way, deferring to Section 6 a more detailed discussion of exactly which q–positive sets can be produced by this procedure. In Theorem 3.6, we find which functions are of the form Φ_A for some q–positive set A and, in Theorem 3.7, we show what happens if we extend the definition of $\Phi_{\{.\}}$ to arbitrary subsets of F.

In Section 4, we discuss subtler results on maximally q–positive sets. Specifically, we give (in Theorem 4.1) a criterion for maximal q–positivity, valid under certain conditions on the basic bilinear form, in terms of an "additive transversal". We also give (in Theorem 4.3) conditions under which Lemma 3.5 can be used to produce *maximally* q–positive sets.

In Section 5, we introduce the iterated operation ${}^{\pi\pi}$ and discuss, (under the name "pre-maximally q–positive sets") those q–positive sets that have a unique maximally q–positive superset.

As promised above, in Section 6 we discuss (under the name "$\mathcal{S}$–q–positive sets") exactly

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which q–positive sets can be produced by the procedure of Lemma 3.5. (The "$\mathcal{S}$" stands for "special".) Given a q–positive set A, we write $\mathrm{cl}_{\mathcal{S}}(A)$ for the smallest $\mathcal{S}$–q–positive superset of A (which always exists). In Theorem 6.5, we give a number of connections between $\mathrm{cl}_{\mathcal{S}}(A)$, Φ_A and (pre)maximally q–positive subsets. In particular, Theorem 6.5(d) shows that, under certain circumstances, $\mathrm{cl}_{\mathcal{S}}(A)$ is the largest q–positive set with the same $\Phi_{\{\cdot\}}$ as A, and Theorem 6.5(b) shows that $\mathrm{cl}_{\mathcal{S}}(A) \subset A^{\pi\pi}$.

Motivated by the result just mentioned, we now consider the fascinating question of when $\mathrm{cl}_{\mathcal{S}}(A) = A^{\pi\pi}$. In Section 7, we give a complete solution of this problem in the finite–dimensional case. We will see in Theorem 7.3 that this depends critically on the "signature" of the basic bilinear form.

In Section 8, we show how the results of Sections 1–3 specialize to the theory of monotone sets for Banach spaces, obtaining many results that appeared in [4] and some new ones. In this case, Φ_A becomes the Fitzpatrick function, φ_A, $^{\pi}$ becomes the operation of monotone relationship, $^{\mu}$, and $\mathrm{cl}_{\mathcal{S}}(A)$ becomes the monotone representable closure of A, $\mathrm{cl}_{\mathcal{R}}(A)$, introduced in [6]. In Theorem 8.2, we find which functions are Fitzpatrick, and in Theorem 8.3 we show what happens if we extend the definition of $\varphi_{\{\cdot\}}$ to arbitrary subsets of $E \times E^*$.

In Section 9, we show how the results of Section 4 specialize to the theory of monotone sets. In Theorem 9.2, we deduce a result on maximal monotonicity that implies Rockafellar's surjectivity theorem, while in Theorem 9.3, we deduce a result that has been used for obtaining sufficient conditions for the sum of maximal monotone multifunctions on a reflexive space to be maximal monotone.

In Section 10, we show how the results of Section 5–6 specialize to the theory of monotone sets. Lemma 10.2, was motivated by and generalizes a result from [6]. The same comment can also be made about part of Theorem 10.5, though this theorem also contains some new results.

In Section 11, we show how the results of Section 7 specialize to the theory of monotone sets. We will see that the critical fact determining whether $\mathrm{cl}_{\mathcal{R}}(A) = A^{\mu\mu}$ in the monotone case is whether the underlying space has finite or infinite dimension. This is in contrast to the situation for the q–positive case, where, as we have already observed, the determining feature is the "signature" of the basic bilinear form. We finish the paper by giving (in Example 11.3) a simple example showing that an $\mathcal{R}$–monotone set (that is a representable monotone set in the notation of [6]) does not have to be premaximally monotone (where, as in [6], a premaximally monotone set is a monotone set with a unique maximally monotone cover).

Certain very special q–positive sets, A, have the property that the basic quadratic form is nonnegative on the effective domain of Φ_A. These are treated in the series of results Theorem 2.2, Theorem 5.5, Lemma 7.1, Lemma 7.2, Corollary 7.4, Theorem 10.3, Theorem 11.1, and Theorem 11.2.

It is explained in Example 8.1 exactly how the theory of positive subsets of a Banach space subsumes the theory of monotone subsets of a Banach space, and the analysis of Sections 8–11 shows the kind of problem that can be handled successfully using this positivity approach. What is lacking in the approach as presented here is an analog of the projection maps of $E \times E^*$ onto E and E^*, respectively, which would seem to preclude applying our techniques to problems involving the "domain" and "range" of a monotone multifunction. However, as pointed out above, Theorem 9.2, implies Rockafellar's surjectivity theorem which is a "range" result, and many of the applications of Theorem 9.3, mentioned above are "domain" results.

Fitzpatrick's seminal work ([4]) on the representation of maximal monotone multifunctions by convex functions was preceded by work of Krauss ([5]), with a representation in terms of saddle functions. In this connection, the formula for q in Section 7 is suggestive. Fitzpatrick's results were rediscovered by Burachik–Svaiter ([2]) and Martínez-Legaz–Théra ([7]). The first people to use Fitzpatrick's techniques to obtain results on monotone multifunctions other than representing them were Penot ([9]) and Zălinescu ([14]).

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1. Global notation

We will adopt the following notation all through this paper. F will be a nonzero real Banach space with topological dual F^* and $b(\cdot,\cdot)\colon\ F\times F\mapsto\mathbb{R}$ will be a continuous, symmetric, bilinear form that separates the points of F. We define the continuous quadratic form q by $q(x) := \frac{1}{2}b(x,x)$, and the injective linear map $\iota\colon\ F\mapsto F^*$ by $\iota(y) := b(\cdot,y)$, so that, for all $x,y\in F$, $b(x,y) = \big\langle x,\iota(y)\big\rangle$.

If $h\colon\ F\mapsto\,]-\infty,\infty]$ then we write $\operatorname{dom} h := \{x\in F\colon\ h(x)\in\mathbb{R}\}$. h is said to be *proper* if $\operatorname{dom} h\neq\emptyset$.

2. Certain bilinear forms on real Banach spaces

We start this section by giving some examples that fall or do not fall under the scope of Section 1.

Examples 2.1. (a) If F is a real Hilbert space with inner product $(x,y)\mapsto\langle x,y\rangle$ we can (not surprisingly) take $b(x,y) := \langle x,y\rangle$. However, we can also take $b(x,y) := -\langle x,y\rangle$. In both these cases, ι is a surjective isometry from F onto F^*.

(b) If $F = \big(\mathbb{R}^3,\|\cdot\|_2\big)$ then we can take the map defined by

$$b\big((x_1,x_2,x_3),(y_1,y_2,y_3)\big) := x_1y_2 + x_2y_1 + x_3y_3.$$

Again, ι is a surjective isometry from $\big(\mathbb{R}^3,\|\cdot\|_2\big)$ onto $\big(\mathbb{R}^3,\|\cdot\|_2\big)^*$.

(c) If $F = \big(\mathbb{R}^3, \|\cdot\|_2\big)$ then we cannot take the map defined by

$$b\big((x_1,x_2,x_3),(y_1,y_2,y_3)\big) := x_1y_2 + x_2y_3 + x_3y_1.$$

(The bilinear form b is not symmetric.)

(d) Let E be a nonzero real Banach space and E^* be its topological dual space. We norm $F := E \times E^*$ by $\big\|(x,x^*)\big\| := \sqrt{\|x\|^2 + \|x^*\|^2}$, and define the bilinear form $b\colon\ F \times F \mapsto \mathbb{R}$ by $b\big((x,x^*),(y,y^*)\big) := \langle x, y^*\rangle + \langle y, x^*\rangle$. Then ι is an isometry from $E \times E^*$ into $(E \times E^*)^*$ and if, further, E is reflexive then ι is surjective. This example will be considered in much more detail in Sections 8–11.

If $h\colon\ F \mapsto]-\infty,\infty]$ is proper, convex and lower semicontinuous and h^* is the *Fenchel conjugate* of h then, for all $y \in F$,

$$h^* \circ \iota(y) = h^*\big(\iota(y)\big) = \sup\nolimits_{x\in F}\big[b(x,y) - h(x)\big] = \sup\nolimits_{x\in F}\big[b(y,x) - h(x)\big], \tag{1}$$

from which $h^* \circ \iota$ is lower semicontinuous. If, further, $\iota\colon\ F \mapsto F^*$ is surjective then, from the *Fenchel–Moreau theorem* ([8]), for all $x \in F$, $h(x) = \sup_{x^*\in F^*}\big[\langle x, x^*\rangle - h^*(x^*)\big] = \sup_{y\in F}\big[\big\langle x, \iota(y)\big\rangle - h^* \circ \iota(y)\big] = \sup_{y\in F}\big[\big\langle y, \iota(x)\big\rangle - h^* \circ \iota(y)\big] = (h^* \circ \iota)^* \circ \iota(x)$. Thus

$$h = (h^* \circ \iota)^* \circ \iota \text{ on } F. \tag{2}$$

The final result in this section, Theorem 2.2, will be used explicitly in Theorem 5.5. As a general point of notation, we will write $\mathbb{I}_C^*$ instead of the more cumbersome $(\mathbb{I}_C)^*$. We will adopt a similar convention with S_C^*, Φ_A^*, etc.

