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Peak Set Crossing all the Circles

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Let $\Omega \subset \mathbb{C}^d$ be a circular, bounded, strictly convex domain with C^2 boundary. We construct a peak set $K \subset \partial\Omega$ which intersects all the circles in $\partial\Omega$ with the center at zero. In particular Hausdorff dimension of K is at least $2d-2$.

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1. Introduction

Let $\Omega \subset\subset \mathbb{C}^d$ be a bounded, circular, strictly convex domain with C^2 boundary. We say that a compact set K is a peak set for $A(\Omega)$ if there exists $f \in A(\Omega)$ such that $|f| < 1$ on $\overline{\Omega} \setminus K$ and $f = 1$ on K . There is also a weaker concept of a maximum modulus set, when $|f| = 1$ on K and $|f| < 1$ on $\overline{\Omega} \setminus K$.

In fact it is well known that for $d > 1$ a holomorphic non constant function $f \in A(\mathbb{B}^d)$ such that $|f(z)| = 1$ for all $z \in \partial\mathbb{B}^d$ does not exist. Therefore maximum modulus sets and peak sets are extensively considered by many authors.

Topologically, peak sets and maximum modulus sets are small in strictly pseudoconvex domains. The real topological dimension of a maximum modulus set is no more than d [6] and for a peak set is no more than $d-1$ [7]. In particular peak set and maximum modulus set must have an empty interior.

However, from the measure-theoretic point of view peak sets and maximum modulus sets no longer have to be small. Stensönes Henriksen has proved [5] that every strictly pseudoconvex domain with C^∞ boundary in $\mathbb{C}^d$ has a peak set with a Hausdorff dimension $2d-1$. In the case where the boundary is only C^2, Løw has proved [4] that a maximum modulus set can have positive $(2d-1)$-dimensional Hausdorff measure.

In this paper we show that it is possible to construct a peak set which crosses all the circles in $\partial\Omega$ with the center at zero. In particular the Hausdorff dimension of our peak set is at least $2d-2$.

The problem described in the paper is close to the one presented by Henriksen [5]; hence we briefly compare the analogous results. Undoubtedly, the best possible dimension of a peak set was given by Henriksen. However, in our paper we assume that the boundary

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of a bounded, circular, strictly convex domain is C^2 class instead of C^∞ class as it was done in Henriksen's paper. Moreover, Henriksen constructs his peak set on the basis of the solution of $\overline{\partial}$ problem. Whereas in our paper, we use exclusively polynomials. Henriksen's peak set in circular domains crosses almost all circles in $\partial\Omega$ with the center at zero, while our set crosses precisely all circles this type.

We need the following fact:

Theorem (see [2, Theorem 3.2]). *There exists a natural number N such that, if $\varepsilon \in (0,1)$, T is a compact subset of Ω, H is a continuous, strictly positive function on $\partial\Omega$, then there exist functions $f_1, ..., f_N \in A(\Omega)$ such that:*

(*1*) $|f_j| < \varepsilon$ on T.
(*2*) $\frac{1}{2}H < \max_{j=1,...,N} |f_j| < H$ *on* $\partial\Omega$.

Now we can prove the first observation:

Lemma 1.1. *There exists a natural number N such that if D is a compact subset of $\partial\Omega$ and T is a compact subset of $\overline{\Omega}$ with $T \cap D = \emptyset$ then for a given $\varepsilon \in (0,1)$ we can choose polynomials $f_1, ..., f_N$ such that:*

(*1*) $\frac{1}{2} < \max_{j=1,...,N} |f_j|$ *on* D;
(*2*) $|f_j| < \varepsilon$ *on* T;
(*3*) $|f_j| < 1$ *on* $\overline{\Omega}$.

Proof. Let N be a natural number from [2, Theorem 3.2]. Since Ω is a balanced bounded domain, every function $f \in A(\Omega)$ can be uniformly approximated by polynomials and therefore it is enough to construct $f_1, ..., f_N \in A(\Omega)$ with the properties (*1*)–(*3*).

There exists U an open subset of $\partial\Omega$ such that $D \cap \overline{U} = \emptyset$ and $T \cap \partial\Omega \subset U$. Since Ω is a strictly convex domain, there exists W an open subset in $\overline{\Omega}$ such that $T \cap \partial\Omega \subset W$ and if $z \in W \setminus \partial\Omega$ then there exists an analytical disc Q with $z \in Q$ and $\partial Q \subset U$. Let $T_0 := T \setminus W$. We may observe that T_0 is a compact subset of Ω.

