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A Version of the Lax-Milgram Theorem for Locally Convex Spaces*

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Dedicated to Stephen Simons on the occasion of his 70th birthday.

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We show an extension of the Lax–Milgram theorem for the context of locally convex spaces. Furthermore we prove that such version of the Lax–Milgram theorem does not admit an analogous generalization for the multilinear case, even though we give a positive partial result.

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0. Introduction

The celebrated theorem of Lax–Milgram [3] asserts that if E is a Hilbert space with inner product $< \cdot, \cdot >$, then every continuous bilinear form a on E that is coercive ($\inf_{\|y\|=1} a(y,y) > 0$) represents E, in the sense that for all $y_0 \in E$ there exists a unique $x_0 \in E$ such that

$$\text{for all } y \in E, \quad a(x_0, y) = < y_0, y > .$$

There are some generalizations of this important result in the context of reflexive Banach spaces (see for instance [1], [4] and [5] or [6]). In this paper we show a proper generalization of all of them, which unifies them and generates new fields of application. Specifically, in the main result of Section 1 we give a Lax–Milgram's type result for locally convex spaces (Theorem 1.2), whose proof is based on the Hahn–Banach–Lagrange theorem, a new version of the Hahn–Banach theorem due to S. Simons [10]. We come to characterize those elements that represent a continuous linear functional through a bilinear form, also providing certain control of such a functional, which in the normed case (Corollary 1.3) entails an estimation of its norm. As a matter of fact, we consider functionals represented on any convex subset, that does not necessarily satisfy any topological condition at all. Neither of our results assume the bilinear form to be continuous unlike those appearing in [1], [4], [5] and [6]. In fact, in Example 1.6 we show how even for reflexive Banach spaces Corollary 1.3 is more general than they are.

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In Section 2 we prove that the Lax–Milgram theorem for locally convex spaces, Theorem 1.2, does not admit a literal extension in terms of multilinear forms, although we state a positive partial result (Theorem 2.2) that generalizes Theorem 1.2.

We finish with Section 3, in which we prove that the Hahn–Banach theorem and the Lax–Milgram theorem for locally convex spaces are equivalent.

The vector spaces will always be considered as real vector spaces.

1. Lax–Milgram for LCS

First we evoke a generalization of the Hahn–Banach theorem, known as the Hahn–Banach–Lagrange theorem, that has encountered numerous applications in functional and convex analysis and monotone multifunctions theory (see [10], [11], [12] and [13]). Let us recall that if E is a real vector space, a function $S : E \longrightarrow \mathbb{R}$ is *sublinear* provided that it is subadditive and positively homogeneous. For such an S, if C is a nonempty convex subset of a vector space then we say that $j : C \longrightarrow E$ is $S-$*convex* if for all $x, y \in C$ and $0 < t < 1$ we have that $S(j(tx + (1-t)y - tj(x) - (1-t)j(y)) \leq 0$. Finally, a convex function $k : C \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to be *proper* when there exists $x \in C$ with $k(x) < \infty$.

Theorem 1.1 (Simons [10], [12]). *Let E be a nontrivial vector space and let $S : E \longrightarrow \mathbb{R}$ be a sublinear function. Assume in addition that C is a nonempty convex subset of a vector space, $k : C \longrightarrow \mathbb{R} \cup \{\infty\}$ is a proper convex function and $j : C \longrightarrow E$ is $S-$convex. Then there exists a linear functional $L : E \longrightarrow \mathbb{R}$ such that $L \leq S$ and*

$$\inf_C (L \circ j + k) = \inf_C (S \circ j + k) .$$

Given a nontrivial real Hausdorff locally convex space E, we shall write E^* to denote its topological dual space. If $n \geq 1$ and $E_1, \ldots, E_n$ are nontrivial real vector spaces, for a $n-$linear form $a : E_1 \times \cdots \times E_n \longrightarrow \mathbb{R}$, $1 \leq k \leq n$ and $(x_{k+1}, \ldots, x_n) \in E_{k+1} \times \cdots \times E_n$, $a(\cdot, \ldots, \cdot, x_{k+1}, \ldots, x_n)$ stands for the $k-$linear form

$$(x_1, \ldots, x_k) \in E_1 \times \cdots \times E_k \longmapsto a(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in \mathbb{R}.$$