Theorem 2.2. *Suppose that ι is surjective and C is a nonempty closed convex subset of F. Define the* indicator function $\mathbb{I}_C\colon\ F \mapsto]-\infty,\infty]$ *by*

$$\mathbb{I}_C(x) := \begin{cases} 0 & \textit{if } x \in C; \\ \infty & \textit{otherwise,} \end{cases}$$

and (using (1)*) the* support functional $S_C\colon\ F \mapsto]-\infty,\infty]$ *by*

$$S_C(y) := \mathbb{I}_C^* \circ \iota(y) = \sup\nolimits_{x\in C} b(x,y) \quad (y \in F). \tag{3}$$

The functions $\mathbb{I}_C$ and S_C are proper, convex and lower semicontinuous. Suppose that

$$q \geq 0 \textit{ on } C \quad \textit{and} \quad \operatorname{dom} S_C \subset C, \tag{4}$$

and let $H := \{y \in C\colon\ q(y) = 0\}$. Then

$$y \in \operatorname{dom} S_C \implies y \in H \implies S_C(-y) \leq 0 \implies -y \in \operatorname{dom} S_C \implies y \in \operatorname{dom} S_C, \tag{5}$$

$$H \textit{ is a subspace of } F, \quad \textit{and} \quad C = H^\perp := \big\{x \in F\colon\ b(x,H) = \{0\}\big\}. \tag{6}$$

Proof. First, let $y \in \operatorname{dom} S_C$. Let $\lambda > 0$. Since S_C is positively homogeneous, $\lambda y \in \operatorname{dom} S_C$. From (4), $\lambda y \in C$, thus $\lambda q(y) = \frac{1}{2}\lambda b(y,y) = \frac{1}{2}b(\lambda y, y) \leq \frac{1}{2}S_C(y) < \infty$. It follows by letting $\lambda \to \infty$ in this that $q(y) \leq 0$ and, using (4) again, that $y \in H$, which gives the first implication in (5). Next, let $y \in H$ and x be an arbitrary element of C. Then, for all $\lambda > 0$, $(x+\lambda y)/(1+\lambda) \in C$, so (4) gives $q(x+\lambda y) \geq 0$. Expanding this out, $q(x) + \lambda b(x,y) \geq 0$. Letting $\lambda \to \infty$, we derive that $b(x,y) \geq 0$, from which $b(x,-y) \leq 0$. Taking the supremum over $x \in C$, $S_C(-y) \leq 0$, which gives the second implication in (5). It is clear that if $S_C(-y) \leq 0$ then $-y \in \operatorname{dom} S_C$. Finally, if $-y \in \operatorname{dom} S_C$ then, replacing y by $-y$ in what we have already proved, we see that $y \in \operatorname{dom} S_C$. This completes the proof of (5).

Since $\operatorname{dom} S_C$ is a cone, (5) implies that $\operatorname{dom} S_C$ is a subspace of F and that $H = \operatorname{dom} S_C$, so H is a subspace of F, as required. (5) also implies that $S_C \leq 0$ on H. However, S_C is real and sublinear on H, consequently $S_C = 0$ on H. Since $S_C = \infty$ on $F \setminus H$, in fact S_C is the indicator function of H, $\mathbb{I}_H$. (3) and (2) now imply that, for all $x \in F$,

$$\mathbb{I}_C(x) = \big(\mathbb{I}_C^* \circ \iota\big)^* \circ \iota(x) = S_C^* \circ \iota(x) = \mathbb{I}_H^* \circ \iota(x) = \sup\nolimits_{y\in H} b(x,y) = \mathbb{I}_{H^\perp}(x),$$

where the last equality depends on the fact that H is a subspace. So, $C = H^\perp$, completing the proof of (6). □

3. Positive sets with respect to certain quadratic forms on Banach spaces

Definition 3.1. Let $\emptyset \neq A \subset F$. We say that A is *q–positive* if

$$a_1, a_2 \in A \implies q(a_1 - a_2) \geq 0.$$

Now let A be q–positive. We define the function $\Phi_A\colon\ F \mapsto]-\infty,\infty]$ associated with A by

$$\Phi_A(x) := \sup\nolimits_{a\in A} \big[b(x,a) - q(a)\big] = q(x) - \inf\nolimits_{a\in A} q(x-a).$$

These definitions imply that Φ_A is proper, convex and lower semicontinuous,

$$\Phi_A = q \text{ on } A, \quad \text{and} \quad y \in F \text{ and } a \in A \implies b(y,a) \leq \Phi_A(y) + q(a). \tag{7}$$

If $a \in A$ then, from (7), for all $y \in F$, $b(a,y) - \Phi_A(y) \leq q(a)$. Taking the supremum over $y \in F$:

$$\Phi_A^* \circ \iota \leq q \text{ on } A. \tag{8}$$

Let $z \in F$, and write $A - z$ for the q–positive set $\{a - z\colon\ a \in A\} \subset F$. By direct computation,

$$x \in F \implies \Phi_{A-z}(x) = \Phi_A(x+z) - b(x+z,z) + q(z) \tag{9}$$

and

$$y \in F \implies \Phi_{A-z}^* \circ \iota(y) = \Phi_A^* \circ \iota(y+z) - b(z,y) - q(z). \tag{10}$$

Let $x \in F$. x is said to be *q–positively related to* A if $a \in A \Longrightarrow q(x-a) \geq 0$ or, equivalently, $A \cup \{x\}$ is q–positive. We write A^π for the set of all elements of F that are q–positively related to A. Then it is clear from the definition of Φ_A and (7) that

$$x \in A^\pi \iff \Phi_A(x) \leq q(x), \quad \text{from which} \quad A \subset A^\pi \subset \operatorname{dom}\Phi_A. \tag{11}$$

We note that if $z \in F$ then $y \in (A-z)^\pi$ if, and only if, $a \in A \Longrightarrow q\big(y - (a-z)\big) \geq 0$ if, and only if, $a \in A \Longrightarrow q(y+z-a) \geq 0$ if, and only if, $y + z \in A^\pi$. Thus

$$(A-z)^\pi = A^\pi - z. \tag{12}$$

A^π is not generally q–positive — see Lemma 5.4 and Example 11.3.

Examples 3.2. We now give some examples of q–positive sets. We first make the elementary observation that if $x \in F$ and $q(x) \geq 0$ then the linear span $\mathbb{R}x$ of $\{x\}$ is q–positive.

In Example 2.1(a), every subset of F is q–positive if $b(x,y) := \langle x, y\rangle$, and the q–positive sets are the singletons if $b(x,y) := -\langle x, y\rangle$.

In Example 2.1(b), If M is any nonempty monotone subset of $\mathbb{R} \times \mathbb{R}$ then $M \times \mathbb{R}$ is a q–positive subset of F. The set $\mathbb{R}(1, -1, 2)$ is a q–positive subset of F which is not contained in a set $M \times \mathbb{R}$ for any monotone subset of $\mathbb{R} \times \mathbb{R}$. The helix $\big\{(\cos\theta, \sin\theta, \theta)\colon \theta \in \mathbb{R}\big\}$ is a q–positive subset of F, but if $0 < \lambda < 1$ then the helix $\big\{(\cos\theta, \sin\theta, \lambda\theta)\colon \theta \in \mathbb{R}\big\}$ is not.

In Example 2.1(d), the q–positive sets have been extensively studied. We will return to this situation in Sections 8–11.

Definition 3.3. We now introduce two very useful pieces of notation. Firstly, let $\mathcal{H}$ consist of all those convex lower semicontinuous functions $h\colon\ F \mapsto]-\infty, \infty]$ such that $h \geq q$ on F. Secondly, if $h \in \mathcal{H}$, let $K(h) := \big\{x \in F\colon\ h(x) = q(x)\big\}$.

Definition 3.4. Let $\emptyset \neq M \subset F$. We say that M is *maximally q–positive* if M is q–positive and not properly contained in any other q–positive set. Since the continuity of q implies that the closure of any q–positive set is q–positive, any maximally q–positive set is closed. Now let M be maximally q–positive. Then $M^\pi \subset M$, so that if $x \in F \setminus M$ then (11) gives $\Phi_M(x) > q(x)$. On the other hand, if $x \in M$ then (7) gives $\Phi_M(x) = q(x)$. Combining these two observations:

$$\Phi_M \in \mathcal{H} \quad \text{and} \quad M = K(\Phi_M). \tag{13}$$

It follows from (11) that if A is nonempty and q–positive then

$$A \text{ is maximally } q\text{–positive} \iff A^\pi \subset A \iff A^\pi = A. \tag{14}$$

We will give a nontrivial characterization of maximally q–positive subsets in certain special but common situations in Theorem 4.1.

Let $x \in F$. Then we see from (1) and (7) that

$$\Phi_A^* \circ \iota(x) \geq \sup\nolimits_{a\in A}\big[b(x,a) - \Phi_A(a)\big] = \sup\nolimits_{a\in A}\big[b(x,a) - q(a)\big] = \Phi_A(x). \tag{15}$$

We now show that

$$\Phi_A^* \circ \iota \in \mathcal{H} \quad \text{and} \quad A \subset K(\Phi_A^* \circ \iota). \tag{16}$$

To see this, we first use Zorn's lemma to find a maximally q–positive set M such that $M \supset A$. Clearly $\Phi_M \geq \Phi_A$ on F, from which $\Phi_A^* \circ \iota \geq \Phi_M^* \circ \iota$ on F. From (15) with A replaced by M, $\Phi_M^* \circ \iota \geq \Phi_M$ on F and, from (13), $\Phi_M \geq q$ on F. The first assertion in (16) follows by combining the last three inequalities, and the second assertion follows by combining the first one with (8).

We next give a simple result which enables us to obtain q–positive sets from certain convex functions. In this result, we use the "parallelogram law". Lemma 3.5 will be used explicitly in Theorem 3.6, Theorem 3.7, Theorem 4.3, Theorem 6.5, Lemma 7.2 and Section 8.

Lemma 3.5. *Let $h \in \mathcal{H}$ and $K(h) \neq \emptyset$. Then $K(h)$ is q–positive.*

Proof. This follows from the observation that if $a_1, a_2 \in K(h)$ then

$$\frac{1}{4}q(a_1 - a_2) = \frac{q(a_1) + q(a_2)}{2} - q\Big(\frac{a_1 + a_2}{2}\Big) \geq \frac{h(a_1) + h(a_2)}{2} - h\Big(\frac{a_1 + a_2}{2}\Big) \geq 0.$$

□

We next give a characterization of functions of the form Φ_A.