We can define a continuous strictly positive function H on $\partial\Omega$ such that $H = 1$ on D, $H \leq 1$ on $\partial\Omega$ and $H < \varepsilon$ on U. Due to [2, Theorem 3.2] there exist functions $f_1, ..., f_N \in A(\Omega)$ such that:

- $|f_j| < \varepsilon$ on T_0.
- $\frac{1}{2}H < \max_{j=1,...,N} |f_j| < H$ on $\partial\Omega$.

The properties (*1*), (*3*) are obvious. Now let $z \in T$. If $z \in T_0$ or $z \in T \cap \partial\Omega$ then $|f_j(z)| < \varepsilon$. So we assume that $z \in T \setminus (T_0 \cup \partial\Omega)$. In particular $z \in W \setminus \partial\Omega$, so there exists an analytical disc Q such that $z \in Q$ and $\partial Q \subset U$. We may estimate $|f_j(z)| \leq \max_{w\in\partial Q} |f_j(w)| \leq \max_{w\in\partial Q} H(w) < \varepsilon$, which finishes the proof. □

We also need the following property of homogeneous polynomials:

Lemma 1.2. *There exists $K \in \mathbb{N}$ such that we can choose $m_0 \in \mathbb{N}$ and a sequence p_m of homogeneous polynomials of m degree which satisfy:*

(*1*) $|p_m(z)| \leq 2$ *for all* $z \in \partial\Omega$, $m > m_0$;
(*2*) $\sum_{i=1}^{K} |p_{m_i}(z)|^2 \geq 0.25$ *for all* $z \in \partial\Omega$, $m_0 < N \leq m_1 < ... < m_K \leq 2N$.

Proof. The case $m_j = mK + j$ was proved in [1, Theorem 2.6]. We used [1, Lemma 2.5], which can be applied also in a more general case: $N \leq m_1 < m_2 < ... < m_K \leq 2N$. Therefore, in order to prove the required result, it is enough to repeat the same arguments as in the proof of [1, Theorem 2.6]. □

2. Peak set

We say that $U \in \tau$ iff U is a non empty and open subset of $\overline{\Omega}$ such that $\partial\Omega \subset \mathbb{S}U$, where $\mathbb{S} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

We start with the following simple property of τ:

Proposition 2.1. *If $U \in \tau$ then there exists $V \in \tau$ such that $\overline{V} \subset U$.*

Proof. For a given $z \in U$ there exists U_z an open neighborhood of z in $\overline{\Omega}$ such that $z \in \overline{U}_z \subset U$. Since $\partial\Omega$ is a compact set and $\partial\Omega \subset \bigcup_{z\in U} \mathbb{S}U_z$ there exist $z_1, ..., z_s \in U$ such that $\partial\Omega \subset \bigcup_{i=1}^{s} \mathbb{S}U_{z_i}$. Now it is enough to define $V := \bigcup_{i=1}^{s} U_{z_i} \in \tau$ and observe that $\overline{V} = \bigcup_{i=1}^{s} \overline{U}_{z_i} \subset U$. □

Lemma 2.2. *There exists $M_0 > 1$ such that for a given $U \in \tau$, $\delta > 0$ and f a continuous function on $\overline{\Omega}$, we can choose $V \in \tau$ and g polynomial such that:*

(*1*) $\overline{V} \subset U$;
(*2*) *If $z \in V$ and $|f(z)| \geq \delta$ then $\Re(gf)(z) < -|f(z)|$;*
(*3*) $|g| < M_0$ *on* $\overline{\Omega}$.

Proof. Due to Proposition 2.1 there exist $V_0, V_1 \in \tau$ such that $\overline{V}_0 \subset V_1 \subset \overline{V}_1 \subset U$. Let

$$T = \left\{ z \in \overline{V}_0 \cap \partial\Omega : |f(z)| \geq \delta \right\}.$$

We can choose $\varepsilon > 0$ such that $n := \frac{\pi}{\varepsilon} \in \mathbb{N}$ and

- $e^{i\varphi}z \in V_1$,
- $|f(e^{i\varphi}z)| \geq \frac{1}{2} \left|f(e^{i\tilde{\varphi}}z)\right|$

for $z \in T$ and $\varphi, \tilde{\varphi} \in [-2\varepsilon, 2\varepsilon]$.