If in addition $E_1, \ldots, E_n$ are Hausdorff locally convex spaces, we denote by $\mathcal{L}^n(E_1, \ldots, E_n)$ the vector space of all continuous $n-$linear forms on $E_1 \times \cdots \times E_n$. In the normed case, ($E_1, \ldots, E_n$ nontrivial real normed spaces), $\mathcal{L}^n(E_1, \ldots, E_n)$ is a Banach space when endowed with its usual norm: for $a \in \mathcal{L}^n(E_1, \ldots, E_n)$

$$\|a\| = \sup\{|a(x_1, \ldots, x_n)| : (x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n, \ \|x_1\| \vee \cdots \vee \|x_n\| \leq 1\}.$$

Now we state the main result of this section, along the lines of [10], [11], [12] and [13]. We characterize in terms of a continuous seminorm on E^* those $x_0^{**} \in E^{**}$ that represent a functional $y_0^* \in F^*$ on a convex subset C by the action of a bilinear form at one variable. More precisely:

Theorem 1.2. *Let E and F be nontrivial real Hausdorff locally convex spaces and let τ_{E^*} be any topology on E^* for which E^* is a real Hausdorff locally convex space. Let $y_0^* \in F^*$ and let $a : E \times F \longrightarrow \mathbb{R}$ be a bilinear form and C be a nonempty convex subset of F such that for all $y \in C$, $a(\cdot, y) \in E^*$, i.e., $a(\cdot, y)$ is continuous on E. Then the variational inequality*

$$\textit{there exists } x_0^{**} \in E^{**} \textit{ such that for all } y \in C, \quad y_0^*(y) \leq x_0^{**}(a(\cdot, y))$$

is equivalent to the existence of a continuous seminorm $p : E^* \longrightarrow \mathbb{R}$ *such that*

$$\text{for all } y \in C, \quad y_0^*(y) \leq p(a(\cdot, y)).$$

Moreover, if these statements are satisfied then we can take x_0^{**} *and* p *with* $x_0^{**} \leq p$.

Proof. We first assume that there exists a continuous seminorm $p : E^* \longrightarrow \mathbb{R}$ such that

$$y \in C \Rightarrow y_0^*(y) \leq p(a(\cdot, y)).$$

We apply the Hahn–Banach–Lagrange theorem (Theorem 1.1) with the sublinear function $S = p$, the proper convex function

$$\begin{aligned} k : \quad & C \longrightarrow \mathbb{R} \\ & y \mapsto -y_0^*(y) \end{aligned}$$

and the $S-$convex mapping

$$\begin{aligned} j : \quad & C \longrightarrow E^* \\ & y \mapsto a(\cdot, y), \end{aligned}$$

obtaining thus that there exists $x_0^{**} : E^* \longrightarrow \mathbb{R}$ linear such that $x_0^{**} \leq p$ (hence the fact that $x_0^{**} \in E^{**}$) and

$$\inf_{y \in C} \left(x_0^{**}(a(\cdot, y)) - y_0^*(y) \right) = \inf_{y \in C} \left(p(a(\cdot, y)) - y_0^*(y) \right).$$

But we have by hypothesis that

$$\inf_{y \in C} \left(p(a(\cdot, y)) - y_0^*(y) \right) \geq 0,$$

so

$$\inf_{y \in C} \left(x_0^{**}(a(\cdot, y)) - y_0^*(y) \right) \geq 0,$$

that is,

$$\text{for all } y \in C, \quad y_0^*(y) \leq x_0^{**}(a(\cdot, y)).$$

And conversely, if for some $x_0^{**} \in E^{**}$ we have that

$$y \in C \Rightarrow y_0^*(y) \leq x_0^{**}(a(\cdot, y)),$$

then it holds that

$$\text{for all } y \in C, \quad y_0^*(y) \leq p(a(\cdot, y))$$

for the continuous seminorm p on E^* given by $p(\cdot) = |x_0^{**}(\cdot)|$.