Theorem 3.6. *Let $f\colon\ F \mapsto]{-\infty}, \infty]$ be proper, convex and lower semicontinuous. Then there exists a nonempty q–positive subset A of F such that $f = \Phi_A$ on F if, and only if,*

$$f^* \circ \iota \in \mathcal{H} \quad \textit{and}, \quad \textit{for all } x \in F,\ f(x) \leq \sup_{z \in K(f^* \circ \iota)} \big[b(x, z) - q(z)\big]. \tag{17}$$

Proof. ($\Longrightarrow$) If A is a nonempty q–positive subset of F then, from (16), $\Phi_A^* \circ \iota \in \mathcal{H}$ and $A \subset K(\Phi_A^* \circ \iota)$. Thus $\Phi_A(x) = \sup_{a \in A} \big[b(x, a) - q(a)\big] \leq \sup_{z \in K(\Phi_A^* \circ \iota)} \big[b(x, z) - q(z)\big]$. This gives the desired result.

($\Longleftarrow$) If f satisfies (17) then clearly $K(f^* \circ \iota) \neq \emptyset$ thus, from Lemma 3.5, $K(f^* \circ \iota)$ is q–positive. If $x \in F$ and $z \in K(f^* \circ \iota)$ then the Fenchel–Young inequality implies that $b(x, z) = \langle x, \iota(z)\rangle \leq f(x) + f^* \circ \iota(z) = f(x) + q(z)$. Consequently, $b(x, z) - q(z) \leq f(x)$. Combining this with (17), we obtain that

$$x \in F \implies f(x) = \sup_{z \in K(f^* \circ \iota)} \big[b(x, z) - q(z)\big] = \Phi_{K(f^* \circ \iota)}(x).$$

The desired result follows by taking $A = K(f^* \circ \iota)$. □

We conclude this section with a strange result that shows what happens if we extend the definition of $\Phi_{\{\cdot\}}$ to arbitrary subsets of F.

Theorem 3.7. *Suppose that A is a nonempty q–positive subset of F and $D \subset F$. For all $x \in F$, let $\Phi_D(x) := \sup_{d \in D} \big[b(x, d) - q(d)\big]$. If $\Phi_D \leq \Phi_A$ on F then D is q–positive.*

Proof. If d is an arbitrary element of D then, by hypothesis,

$$x \in F \implies b(x, d) - q(d) \leq \Phi_A(x) \implies b(x, d) - \Phi_A(x) \leq q(d),$$

from which

$$\Phi_A^* \circ \iota(d) = \sup_{x \in F} \big[b(x, d) - \Phi_A(x)\big] \leq q(d).$$

Thus, from (16), $d \in K(\Phi_A^* \circ \iota)$. Consequently, we have proved that $D \subset K(\Phi_A^* \circ \iota)$, and the result follows from Lemma 3.5. □

4. An additive transversal for maximally positive sets

For the analysis in this section, it seems necessary to assume that the map ι is an isometry. This condition is true in Examples 2.1(a,b,d). We note then that, for all $x \in F$,

$$|q(x)| = \tfrac{1}{2}\big|\big\langle x, \iota(x)\big\rangle\big| \leq \tfrac{1}{2}\|x\|\|\iota(x)\| = \tfrac{1}{2}\|x\|^2. \tag{18}$$

We then define

$$G = \big\{x \in F\colon\ q(x) = -\tfrac{1}{2}\|x\|^2\big\}. \tag{19}$$

Theorem 4.1. *Suppose that ι is a surjective isometry and A is a nonempty q–positive subset of F. Then*

$$A \text{ is maximally } q\text{–positive} \iff A + G = F.$$

Proof. ($\Longleftarrow$) Suppose that $y \in A^\pi$. By hypothesis, there exists $a \in A$ such that $y - a \in G$. But then $q(y-a) \geq 0$, from which $\frac{1}{2}\|y-a\|^2 \leq 0$, and so $y = a$. Thus we have proved that $A^\pi \subset A$, and (14) now implies that A is maximally q–positive.

($\Longrightarrow$) Suppose, conversely, that A is maximally q–positive and y is an arbitrary element of F. Let $M = A - y$. Clearly, M is maximally q–positive. From (13),

$$x \in F \implies \Phi_M(x) + \tfrac{1}{2}\|x\|^2 \geq \tfrac{1}{2}\|x\|^2 + q(x) \geq 0.$$

Thus, from Rockafellar's version of the Fenchel duality theorem (see Rockafellar, [11, Theorem 1, pp. 82–83] for the original version and Zălinescu, [18, Theorem 2.8.7, pp. 126–127] for more general results), there exists $z \in F$ such that $\Phi_M^*\big(\iota(z)\big) + \big\|-\iota(z)\big\|^2 \leq 0$, that is to say $\Phi_M^* \circ \iota(z) + \|z\|^2 \leq 0$. Combining this with (18), (13) and (15), we have

$$0 \leq \tfrac{1}{2}\|z\|^2 + q(z) \leq \Phi_M(z) + \tfrac{1}{2}\|z\|^2 \leq \Phi_M^* \circ \iota(z) + \tfrac{1}{2}\|z\|^2 \leq 0,$$

from which we derive that $\frac{1}{2}\|z\|^2 + q(z) = 0$ and $\Phi_M(z) = q(z)$. Thus $z \in G = -G$ and, using (13) again, $z \in M$, that is to say, $y + z \in A$. But then $y = (y+z) - z \in A + G$. $\square$

Lemma 4.2. *Suppose that $\iota\colon\ F \mapsto F^*$ is an isometry, $y \in F$ and $g\colon\ F \mapsto \mathbb{R}$ is defined by $g(x) := q(x-y) + \frac{1}{2}\|x-y\|^2 - q(x)$. Then:*

(*a*) $g + q \geq 0$ *on* F.
(*b*) *The function g is convex and continuous on F.*
(*c*) *Let $z \in F$. Then $g^* \circ \iota(-z) = g(z)$.*

Proof. (*a*) is immediate from (18).

(*b*) follows from the identity $g(x) = -b(x,y) + q(y) + \frac{1}{2}\|x-y\|^2$.

(*c*) Using the formula in (*b*) and the fact that $\left(\frac{1}{2}\|\cdot\|^2\right)^* = \frac{1}{2}\|\cdot\|^2$,

$$\begin{aligned} g^* \circ \iota(-z) &= \sup\nolimits_{x \in F}\left[b(x,-z) + b(x,y) - q(y) - \tfrac{1}{2}\big\|x-y\big\|^2\right] \\ &= \sup\nolimits_{w \in F}\left[b(w+y,-z) + b(w+y,y) - q(y) - \tfrac{1}{2}\big\|w\big\|^2\right] \\ &= \sup\nolimits_{w \in F}\left[b(w,y-z) - \tfrac{1}{2}\big\|w\big\|^2\right] - b(y,z) + q(y) \\ &= \tfrac{1}{2}\big\|y-z\big\|^2 - b(y,z) + q(y) = g(z). \end{aligned}$$

$\square$

Theorem 4.3. *Suppose that $\iota\colon\ F \mapsto F^*$ is a surjective isometry, $h\colon F \mapsto]-\infty,\infty]$ is proper and convex, $h \geq q$ on F and $h^* \circ \iota \geq q$ on F. Then:*

(*a*) *$K(h^* \circ \iota)$ is maximally q–positive.*
(*b*) *If h is lower semicontinuous then $K(h)$ is maximally q–positive.*

Proof. (*a*) Let y be an arbitrary element of F. Then, using Lemma 4.2(a),

$$h + g \geq q + g \geq 0 \text{ on } F.$$

Thus, from Lemma 4.2(b) and Rockafellar's version of the Fenchel duality theorem again, there exists $z \in F$ such that $h^* \circ \iota(z) + g^* \circ \iota(-z) \leq 0$ and so, from Lemma 4.2(c), $h^* \circ \iota(z) + g(z) \leq 0$. Using (18) and the fact that $h^* \circ \iota \in \mathcal{H}$, we have

$$0 \leq q(z - y) + \tfrac{1}{2}\|z - y\|^2 = q(z) + g(z) \leq h^* \circ \iota(z) + g(z) \leq 0.$$

All the inequalities in the line above must be equalities, and so $h^* \circ \iota(z) = q(z)$, that is to say $z \in K(h^* \circ \iota)$, and $q(z - y) + \frac{1}{2}\|z - y\|^2 = 0$. Thus $K(h^* \circ \iota) \neq \emptyset$ and so, from Lemma 3.5, $K(h^* \circ \iota)$ is q–positive. Now suppose that $y \in K(h^* \circ \iota)^\pi$. In this case, $q(y - z) \geq 0$, from which $\frac{1}{2}\|y - z\|^2 \leq 0$, and so $y = z$. Thus we have proved that $K(h^* \circ \iota)^\pi \subset K(h^* \circ \iota)$, and (14) now implies that A is maximally q–positive.

(b) If h is lower semicontinuous then the result follows from (2) by applying (a) with h replaced by $(h^* \circ \iota)^*$. □

5. The iterated operation $^{\pi\pi}$ and premaximally q–positive sets

We write $y \in A^{\pi\pi}$ if $\quad x \in A^\pi \Longrightarrow q(y - x) \geq 0.$ It is immediate from this definition that $A^{\pi\pi} \supset A$ and, from (11) and the fact that the operation $^\pi$ is inclusion–reversing, that $A^{\pi\pi} \subset A^\pi$ — which implies that $A^{\pi\pi}$ is q–positive. To sum up:

$$A \subset A^{\pi\pi} \subset A^\pi \quad \text{and} \quad A^{\pi\pi} \text{ is } q\text{–positive.} \tag{20}$$

It is easy to see from an argument similar to that used in (12) that if $z \in F$ then

$$(A - z)^{\pi\pi} = A^{\pi\pi} - z. \tag{21}$$

In Section 7, we will consider the nontrivial problem of finding a representation for the set $A^{\pi\pi}$. The results contained in Lemma 5.1 below will be critical. Lemma 5.1(c) will be used explicitly in Theorem 5.5. As will become evident in Section 8, Lemma 5.1(b) generalizes [6, Proposition 28], Lemma 5.1(c) generalizes [6, Proposition 26] (for monotone A), and Lemma 5.1(d) (which is a special case of Theorem 5.2(b)) generalizes [6, Proposition 29].