Due to Lemma 1.2 there exist $K, N_0 \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ with $N_0 \leq mn < (m+1)n < ... < (m+K)n < 2mn$ there exist homogeneous polynomials $p_{(m+1)n}, ..., p_{(m+K)n}$ of degree $(m+1)n, ..., (m+K)n$ respectively such that $\frac{1}{4} \leq \sum_{j=1}^{K} \left|p_{(m+j)n}\right|^2$ and $\left|p_{(m+j)n}\right| \leq 2$ on $\partial\Omega$. Let us choose $g_m := 10 \sum_{j=1}^{K} p_{(m+j)n}$ and $M_0 := 20K + 1$. First let us observe that $|g_m| < M_0$ on $\overline{\Omega}$. We show that it is enough to shrink V_1 and define $g = g_m$ for m large enough.

Since $\int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \exp(-itkn)\, dt = 0$ for $k \in \mathbb{Z} \setminus \{0\}$ we may estimate

$$\begin{aligned} \sqrt{2\varepsilon} \max_{-\varepsilon\leq\varphi\leq\varepsilon} \left|g_m(e^{i\varphi}z)\right| &\geq \sqrt{\int_{-\varepsilon}^{\varepsilon} \left|g_m(e^{i\varphi}z)\right|^2 d\varphi} = \sqrt{\int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} 100 \sum_{j=1}^{K} \left|p_{(m+j)n}(z)\right|^2 d\varphi} \\ &\geq \sqrt{\frac{50\pi}{n}} \geq 5\sqrt{2\varepsilon} > 4\sqrt{2\varepsilon} \end{aligned}$$

for $z \in \partial\Omega$. Moreover there exists $\eta_z \in [-\varepsilon, \varepsilon]$ such that

$$\left|g_m(e^{i\eta_z} z)\right| = \max_{-\varepsilon \leq \varphi \leq \varepsilon} \left|g_m(e^{i\varphi} z)\right|.$$

In particular $|g_m(e^{i\eta_z} z)| \geq 4$ for $z \in \partial\Omega$.

We now show the following inequality

$$I(m, z, \varphi) := \left|(f g_m)(e^{i\varphi} z) - e^{imn\varphi}(f g_m)(z)\right| < \frac{\delta}{4}$$

for some $\varepsilon_1 \in (0, 1)$, all $m \in \mathbb{N}$, $z \in \overline{\Omega}$ and $\varphi \in [-\varepsilon_1, \varepsilon_1]$.

In fact since f is a continuous function on $\overline{\Omega}$ there exists $\varepsilon_1 \in (0, \varepsilon)$ such that we may estimate

$$\begin{aligned}
I(m, z, \varphi) &\leq 10 \sum_{j=1}^{K} \left|e^{i(m+j)n\varphi} f(e^{i\varphi} z) p_{(m+j)n}(z) - e^{imn\varphi} f(z) p_{(m+j)n}(z)\right| \\
&\leq 10 \sum_{j=1}^{K} \left|p_{(m+j)n}(z)\right| \left|e^{ijn\varphi} f(e^{i\varphi} z) - f(z)\right| \\
&\leq 20 \sum_{j=1}^{K} \left|e^{ijn\varphi} f(e^{i\varphi} z) - f(z)\right| < \frac{\delta}{4}
\end{aligned}$$

for all $m \in \mathbb{N}$, $z \in \overline{\Omega}$ and $\varphi \in [-\varepsilon_1, \varepsilon_1]$.

Assume that m is so large that $mn\varepsilon_1 > \pi$ and let us define $g := g_m$. We may observe that there exists $\varphi_z \in [-\varepsilon_1, \varepsilon_1]$ such that

$$e^{imn\varphi_z}(fg)(e^{i\eta_z} z) = -\left|(fg)(e^{i\eta_z} z)\right|.$$

In particular for $z \in T$ we may observe that $\eta_z, \varphi_z \in [-\varepsilon, \varepsilon]$ and estimate

$$\begin{aligned}
\Re\left((fg)(e^{i(\eta_z+\varphi_z)} z)\right) &\leq \frac{\delta}{4} + \Re\left(e^{imn\varphi_z}(fg)(e^{i\eta_z} z)\right) \leq \frac{\delta}{4} - \left|(fg)(e^{i\eta_z} z)\right| \\
&\leq \frac{\delta}{4} - \frac{1}{2}\left|f(e^{i(\eta_z+\varphi_z)} z)\right| \left|g(e^{i\eta_z} z)\right| \leq \frac{\delta}{4} - 2\left|f(e^{i(\eta_z+\varphi_z)} z)\right| \\
&\leq \frac{1}{8}\left|f(e^{i(\eta_z+\varphi_z)} z)\right| - 2\left|f(e^{i(\eta_z+\varphi_z)} z)\right| < -\left|f(e^{i(\eta_z+\varphi_z)} z)\right|.
\end{aligned}$$