Lemma 5.1. *Suppose that A is a nonempty q–positive subset of F. Then:*

(a) $\Phi_A \leq q$ *on* $A^{\pi\pi}$.

(b) *If* $0 \in A^{\pi\pi}$, $x \in F$ *with* $q(x) < 0$, *and* $\kappa \in]0, 1[$ *then*

$$\kappa^2 q(x) < \kappa\Phi_A(x) + (1 - \kappa)\Phi_A(0) \quad \textit{and} \quad \Phi_A(x) \geq 0. \tag{22}$$

In particular,

$$\Phi_A \vee q \geq 0 \ \textit{on} \ F. \tag{23}$$

(c) *If* $0 \in A^{\pi\pi}$ *and* $\Phi_A(0) < 0$ *then* $q \geq 0$ *on* $\operatorname{dom}\Phi_A$.

(d) *If* $0 \in A^{\pi\pi}$ *and* $\Phi_A(0) \geq 0$ *then* $\Phi_A \geq 0$ *on* F. *Consequently,* $\Phi_A^*(0) \leq 0$.

Proof. (a) is immediate from (20) and (11).

(b) Write $w = \kappa x = \kappa x + (1-\kappa)0$. Then $q(w) = \kappa^2 q(x) < 0$. Since $0 \in A^{\pi\pi}$, it follows that $w \notin A^{\pi}$, and (11) then implies that $q(w) < \Phi_A(w)$. We now obtain the first part of (22) from the convexity of Φ_A. (a) now gives us that $\kappa^2 q(x) < \kappa\Phi_A(x)$, and so $\kappa q(x) < \Phi_A(x)$, and the second part of (22) follows by letting $\kappa \to 0$.

(c) Suppose that there exists $x \in \operatorname{dom}\Phi_A$ such that $q(x) < 0$. We derive the contradiction $\Phi_A(0) \geq 0$ by letting $\kappa \to 0+$ in the first assertion in (22).

(d) Let $y \in F$ and $\lambda \in]0,1[$. If $a \in A$, write $x = \lambda y + (1-\lambda)a$. Then, from (7),

$$\Phi_A(x) \leq \lambda\Phi_A(y) + (1-\lambda)\Phi_A(a) = \lambda\Phi_A(y) + (1-\lambda)q(a).$$

On the other hand, noting from (7) (again) that $b(y,a) \leq \Phi_A(y) + q(a)$, we have

$$\begin{array}{rcl} q(x) & = & q\big(\lambda y + (1-\lambda)a\big) = \lambda^2 q(y) + \lambda(1-\lambda)b(y,a) + (1-\lambda)^2 q(a) \\ & \leq & \lambda^2 q(y) + \lambda(1-\lambda)\Phi_A(y) + \lambda(1-\lambda)q(a) + (1-\lambda)^2 q(a) \\ & = & \lambda^2 q(y) + \lambda(1-\lambda)\Phi_A(y) + (1-\lambda)q(a). \end{array}$$

(23) now implies that

$$\big[\lambda\Phi_A(y) + (1-\lambda)q(a)\big] \vee \big[\lambda^2 q(y) + \lambda(1-\lambda)\Phi_A(y) + (1-\lambda)q(a)\big] \geq 0,$$

and so $\lambda\Phi_A(y) \vee \big[\lambda^2 q(y) + \lambda(1-\lambda)\Phi_A(y)\big] \geq -(1-\lambda)q(a)$. However, $\sup_{a\in A}\big[-q(a)\big] = \Phi_A(0) \geq 0$ thus, taking the supremum of the right hand side over $a \in A$,

$$\lambda\Phi_A(y) \vee \big[\lambda^2 q(y) + \lambda(1-\lambda)\Phi_A(y)\big] \geq 0.$$

Consequently, $\Phi_A(y) \vee \big[\lambda q(y) + (1-\lambda)\Phi_A(y)\big] \geq 0$, and letting $\lambda \to 0$ implies that $\Phi_A(y) \geq 0$, as required. The final observation follows since $\Phi_A^*(0) = \sup_{y\in F}\big[-\Phi_A(y)\big]$. $\square$

Theorem 5.2(b) will be used explicitly in Theorem 6.5(b), and both parts of Theorem 5.2 will be used in Section 8.

Theorem 5.2. *Suppose that A is a nonempty q–positive subset of F. Then:*

(a) $\Phi_A \vee q \geq \Phi_{A^{\pi\pi}}$ *on* F.
(b) *If* $z \in A^{\pi\pi}$ *and* $\Phi_A(z) \geq q(z)$ *then* $\Phi_A \geq \iota(z) - q(z)$ *on* F *and* $\Phi_A^* \circ \iota(z) \leq q(z)$.

Proof. (a) Let $w \in F$ and $z \in A^{\pi\pi}$. From (21), $0 \in (A-z)^{\pi\pi}$ and so (23) gives us that $\Phi_{A-z}(w-z) \vee q(w-z) \geq 0$. Using (9), we can rewrite this conclusion as: $\big[\Phi_A(w) - b(w,z) + q(z)\big] \vee \big[q(w) - b(w,z) + q(z)\big] \geq 0$, that is to say, $\Phi_A(w) \vee q(w) \geq b(w,z) - q(z)$. (a) follows since $\sup_{z\in A^{\pi\pi}}\big[b(w,z) - q(z)\big] = \Phi_{A^{\pi\pi}}(w)$.

(b) As in (a), $0 \in (A-z)^{\pi\pi}$ and $\Phi_{A-z}(0) \geq 0$, and so Lemma 5.1(d) implies that $\Phi_{A-z} \geq 0$ on F and $\Phi_{A-z}^*(0) \leq 0$. It now follows from (9) and (10) that $\Phi_A \geq \iota(z) - q(z)$ on F and $\Phi_A^* \circ \iota(z) \leq q(z)$, as required. $\square$

As will become evident in Section 8, Definition 5.3 below is a generalization of [6, Definition 35], and Lemma 5.4 is a generalization of most of [6, Proposition 36]. Lemma 5.4 will be used explicitly in Theorem 5.5, Theorem 6.5(c), and Corollary 7.4.

Definition 5.3. Suppose that A is a nonempty q–positive subset of F. We say that M is a *maximally q–positive cover for A* if M is maximally q–positive and $M \supset A$. We say that A is *premaximally q–positive* if A has a unique maximally q–positive cover.

Lemma 5.4. *Suppose that A is a nonempty q–positive subset of F. Then the conditions* (24)–(28) *are equivalent.*

$$A \textit{ is premaximally } q\textit{–positive,} \tag{24}$$

$$A^{\pi} = A^{\pi\pi}, \tag{25}$$

$$A^{\pi} \textit{ is } q\textit{–positive,} \tag{26}$$

$$A^{\pi} \textit{ is maximally } q\textit{–positive,} \tag{27}$$

$$A^{\pi\pi} \textit{ is maximally } q\textit{–positive.} \tag{28}$$

Furthermore, in this case $A^{\pi\pi} = A^{\pi}$ is the unique maximally q–positive cover for A.

Proof. If (24) is true, let M be the unique maximally q–positive cover of A. Now let x be an arbitrary element of A^{π}. Since $A \cup \{x\}$ is q–positive, Zorn's lemma gives a maximally q–positive cover M' of $A \cup \{x\}$. Since M' is then a maximally q–positive cover of A, the uniqueness of M implies that $M' = M$. So we have proved that $A^{\pi} \subset M$. It follows from this that $M^{\pi} \subset A^{\pi\pi}$, and so (20) (applied to both A and M) gives (25).

It is immediate from (20) that (25) implies (26).

If (26) is true then (14) (applied to A^{π}) and (20) give (27).

If (27) is true then $A^{\pi\pi} = A^{\pi}$, from which (28) is immediate.

If, finally, (28) is true, suppose that M is a maximally q–positive cover of A. Then $M^{\pi\pi} \supset A^{\pi\pi}$, and the maximality of $A^{\pi\pi}$ gives $M^{\pi\pi} = A^{\pi\pi}$. However, two applications of (14) imply that $M^{\pi\pi} = M$, and so M is uniquely determined, which gives (24), and the final observation that $M = A^{\pi\pi} = A^{\pi}$. □

Theorem 5.5 will be used explicitly in Lemma 7.1, Lemma 7.2, Corollary 7.4 and Theorem 10.3.

Theorem 5.5. *Suppose that ι is surjective and A is a nonempty q–positive subset of F. Consider the condition*

$$q \geq 0 \textit{ on } \operatorname{dom} \Phi_A, \tag{29}$$

and write $H = \{y \in \operatorname{dom} \Phi_A\colon\ q(y) = 0\}$.

(*a*) *If* (29) *is satisfied then A is premaximally q–positive and* $\operatorname{dom} \Phi_A$ *is the unique maximally q–positive cover of A. Further,* $\operatorname{dom} \Phi_A$ *and H are both subspaces of F and* $\operatorname{dom} \Phi_A = H^{\perp}$.

(*b*) *If $0 \in A^{\pi\pi}$ and $\Phi_A(0) < 0$ then* (29) *is satisfied, $q \geq 0$ on $H^{\perp}$ and $H \cap A = \emptyset$.*

Proof. (*a*) Let $C = \overline{\operatorname{dom} \Phi_A}$. It follows from (29) and the continuity of the map q that the first assertion in (4) is satisfied. Further, if $y \in \operatorname{dom} S_C$ then, using the first assertion in (4) and (7),

$$\infty > S_C(y) = \sup\nolimits_{x \in C} b(x, y) \geq \sup\nolimits_{x \in C} \big[b(x, y) - q(x)\big] \geq \sup\nolimits_{a \in A} \big[b(a, y) - q(a)\big] = \Phi_A(y),$$

so $\operatorname{dom} S_C \subset \operatorname{dom} \Phi_A \subset C$, and thus all of (4) is satisfied. We now deduce from Theorem 2.2 that C is a subspace of F. If now $d_1, d_2 \in C$ then $d_1 - d_2 \in C$, and so, from (4), C is q–positive. Now (11) implies that $A^\pi \subset \operatorname{dom} \Phi_A \subset C$, and so A^π is q–positive. From Lemma 5.4, A is premaximally q–positive and A^π is the unique maximally q–positive cover of A. The inclusion $A^\pi \subset \operatorname{dom} \Phi_A \subset C$ now implies that $A^\pi = \operatorname{dom} \Phi_A = C$. The remaining assertions follow by making the substitution $C = \operatorname{dom} \Phi_A$ in (6).

(b) is immediate from Lemma 5.1(c), (a) and the fact that $\Phi_A(0) = \sup_{a \in A} \big[-q(a)\big]$. □

6. $\mathcal{S}$–q–positive sets

In this section, we investigate those q–positive sets that are produced by the procedure of Lemma 3.5.

Lemma 6.1.

(a) *Suppose that A is a nonempty q–positive subset of F, and define $\theta_A\colon\ F \mapsto]-\infty, \infty]$ by*

$$\theta_A = \sup\{h\colon\ h \in \mathcal{H},\ K(h) \supset A\}.$$

Then $\theta_A \in \mathcal{H}$ and $K(\theta_A) \supset A$. If $h \in \mathcal{H}$ and $K(h) \supset A$ then $h \le \theta_A$ on F. Furthermore, $\theta_A \ge \Phi_A^* \circ \iota$ on F.

(b) *If ι is surjective and $A \subset F$ is nonempty and q–positive then $\theta_A = \Phi_A^* \circ \iota$.*

Proof. (a) is immediate from the definitions and (16).

(b) If $h \in \mathcal{H}$ and $K(h) \supset A$ then $h = q$ on A. Thus, from (1), for all $y \in F$,

$$h^* \circ \iota(y) \ge \sup_{a \in A} \big[b(y, a) - h(a)\big] = \sup_{a \in A} \big[b(y, a) - q(a)\big] = \Phi_A(y),$$

and (2) now implies that $h = (h^* \circ \iota)^* \circ \iota \le \Phi_A^* \circ \iota$ on F. Taking the supremum over h, $\theta_A \le \Phi_A^* \circ \iota$ on F. The opposite inequality was established in (a). □

Definition 6.2. Suppose that A is a nonempty q–positive subset of F. We say that A is *$\mathcal{S}$–q–positive* if there exists $h \in \mathcal{H}$ such that $A = K(h)$.

Theorem 6.3.

(a) *Every maximally q–positive subset of F is $\mathcal{S}$–q–positive.*

(b) *If $A \subset F$ is nonempty and q–positive then $K(\theta_A)$ is the smallest $\mathcal{S}$–q–positive superset of A.*

Proof. This is immediate from (13) and Lemma 6.1(a). □

Definition 6.4. Suppose that A is a nonempty q–positive subset of F. We write $\operatorname{cl}_{\mathcal{S}}(A)$ for the smallest $\mathcal{S}$–q–positive subset of F containing A, that is to say, $\operatorname{cl}_{\mathcal{S}}(A) = K(\theta_A)$. $\operatorname{cl}_{\mathcal{S}}(\cdot)$ can be thought of as an abstract closure operation. (16) implies that $\operatorname{cl}_{\mathcal{S}}(A) \subset K(\Phi_A^* \circ \iota)$, and Lemma 6.1$(b)$ that if ι is surjective then $\operatorname{cl}_{\mathcal{S}}(A) = K(\Phi_A^* \circ \iota)$ — in this case, Theorem 6.5(a) below gives another characterization of $\operatorname{cl}_{\mathcal{S}}(A)$.

Suppose that $z \in F$ and A is a nonempty q–positive subset of F. It is easy to see from the definitions that, for all $x \in F$, $\theta_{A-z}(x) = \theta_A(x+z) - b(x, z) - q(z)$, thus $y \in \operatorname{cl}_{\mathcal{S}}(A-z)$

if, and only if, $\theta_A(y+z) - b(y,z) - q(z) = q(y)$ if, and only if, $\theta_A(y+z) = q(y+z)$, if and only if $y+z \in \mathrm{cl}_{\mathcal{S}}(A)$. Consequently,

$$\mathrm{cl}_{\mathcal{S}}(A-z) = \mathrm{cl}_{\mathcal{S}}(A) - z.$$

Theorem 6.5 will be used explicitly in Theorem 7.3, Corollary 7.5 and Theorem 10.5.

Theorem 6.5. *Suppose that A is a nonempty q–positive subset of F. Then:*

(*a*) $\Phi_{K(\Phi_A^*\circ\iota)} = \Phi_A$ *on F and $K(\Phi_A^*\circ\iota)$ is the largest q–positive subset B of F such that $\Phi_B = \Phi_A$ on F.*

(*b*) $\{z \in A^{\pi\pi}\colon\ \Phi_A(z) \geq q(z)\} \subset K(\Phi_A^*\circ\iota) \subset A^{\pi\pi}$. *Consequently,* $\mathrm{cl}_{\mathcal{S}}(A) \subset A^{\pi\pi}$.

(*c*) *If $K(\Phi_A^*\circ\iota)$ is maximally q–positive then $\Phi_A \in \mathcal{H}$. If, conversely, $\Phi_A \in \mathcal{H}$ then $K(\Phi_A^*\circ\iota) = A^{\pi\pi}$, A is premaximally q–positive, and $K(\Phi_A^*\circ\iota)$ is the unique maximally q–positive cover of A.*

(*d*) $\Phi_{\mathrm{cl}_{\mathcal{S}}(A)} = \Phi_A$ *on F. If $\mathrm{cl}_{\mathcal{S}}(A)$ is maximally q–positive then $\Phi_A \in \mathcal{H}$. If ι is surjective then $\mathrm{cl}_{\mathcal{S}}(A)$ is the largest q–positive subset B of F such that $\Phi_B = \Phi_A$ on F and if, further, $\Phi_A \in \mathcal{H}$ then $\mathrm{cl}_{\mathcal{S}}(A)$ is the unique maximally q–positive cover of A.*

Proof. If $x \in F$ and $z \in K(\Phi_A^*\circ\iota)$ then the Fenchel–Young inequality implies that $b(x,z) = \langle x, \iota(z)\rangle \leq \Phi_A(x) + \Phi_A^*\circ\iota(z) = \Phi_A(x) + q(z)$. Consequently,

$$x \in F \implies \Phi_{K(\Phi_A^*\circ\iota)}(x) = \sup\nolimits_{z\in K(\Phi_A^*\circ\iota)} \big[b(x,z) - q(z)\big] \leq \Phi_A(x) \tag{30}$$

and, from (11),

$$\left.\begin{aligned} z \in K(\Phi_A^*\circ\iota) \text{ and } x \in A^\pi &\implies b(x,z) \leq \Phi_A(x) + q(z) \leq q(x) + q(z)\\ &\implies q(z-x) \geq 0. \end{aligned}\right\} \tag{31}$$

(*a*) Since $K(\Phi_A^*\circ\iota) \supset A$, it is obvious that $\Phi_{K(\Phi_A^*\circ\iota)} \geq \Phi_A$ on F, and so (30) implies that $\Phi_{K(\Phi_A^*\circ\iota)} = \Phi_A$ on F. If, on the other hand, $B \subset F$ is nonempty and q–positive and $\Phi_B = \Phi_A$ on F then $\Phi_B^*\circ\iota = \Phi_A^*\circ\iota$ on F, and so $B \subset K(\Phi_B^*\circ\iota) = K(\Phi_A^*\circ\iota)$.

(*b*) It is clear from (31) that $K(\Phi_A^*\circ\iota) \subset A^{\pi\pi}$. If $z \in A^{\pi\pi}$ and $\Phi_A(z) \geq q(z)$ then, from Theorem 5.2(*b*), $\Phi_A^*\circ\iota(z) \leq q(z)$. On the other hand, (15) implies that $\Phi_A^*\circ\iota(z) \geq \Phi_A(z) \geq q(z)$. Thus $\Phi_A^*\circ\iota(z) = q(z)$, and so $z \in K(\Phi_A^*\circ\iota)$.

(*c*) The first assertion is immediate from (*a*), and (13) with $M = K(\Phi_A^*\circ\iota)$. As for the converse, it is clear from (*b*) that $K(\Phi_A^*\circ\iota) = A^{\pi\pi}$. On the other hand, $\Phi_A \in \mathcal{H}$ and, from (11), $A^\pi \subset K(\Phi_A)$. Thus Lemma 3.5 implies that A^π is q–positive and so, from Lemma 5.4, $A^{\pi\pi} = K(\Phi_A^*\circ\iota)$ is the unique maximally q–positive cover of A.

(*d*) This is immediate from (*a*)–(*c*) and the remarks in Definition 6.4. □

Remark 6.6. In general, it is not true that $\mathrm{cl}_{\mathcal{S}}(A) = A^{\pi\pi}$. For instance, let F be a nonzero Hilbert space, b be the inner product of F, $z \in H$ and $h\colon\ F \mapsto \mathbb{R}$ be defined by $h(x) := q(x) + q(x-z)$. Then $h \in \mathcal{H}$ and $K(h) = \{z\}$. So $\{z\}$ is $\mathcal{S}$–q–positive and, consequently, $\mathrm{cl}_{\mathcal{S}}(\{z\}) = \{z\}$. On the other hand, $\{z\}^\pi = F$, and so $\{z\}^{\pi\pi} = F$. When can we assert that $\mathrm{cl}_{\mathcal{S}}(A) = A^{\pi\pi}$? Even in the finite–dimensional case, this is a very interesting question, which will be the subject of Section 7.