Now we define

$$V := \{z \in V_1 : |f(z)| < \delta \text{ or } \Re(gf)(z) < -|f(z)|\}.$$

We may observe that $\overline{V} \subset U$. Suppose that there exists $z \in \partial\Omega$ such that $\mathbb{S}z \cap V = \emptyset$. Since $\mathbb{S}z \cap V_1 \neq \emptyset$ we can easily conclude that $|f(e^{i\varphi} z)| \geq \delta$ for $\varphi \in \mathbb{R}$. Due to $\mathbb{S}z \cap V_0 \neq \emptyset$ there exists φ_0 such that $e^{i\varphi_0} z \in T$. Now we can easily see that there exists $\varphi \in [-2\varepsilon, 2\varepsilon]$ such that $\Re(gf)(e^{i(\varphi_0+\varphi)} z) < -|f(e^{i(\varphi_0+\varphi)} z)|$ and $e^{i(\varphi_0+\varphi)} z \in V_1$. In particular $e^{i\varphi_0+\varphi} z \in V$, which gives a contradiction.

We have just proved that $V \in \tau$, which finishes the proof. □

Lemma 2.3. *There exists $M > 0$ such that for a given $U \in \tau$ and $\tilde{\varepsilon} > 0$, we can choose $V \in \tau$ and a polynomial p such that:*

(*1*) $\overline{V} \subset U$;
(*2*) $\Re p < -1$ *on* V;
(*3*) $|p| < M$ *on* $\overline{\Omega}$;
(*4*) $|p| < \tilde{\varepsilon}$ *on* $\overline{\Omega} \setminus U$.

Proof. Let $N \in \mathbb{N}$ be a number from Lemma 1.1 and $M_0 > 1$ be a constant from Lemma 2.2. We can define $M := 4NM_0$.

Due to Proposition 2.1 there exists $W \in \tau$ such that $\overline{W} \subset U$. Let us denote $D := \overline{W} \cap \partial\Omega$, $T := \overline{\Omega} \setminus U$. Due to Lemma 1.1 there exist polynomials $f_1, ..., f_N$ such that

- $\frac{1}{2} < \max_{j=1,...,N} |f_j|$ on $D = \overline{W} \cap \partial\Omega$;
- $|f_j| < \frac{\tilde{\varepsilon}}{4NM_0}$ on $T = \overline{\Omega} \setminus U$;
- $|f_j| < 1$ on $\overline{\Omega}$.

Now we can observe that there exists $V_0 \in \tau$ such that $\overline{V}_0 \subset U$ and

- $\frac{1}{2} < \max_{j=1,...,N} |f_j|$ on V_0.

Let $\delta := \frac{1}{4NM_0}$. Due to Lemma 2.2 there exist $V_1, ..., V_N \in \tau$ and polynomials $g_1, ..., g_N$ such that:

- $\overline{V}_m \subset V_{m-1}$;
- If $z \in V_m$ and $|f_m(z)| \geq \delta$ then $\Re(g_m f_m)(z) < -|f_m(z)|$;
- $|g_m| < M_0$ on $\overline{\Omega}$.

Let us denote $V = \bigcap_{m=1}^N V_m$ and $p = 4\sum_{m=1}^N f_m g_m$. The property (*1*) follows from the definition of V. The property (*3*) is also obvious: $|p| \leq 4\sum_{m=1}^N |f_m g_m| < 4\sum_{m=1}^N M_0 \leq 4NM_0 = M$ on $\overline{\Omega}$. In a similar way we conclude the property (*4*): $|p| \leq 4\sum_{m=1}^N |f_m g_m| < 4\sum_{m=1}^N \frac{\tilde{\varepsilon} M_0}{4NM_0} \leq \tilde{\varepsilon}$ on $\overline{\Omega} \setminus U$.