7. $\mathrm{cl}_{\mathcal{S}}(A)$ and $A^{\pi\pi}$ in the finite–dimensional case

We suppose throughout this section that $m \ge 1$, $p+n = m$, $F = \mathbb{R}^m$ and $b(\cdot,\cdot)\colon\ F\times F \mapsto \mathbb{R}$ is defined by $b\big((x_1,\ldots,x_m),(y_1,\ldots,y_m)\big) := \sum_{i=1}^p x_iy_i - \sum_{i=p+1}^m x_iy_i$. Then $\iota\colon\ \mathbb{R}^m \mapsto \mathbb{R}^m$ is defined by $\iota(y_1,\ldots,y_m) := (y_1,\ldots,y_p,-y_{p+1},\ldots,-y_m)$, and ι is surjective. The classical theorem of Sylvester tells us that any symmetric bilinear form on $\mathbb{R}^m\times\mathbb{R}^m$ that separates points can always be put in the above form by a suitable change of variable. Define the linear maps $P\colon\ \mathbb{R}^m \mapsto \mathbb{R}^p$ and $N\colon\ \mathbb{R}^m \mapsto \mathbb{R}^n$ by $P(x_1,\ldots,x_m) := (x_1,\ldots,x_p)$ and $N(x_1,\ldots,x_m) := (x_{p+1},\ldots,x_m)$. Then, for all $x\in\mathbb{R}^m$, $\|P(x)\|^2 = \frac12\|x\|^2 + q(x)$. A consequence of this (which we will not use) is that, if G is as defined in (19), then $G = \{x\in F\colon\ P(x) = 0\}$, and so G is a subspace of F.

Lemma 7.1. *Suppose that $p\le n$ and A is a nonempty q–positive subset of $\mathbb{R}^m$. Then $\Phi_A \ge q$ on $A^{\pi\pi}$.*

Proof. By virtue of (9) and (21), it suffices to prove that $0\in A^{\pi\pi} \implies \Phi_A(0)\ge 0$. Suppose, on the contrary, that $0\in A^{\pi\pi}$ and $\Phi_A(0) < 0$. Let $H = \{y\in\operatorname{dom}\Phi_A\colon\ q(y) = 0\}$. From Theorem 5.5, H is a subspace of $\mathbb{R}^m$, $q\ge 0$ on $H^\perp = \operatorname{dom}\Phi_A$ and $H\cap A = \emptyset$. We now prove that

$$z\in\mathbb{R}^n \text{ and } \big\langle N(H), z\big\rangle = \{0\} \implies z = 0. \tag{32}$$

To this end, let $z = (z_1,\ldots,z_n)$ and write $x = (0,\ldots,0,-z_1,\ldots,-z_n)\in\mathbb{R}^n$. Then, by direct computation, for all $y\in H$, $b(x,y) = \big\langle N(y), z\big\rangle = 0$, thus $x\in H^\perp$. Since $q\ge 0$ on $H^\perp$, $q(x)\ge 0$, from which $\frac12\|x\|^2 \le \|P(x)\|^2 = 0$. Thus $x = 0$, and so $z = 0$, which gives (32). It follows from (32) that $N(H) = \mathbb{R}^n$, so $\dim H \ge n$.

It is clear that if $y\in\operatorname{dom}\Phi_A$ and $P(y) = 0$ then $\frac12\|y\|^2 = \|P(y)\|^2 - q(y)\le 0$, from which $y = 0$, consequently P is injective on $\operatorname{dom}\Phi_A$. Since $A\subset\operatorname{dom}\Phi_A$, $H\subset\operatorname{dom}\Phi_A$ and $H\cap A = \emptyset$, we must have $P(H)\cap P(A) = \emptyset$, and so $P(H)$ is a *proper* subspace of $\mathbb{R}^p$, from which $\dim H = \dim P(H) < p \le n$. This contradiction to the result of the previous paragraph completes the proof of Lemma 7.1. $\square$

Lemma 7.2. *Let $p > n$. Write $e_1,\ldots,e_m$ for the usual basis elements of $\mathbb{R}^m$, define $f\colon\ \mathbb{R}^n\mapsto\mathbb{R}^m$ by $f(v) := e_1 + \sum_{i=1}^n v_i(e_{1+i} + e_{p+i})$ for $v = (v_1,\ldots,v_n)\in\mathbb{R}^n$, let $A = f(\mathbb{R}^n)\subset\mathbb{R}^m$, and write $H = \{y\in\operatorname{dom}\Phi_A\colon\ q(y) = 0\}$. Then A is $\mathcal{S}$–q–positive, $0\in A^{\pi\pi}$, $q\ge 0$ on $\operatorname{dom}\Phi_A$, A is premaximally q–positive, $\operatorname{dom}\Phi_A$ is the unique maximally q–positive cover of A, $\operatorname{dom}\Phi_A$ and H are both subspaces of F, and $\operatorname{dom}\Phi_A = H^\perp$.*

Proof. A is clearly a closed convex subset of $\mathbb{R}^m$. If $v = (v_1,\ldots,v_n)\in\mathbb{R}^n$ then

$$q\big(f(v)\big) = \tfrac12\big[1 + \textstyle\sum_{i=1}^n v_i^2 - \sum_{i=1}^n v_i^2\big] = \tfrac12.$$

Define $h\colon\ \mathbb{R}^m \mapsto]-\infty,\infty]$ by

$$h(x) := \begin{cases} \frac12 & \text{if } x\in A;\\ \infty & \text{otherwise.}\end{cases}$$

Then, in the notation of Definition 3.3, $h\in\mathcal{H}$ and $A = K(h)$. Lemma 3.5 now implies that A is $\mathcal{S}$–q–positive. Now let $x\in A^\pi$. From (11), $\Phi_A(x) < \infty$. But

$$\Phi_A(x) = \sup\nolimits_{v\in\mathbb{R}^n}\big[b(x,f(v)) - \tfrac12\big] = x_1 - \tfrac12 + \sup\nolimits_{v\in\mathbb{R}^n}\textstyle\sum_{i=1}^n (x_{1+i} - x_{p+i})v_i.$$

Consequently, for all $i = 1, \ldots, n$, $x_{1+i} - x_{p+i} = 0$, that is to say, $x_{p+i} = x_{1+i}$, and so $q(x) = \frac{1}{2}\big[x_1^2 + \sum_{i=1}^n x_{1+i}^2 + \sum_{i=n+2}^p x_i^2 - \sum_{i=1}^n x_{1+i}^2\big] = \frac{1}{2}\big[x_1^2 + \sum_{i=n+2}^p x_i^2\big] \ge 0$. Thus we have proved that $x \in A^\pi \Longrightarrow q(x) \ge 0$, and so $0 \in A^{\pi\pi}$. The above computation also shows that $\Phi_A(0) = -\frac{1}{2} < 0$, and the result follows from Theorem 5.5. $\square$

Theorem 7.3. $\mathrm{cl}_{\mathcal{S}}(A) = A^{\pi\pi}$ *for every nonempty q–positive subset, A, of $\mathbb{R}^m$ if, and only if, $p \le n$.*

Proof. Suppose first that $p \le n$ and A is a nonempty q–positive subset of $\mathbb{R}^m$. Then it follows from Lemma 7.1 and Theorem 6.5(b) that $\mathrm{cl}_{\mathcal{S}}(A) = A^{\pi\pi}$.

Conversely, suppose that $p > n$. Consider the example A of Lemma 7.2. Then $\mathrm{cl}_{\mathcal{S}}(A) = A \not\ni 0$ and $A^{\pi\pi} \ni 0$. So $\mathrm{cl}_{\mathcal{S}}(A) \ne A^{\pi\pi}$. $\square$

Corollary 7.4. $\mathrm{cl}_{\mathcal{S}}(A)$ *is a maximally q–positive subset of $\mathbb{R}^m$ (or, equivalently, $\Phi_A \in \mathcal{H}$) for every premaximally q–positive subset, A, of $\mathbb{R}^m$ if, and only if, $p \le n$.*

Proof. Suppose first that $p \le n$ and $A \subset \mathbb{R}^m$ is premaximally q–positive. Then Theorem 7.3 and Lemma 5.4 give us that $\mathrm{cl}_{\mathcal{S}}(A) = A^{\pi\pi}$ and that $A^{\pi\pi}$ is maximally q–positive.

Conversely, suppose that $p > n$. Then, since $\mathrm{cl}_{\mathcal{S}}(A) \ne A^{\pi\pi}$ in the example of Lemma 7.2, $\mathrm{cl}_{\mathcal{S}}(A)$ cannot be maximally q–positive. On the other hand, Theorem 5.5 implies that A is premaximally q–positive. $\square$

Corollary 7.5. *Suppose that $p \le n$ and A is a nonempty q–positive subset of $\mathbb{R}^m$. Then A is premaximally q–positive if, and only if, $\Phi_A \in \mathcal{H}$.*

Proof. This is immediate from Corollary 7.4 and Theorem 6.5(d). $\square$

8. The monotone case

Example 8.1. We now assume that E is a nonzero real Banach space and E^* is its topological dual space. As in Examples 2.1(d), we norm $F = E \times E^*$ by $\big\|(x, x^*)\big\| := \sqrt{\|x\|^2 + \|x^*\|^2}$, and define the bilinear form b: $F \times F \mapsto \mathbb{R}$ by $b\big((x, x^*), (y, y^*)\big) := \langle x, y^*\rangle + \langle y, x^*\rangle$. Then ι is an isometry from $E \times E^*$ into $(E \times E^*)^*$ and if, further, E is reflexive then ι is surjective. We define c: $E \times E^* \mapsto \mathbb{R}$ by $c(x, x^*) := \langle x, x^*\rangle$ — then $q = c$ on $E \times E^*$. $\big($$c$ is identical with the function π as defined in [6].$\big)$ Any of the examples in Definition 2.1(a,b) that are of finite odd dimension cannot be of the special form discussed here. Let $\emptyset \ne A \subset E \times E^*$. Then A is q–positive (resp. maximally q–positive) exactly when A is monotone (resp. maximally monotone) in the classical sense. Let A be nonempty and monotone. Let φ_A be the *Fitzpatrick function* associated with A, defined by

$$\varphi_A(x, x^*) := \sup_{(a,a^*)\in A} \big[\langle a, x^*\rangle + \langle x, a^*\rangle - \langle a, a^*\rangle\big].$$

(The function φ_A was introduced by Fitzpatrick in [4, Definition 3.1, p. 61] under the notation L_A.) φ_A is identical with Φ_A as defined in Definition 3.1. (7), (15) and (16) imply that

$$\varphi_A = \varphi_A^* \circ \iota = c \text{ on } A \quad \text{and} \quad \varphi_A^* \circ \iota \ge \varphi_A \vee c \text{ on } E \times E^*.$$

We write $\mathcal{F}$ for the set of all those convex lower semicontinuous functions f: $E \times E^* \mapsto\ ]-\infty, \infty]$ such that $f \ge c$ on $E \times E^*$ and, if $f \in \mathcal{F}$, we define

$$L(f) := \big\{(x, x^*) \in E \times E^*\colon\ f(x, x^*) = \langle x, x^*\rangle\big\}.$$

Then $\mathcal{F}$ and $L(\cdot)$ are identical with $\mathcal{H}$ and $K(\cdot)$ as defined in Definition 3.3. It is clear from (13) that if M is maximally monotone then

$$\varphi_M \in \mathcal{F} \quad \text{and} \quad M = L(\varphi_M)$$

(see [4, Corollary 3.9, p. 62]). Lemma 3.5 implies that if $f \in \mathcal{F}$ and $L(f) \neq \emptyset$ then $L(f)$ is monotone.

Theorem 3.6 gives the following characterization of Fitzpatrick functions.

Theorem 8.2. *Let $f\colon\ E \times E^* \mapsto]-\infty, \infty]$ be proper, convex and lower semicontinuous. Then there exists a nonempty monotone subset A of $E \times E^*$ such that $f = \varphi_A$ on $E \times E^*$ if, and only if, $f^* \circ \iota \in \mathcal{F}$ and*

$$\text{for all } (x, x^*) \in E \times E^*,\ f(x, x^*) \le \sup\nolimits_{(z,z^*)\in K(f^*\circ\iota)} \big[\langle z, x^*\rangle + \langle x, z^*\rangle - \langle z, z^*\rangle\big].$$

Theorem 3.7 gives the following strange result generalizing [6, Proposition 24], in which "=" was assumed rather than "$\le$". Theorem 8.3 shows what happens if we extend the definition of $\varphi_{\{.\}}$ to arbitrary subsets of $E \times E^*$.

Theorem 8.3. *Let E be a nonzero real Banach space and A be a nonempty monotone subset of $E \times E^*$. Let $D \subset E \times E^*$ and, for all $(x, x^*) \in E \times E^*$, let $\varphi_D(x, x^*) := \sup_{(d,d^*)\in D} \big[\langle d, x^*\rangle + \langle x, d^*\rangle - \langle d, d^*\rangle\big]$. If $\varphi_D \le \varphi_A$ on $E \times E^*$ then D is monotone.*

Let $(x, x^*) \in E \times E^*$. We write $(x, x^*) \in A^\mu$ if (x, x^*) is *monotonically related to* A, that is to say,

$$(a, a^*) \in A \implies \langle x - a, x^* - a^*\rangle \ge 0.$$

This concept goes back at least as far as [10]. The set A^μ is identical with the set A^π as defined in Section 3. Then it is clear from (11) and (20) that

$$(x, x^*) \in A^\mu \iff \varphi_A(x, x^*) \le \langle x, x^*\rangle, \quad A \subset A^\mu \subset \operatorname{dom}\varphi_A, \tag{33}$$

$$A \subset A^{\mu\mu} \subset A^\mu, \quad \text{and} \quad A^{\mu\mu} \text{ is a monotone subset of } E \times E^*.$$

Theorem 5.2 implies that

$$\varphi_A \vee c \ge \varphi_{A^{\mu\mu}} \text{ on } E \times E^*,$$

and

$$(z, z^*) \in A^{\mu\mu} \text{ and } \varphi_A(z, z^*) \ge \langle z, z^*\rangle \implies \varphi_A^* \circ \iota(z, z^*) \le \langle z, z^*\rangle.$$

The transformation $A \mapsto A^\mu$ was studied as a *polarity operation* in [6, Section 4].

9. Applications of the additive transversal to the monotone case

Remark 9.1. Let E be a nonzero reflexive real Banach space and the notation be as in Example 8.1. With $F = E \times E^*$, we now identify the set G introduced in (19). Indeed,

$$(x, x^*) \in G \iff \langle x, x^*\rangle = -\tfrac{1}{2}\big\|(x, x^*)\big\|^2 \iff \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 + \langle x, x^*\rangle = 0,$$

thus, writing $J\colon\ E \rightrightarrows E^*$ for the *duality map*, $G = \operatorname{Graph}(-J)$. So Theorem 4.1 implies the following result, which first appeared in [13, Theorem 10.6, p. 37], and which implies Rockafellar's surjectivity theorem (see [13, Theorem 10.7, p. 38]):

Theorem 9.2. *Let E be a nonzero reflexive real Banach space and A be a monotone subset of $E \times E^*$. Then*

$$A \text{ is maximally monotone} \iff A + \operatorname{Graph}(-J) = E \times E^*.$$

Theorem 4.3 implies the following result (see [15, Theorem 1.4]). [15, Theorem 1.4(a)] was used to give many sufficient conditions for the sum of maximal monotone multifunctions on a reflexive Banach space to be maximal monotone. [15, Theorem 1.4(b)] was first proved by Burachik and Svaiter in [3, Theorem 3.1]. The approach given here shows how [3, Theorem 3.1] can be established without (indirectly) having to use a renorming theorem. The interest of Theorem 9.3(a) is that the function h is not required to be lower semicontinuous, a fact which was very useful in the applications in [15].

Theorem 9.3. *Let E be a nonzero reflexive real Banach space, $f\colon\ E \times E^* \mapsto]-\infty, \infty]$ be proper and convex, and $f \geq c$ and $f^* \circ \iota \geq c$ on $E \times E^*$. Then:*

(a) *$L(f^* \circ \iota)$ is a maximal monotone subset of $E \times E^*$.*

(b) *If f is lower semicontinuous then $L(f)$ is a maximal monotone subset of $E \times E^*$.*

10. Premaximally monotone sets

The idea of premaximally monotone sets goes back to [12, Theorem 19, pp. 189–190], though the first systematic study of them was made in [6]. Lemma 10.2 below appears in [6, Proposition 36]. Lemma 10.2 and Theorem 10.3 are immediate from Lemma 5.4 and Theorem 5.5, respectively.

Definition 10.1. Let E be a nonzero real Banach space and A be a nonempty monotone subset of $E \times E^*$. We say that M is a *maximally monotone cover for A* if M is a maximally monotone subset of $E \times E^*$ and $M \supset A$. We say that A is *premaximally monotone* if A has a unique maximally monotone cover.

Lemma 10.2. *Let E be a nonzero real Banach space and A be a nonempty monotone subset of $E \times E^*$. Then the conditions* (34)–(38) *are equivalent.*

$$A \text{ is premaximally monotone,} \tag{34}$$

$$A^\mu = A^{\mu\mu}, \tag{35}$$

$$A^\mu \text{ is monotone,} \tag{36}$$

$$A^\mu \text{ is maximally monotone,} \tag{37}$$

$$A^{\mu\mu} \text{ is maximally monotone.} \tag{38}$$

Furthermore, in this case $A^{\mu\mu} = A^\mu$ is the unique maximally monotone cover for A.

Theorem 10.3. *Let E be a nonzero reflexive real Banach space and A be a nonempty monotone subset of $E \times E^*$. Consider the condition*

$$c \geq 0 \text{ on } \operatorname{dom} \varphi_A \tag{39}$$

and write $H = \{(x, x^*) \in \operatorname{dom} \varphi_A\colon\ \langle x, x^* \rangle = 0\}$.

(*a*) *If* (39) *is satisfied then* A *is premaximally monotone and* $\operatorname{dom}\varphi_A$ *is the unique maximally monotone cover of* A*. Further,* $\operatorname{dom}\varphi_A$ *and* H *are both subspaces of* $E \times E^*$ *and*

$$\operatorname{dom}\varphi_A = H^\perp = \big\{(y,y^*) \in E \times E^*\colon \quad (x,x^*) \in H \Longrightarrow \langle x,y^*\rangle + \langle y,x^*\rangle = 0\big\}.$$

(*b*) *If* $(0,0) \in A^{\mu\mu}$ *and* $\varphi_A(0,0) < 0$ *then* (39) *is satisfied and* $H \cap A = \emptyset$*.*

Remark 10.4. Example 11.3 shows what can happen if (39) is not satisfied.

Most of the results in this paragraph, which follow immediately from Lemma 6.1 – Definition 6.4, appear in [6, Section 3], with minor differencies in notation. We say that A is $\mathcal{R}$*–monotone* if there exists $f \in \mathcal{F}$ such that $A = L(f)$. This concept was studied in [6] under the name "representable (monotone)". $\mathcal{R}$–monotonicity is identical with $\mathcal{S}$–q–positivity as defined in Definition 6.2. Every maximally monotone subset of F is $\mathcal{R}$–monotone. If A is any nonempty monotone subset of $E \times E^*$, we write $\operatorname{cl}_{\mathcal{R}}(A)$ for the smallest $\mathcal{R}$–monotone subset of F that contains A. $\operatorname{cl}_{\mathcal{R}}(\cdot)$ is identical with $\operatorname{cl}_{\mathcal{S}}(\cdot)$ as defined in Definition 6.4. Then $L(\varphi_A^* \circ \iota) \supset \operatorname{cl}_{\mathcal{R}}(A)$, with equality if E is reflexive. The transformation $A \mapsto \operatorname{cl}_{\mathcal{R}}(A)$ was studied as a *closure operation* in [6].

Theorem 6.5 gives us the following result. The characterization of $\operatorname{cl}_{\mathcal{R}}(A)$ in the reflexive case given by Theorem 10.5(a) is interesting, since it shows that $\operatorname{cl}_{\mathcal{R}}(A)$ can be defined by a maximizing condition, as well as by the minimizing condition used in [6]. Theorem 10.5(c) gives [6, Lemma 37 and Proposition 39] and part of [6, Lemma 38].