Let now $z \in V$. There exists $k \in \{1, ..., N\}$ such that

$$|f_k(z)| = \max_{j=1,...,N} |f_j(z)| > \frac{1}{2}.$$

Now for a given m we have two cases: $|f_m(z)| < \delta$ or $|f_m(z)| \geq \delta$. First case immediately implies $\Re(f_m g_m)(z) \leq \delta M_0 = \frac{1}{4N}$. The second case implies $\Re(f_m g_m)(z) < -|f_m(z)| \leq 0 \leq \delta M_0 = \frac{1}{4N}$.

In particular $\Re(f_m g_m)(z) \leq \delta M_0 = \frac{1}{4N}$ and we may conclude the property (*2*):

$$\begin{aligned} \Re p(z) &= 4\Re(f_k g_k)(z) + 4 \sum_{m \in \{1,...,N\} \setminus \{k\}} \Re(f_m g_m)(z) \\ &< -4|f_k(z)| + 4\sum_{m=1}^{N-1} \frac{1}{4N} < -2 + 1 \leq -1. \end{aligned}$$

□

Theorem 2.4. *There exists a compact set $K \subset \partial\Omega$ and a function $f \in A(\Omega)$ such that:*

- $\mathbb{S}K = \partial\Omega$;
- $f = 1$ *on* K;
- $0 < |f| < 1$ *on* $\overline{\Omega} \setminus K$.

Proof. Let $M > 0$ be from Lemma 2.3. First we construct a sequence $U_m \in \tau$ and polynomials p_m with the following properties:

(*1*) $\overline{U}_{m+1} \subset U_m$;
(*2*) $\Re p_m < -1$ on U_{m+1};
(*3*) $|p_m| < M$ on $\overline{\Omega}$;
(*4*) $|p_m| < 2^{-m}$ on $\overline{\Omega} \setminus U_m$;
(*5*) $U_{m+1} \cap (1 - 2^{-m})\Omega = \emptyset$.

Let $U_1 = \overline{\Omega}$. In fact to construct $U_2, ..., U_{m+1}$ and $p_1, ..., p_m$ it is enough to use the Lemma 2.3 and slightly decrease $U_2, ..., U_{m+1}$ so that (*5*) is also fulfilled.

Now we define $g := -M - 2 + \sum_{m=1}^{\infty} p_m$ and $K := \bigcap_{m=1}^{\infty} \overline{U}_m$. Since $U_m \in \tau$, the properties (*1*), (*5*) imply $K = \bigcap_{m=1}^{\infty} U_m$ and $\mathbb{S}K = \partial\Omega$. Due to properties (*1*), (*4*)–(*5*) we also have $g \in \mathcal{O}(\Omega) \cap C(\overline{\Omega} \setminus K)$.

If $z \in \overline{\Omega} \setminus U_1$ then

$$\Re g(z) = -M - 2 + \sum_{m=1}^{\infty} \Re p_m(z) \leq -M - 2 + \sum_{m=1}^{\infty} 2^{-m} < 0.$$

Let now $z \in U_N \setminus U_{N+1}$ for $N = 1, 2, ...$ We may observe that $z \in \overline{\Omega} \setminus U_m$ for $m \geq N+1$. In particular we can estimate

$$\begin{aligned} \Re g(z) &= -M - 2 + \sum_{m=1}^{\infty} \Re p_m(z) = -M - 2 + \sum_{m=1}^{N-1} \Re p_m + \Re p_N + \sum_{m=N+1}^{\infty} \Re p_m \\ &< -M - 2 - (N-1) + M + \sum_{m=N+1}^{\infty} 2^{-m} < -N. \end{aligned}$$

Last inequalities imply that

$$\lim_{z \to w} \Re g(w) = -\infty \quad \text{for } w \in K$$
$$\Re g(z) < 0 \quad \text{for } z \in \overline{\Omega} \setminus K.$$

Now we can define

$$f = \exp\left(\frac{1}{g}\right).$$

Since $\Re \frac{1}{g} = \frac{\Re \overline{g}}{|g|^2} = \frac{\Re g}{|g|^2} < 0$ on $\overline{\Omega} \setminus K$ we may easily observe that $0 < |f| < 1$ on $\overline{\Omega} \setminus K$. Additionally due to $\lim_{z \to w} \frac{1}{|g(z)|} = 0$ for $w \in K$ we have $f = 1$ on K and $f \in A(\Omega)$. □

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