Theorem 10.5. *Let* E *be a nonzero real Banach space and* A *be a nonempty monotone subset of* $E \times E^*$*. Then:*

(*a*) $\varphi_{\operatorname{cl}_{\mathcal{R}}(A)} = \varphi_A$ *on* $E \times E^*$ *and if* E *is reflexive then* $\operatorname{cl}_{\mathcal{R}}(A)$ *is the largest monotone subset* B *of* $E \times E^*$ *such that* $\varphi_B = \varphi_A$ *on* $E \times E^*$*.*

(*b*) *If* E *is reflexive then* $\big\{(z,z^*) \in A^{\mu\mu}\colon\ \varphi_A(z,z^*) \ge \langle z,z^*\rangle\big\} \subset \operatorname{cl}_{\mathcal{R}}(A) \subset A^{\mu\mu}$*.*

(*c*) *If* $\operatorname{cl}_{\mathcal{R}}(A)$ *is a maximally monotone subset of* $E \times E^*$ *then* $\varphi_A \in \mathcal{F}$*. If* $\varphi_A \in \mathcal{F}$ *then* A *is premaximally monotone and* $L(\varphi_A^* \circ \iota)$ *is the unique maximally monotone cover of* A*. If* $\operatorname{cl}_{\mathcal{R}}(A)$ *is a maximally monotone subset of* $E \times E^*$*, or* E *is reflexive and* $\varphi_A \in \mathcal{F}$*, then* $\operatorname{cl}_{\mathcal{R}}(A)$ *is the unique maximally monotone cover of* A*.*

Remark 10.6. The condition "$\varphi_A \in \mathcal{F}$" that appears in Theorem 10.5(c) can be rewritten in the form "for all $(x,x^*) \in E \times E^*$, $\inf_{(a,a^*)\in A}\langle x-a, x^*-a^*\rangle \le 0$". When E is reflexive, this is a specialization of a condition that has proved very useful in studying maximal monotone multifunctions on general Banach spaces, namely that A be of "type (NI)". See [12, Definition 10, p. 183], [13, Definition 25.5, p. 99], and, more recently, [16].

11. $\operatorname{cl}_{\mathcal{R}}(A)$ and $A^{\mu\mu}$

Theorem 11.1 should be compared with [6, Theorem 31 and Proposition 40], the proofs of which rely on totally different techniques.

Theorem 11.1. *Let* E *be a nonzero finite–dimensional real Banach space and* A *be a nonempty monotone subset of* $E \times E^*$*. Then* $\operatorname{cl}_{\mathcal{R}}(A) = A^{\mu\mu}$*. If, further,* A *is premaximally monotone then* $\operatorname{cl}_{\mathcal{R}}(A)$ *is the unique maximally monotone cover of* A*. Finally,* A *is premaximally monotone if, and only if,* $\varphi_A \in \mathcal{F}$*, that is to say,* A *is of type (NI).*

Proof. For convenience, we renorm E as the Euclidean space $\mathbb{R}^k$. If $x = (x_1, \ldots, x_k)$, $x^* = (x_1^*, \ldots, x_k^*)$, $y = (y_1, \ldots, y_k)$ and $y^* = (y_1^*, \ldots, y_k^*) \in \mathbb{R}^k$ then, according to the convention of Example 8.1, $b\big((x, x^*), (y, y^*)\big) = \sum_{i=1}^k x_i y_i^* + \sum_{i=1}^k y_i x_i^*$. Let us represent the pair (x, x^*) as the single element $(x_1, x_1^*, \ldots, x_k, x_k^*)$ of $\mathbb{R}^{2k}$. Then

$$\begin{aligned} b\big((x_1, x_1^*, \ldots, x_k, x_k^*), (y_1, y_1^*, \ldots, y_k, y_k^*)\big) &= \textstyle\sum_{i=1}^k (x_i y_i^* + x_i^* y_i) \\ &= \sum_{i=1}^k \frac{x_i + x_i^*}{\sqrt{2}} \frac{y_i + y_i^*}{\sqrt{2}} - \sum_{i=1}^k \frac{x_i - x_i^*}{\sqrt{2}} \frac{y_i - y_i^*}{\sqrt{2}}. \end{aligned}$$

Thus, after the appropriate changes of variable, we are in the situation of Section 7 with $p = n = k$. The results now follow from Theorem 7.3, Corollary 7.4 and Corollary 7.5. □

The example given in Theorem 11.2 is based rather loosely on an example shown to the author by Dr Benar Fux Svaiter [17].

Theorem 11.2. *Let E be an infinite–dimensional Hilbert space, T be a linear isometry of E onto a proper subspace of E and $p \in T(E)^\perp$ with $\|p\| = 1$. Let*

$$A = \big\{(p + T(v) + v, p + T(v) - v)\colon\ v \in E\big\}$$

and write $H = \big\{(x, x^*) \in \operatorname{dom}\varphi_A\colon\ \langle x, x^*\rangle = 0\big\}$. Then A is a $\mathcal{R}$–monotone subset of $E \times E$, $(0, 0) \in A^{\mu\mu}$, $c \geq 0$ on $\operatorname{dom}\varphi_A$, A is premaximally monotone, $\operatorname{dom}\varphi_A$ is the unique maximally monotone cover of A, $\operatorname{dom}\varphi_A$ and H are both subspaces of $E \times E^*$, and $\operatorname{dom}\varphi_A = \big\{(y, y^*) \in E \times E^*\colon\quad (x, x^*) \in H \Longrightarrow \langle x, y^*\rangle + \langle y, x^*\rangle = 0\big\}$. Last, but not least, $\operatorname{cl}_{\mathcal{R}}(A) \neq A^{\mu\mu}$.

Proof. A is clearly a nonempty convex subset of $E \times E$, and it is not hard to see that A is closed. If $v \in E$ then, since $p \in T(E)^\perp$ and T is an isometry,

$$\big\langle p + T(v) + v, p + T(v) - v\big\rangle = \big\|p + T(v)\big\|^2 - \|v\|^2 = 1 + \big\|T(v)\big\|^2 - \|v\|^2 = 1.$$

We define $f\colon\ E \times E \mapsto]-\infty, \infty]$ by

$$f(x, x^*) := \begin{cases} 1 & \text{if } (x, x^*) \in A; \\ \infty & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{F}$ and $A = L(f)$, thus A is $\mathcal{R}$–monotone. Now let $(x, x^*) \in A^\mu$. Then (33) implies that $\varphi_A(x, x^*) < \infty$. But (writing T^* for the adjoint of T),

$$\begin{aligned} \varphi_A(x, x^*) &= \sup\nolimits_{(a,a^*)\in A} \big[\langle a, x^*\rangle + \langle x, a^*\rangle - \langle a, a^*\rangle\big] \\ &= \sup\nolimits_{v\in E} \big[\big\langle p + T(v) + v, x^*\big\rangle + \big\langle x, p + T(v) - v\big\rangle - 1\big] \\ &= \langle p, x^*\rangle + \langle x, p\rangle - 1 + \sup\nolimits_{v\in E} \big\langle v, x^* - x + T^*(x + x^*)\big\rangle. \end{aligned}$$

So $x^* - x + T^*(x + x^*) = 0$, that is to say, $x - x^* = T^*(x + x^*)$. In this case, noting that $\|T^*\| = \|T\| = 1$,

$$\langle x, x^*\rangle = \tfrac14\big[\|x + x^*\|^2 - \|x - x^*\|^2\big] = \tfrac14\Big[\|x + x^*\|^2 - \big\|T^*(x + x^*)\big\|^2\Big] \geq 0.$$

Thus we have proved that $(x, x^*) \in A^\mu \Longrightarrow \langle x, x^*\rangle \geq 0$, from which $(0, 0) \in A^{\mu\mu}$. The above argument also shows that $\varphi_A(0, 0) = -1 < 0$, and the result now follows from Theorem 10.3 and the fact that, since $p \neq 0$, $(0, 0) \notin A = \operatorname{cl}_{\mathcal{R}}(A)$. □

Example 11.3. Let $E = \mathbb{R}$, D be the doubleton $\{(-1,0),(1,0)\}$ and A be the line–segment $[(-1,0),(1,0)]$ in $E \times E^* = \mathbb{R}^2$. Both D and A are monotone. By direct computation,

$$D^\mu = \big(\,]-\infty,-1\,]\times\,]-\infty,0\,]\,\big) \cup A \cup \big(\,[\,1,\infty[\,\times\,[\,0,\infty[\,\big),$$

and $D^{\mu\mu} = A$. Thus, from Theorem 11.1, $\mathrm{cl}_{\mathcal{R}}(D) = A$. In particular, A is $\mathcal{R}$–monotone. This can also be seen directly, since $\mathbb{I}_A \in \mathcal{F}$ and $A = L(\mathbb{I}_A)$. On the other hand, A is obviously not premaximally monotone. For instance, $\mathbb{R} \times \{0\}$ and

$$\big(\{-1\}\times\,]-\infty,0\,]\,\big) \cup A \cup \big(\{1\} \times\, [\,0,\infty[\,\big)$$

are different maximally monotone covers of A.

Added in proof. The author is grateful to Dr Martínez-Legaz for pointing out that the proof of Theorem 11.2 can be modified to give the result stated below. This has two advantages. Firstly, A is actually a closed subspace of E, and secondly, it provides some explicit elements of $A^{\mu\mu} \setminus A$.

Theorem. *Let E be an infinite-dimensional Hilbert space, T be a linear isometry of E onto a proper subspace of E and*

$$A = \big\{(T(v)+v, T(v)-v)\colon\ v \in E\big\}.$$

Then A is a $\mathcal{R}$–monotone subset of $E \times E$ and $\big\{(p,p)\colon\ p \in T(E)^\perp \setminus \{0\}\big\} \subset A^{\mu\mu} \setminus A$. Furthermore, A is premaximally monotone. In fact, the unique maximal monotone superset of A is the set $\big\{(x,x^*) \in E \times E\colon\ x - x^* = T^*(x+x^*)\big\}$.

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