Note that in both directions we arrive at $x_0^{**} \leq p$. □

Let us point out that C is nothing more than a convex subset of F and we do not assume any additional topological condition on it, not even being closed.

For $\alpha \in \mathbb{R}$ we write $(\alpha)_+ := \max\{\alpha, 0\}$.

In the normed case we can obtain besides the previous characterization a precise estimation of the norm of those x_0^{**} that represent y_0^* by means of the bilinear form a. This is done in Corollary 1.3 below where for the vector space E the topology τ_{E^*} (see Theorem 1.2) is taken to be the strong topology of E^*, i.e., the topology associated with the canonical norm of E^*. Throughout the paper, this will be always the case whenever E is a normed space.

Corollary 1.3. *Assume that E and F are nontrivial real normed spaces, $y_0^* \in F^*$, $a : E \times F \longrightarrow \mathbb{R}$ is bilinear, and C is a nonempty convex subset of F such that for all $y \in C$ we have that $a(\cdot, y) \in E^*$. Then*

$$\text{there exists } x_0^{**} \in E^{**} \text{ such that for all } y \in C, \quad y_0^*(y) \le x_0^{**}(a(\cdot, y))$$
$$\Updownarrow$$
$$\text{there exists } \alpha \ge 0 \text{ such that for all } y \in C, \quad y_0^*(y) \le \alpha \|a(\cdot, y)\|.$$

In addition, if one of these equivalent conditions is satisfied and there exists $y \in C$ such that $a(\cdot, y) \neq 0$, then

$$\begin{aligned} &\min\{\|x_0^{**}\| : x_0^{**} \in E^{**} \text{ and for all } y \in C,\ y_0^*(y) \le x_0^{**}(a(\cdot, y))\} \\ &= \left(\sup_{y \in C,\ a(\cdot,y) \neq 0} \frac{y_0^*(y)}{\|a(\cdot, y)\|} \right)_+ . \end{aligned}$$

Proof. The equivalence follows from Theorem 1.2, by using no more than the well–known fact that if p is a continuous seminorm on a normed space, then p is bounded above by a (positive) multiple of the norm. Therefore, if one of the two equivalent conditions is satisfied, which we assume from now on in this proof, then the set

$$\mathcal{R}(a, y_0^*) := \{x^{**} \in E^{**} : \text{ for all } y \in C,\ y_0^*(y) \le x^{**}(a(\cdot, y))\}$$

is nonempty. On the one hand, we clearly have that

$$x^{**} \in \mathcal{R}(a, y_0^*) \Rightarrow \|x^{**}\| \ge \left(\sup_{y \in C,\ a(\cdot,y) \neq 0} \frac{y_0^*(y)}{\|a(\cdot, y)\|} \right)_+ .$$

On the other hand, if $y \in C$ then

$$y_0^*(y) \le \left(\sup_{v \in C,\ a(\cdot,v) \neq 0} \frac{y_0^*(v)}{\|a(\cdot, v)\|} \right)_+ \|a(\cdot, y)\|,$$

seeing that this inequality is trivially satisfied for $y \in C$ with $a(\cdot, y) \neq 0$, while for $y \in C$ such that $a(\cdot, y) = 0$, the fact that $\mathcal{R}(a, y_0^*)$ is nonempty also implies it. Since by the preceding reasoning we have that

$$\left(\sup_{v \in C,\ a(\cdot,v) \neq 0} \frac{y_0^*(v)}{\|a(\cdot, v)\|} \right)_+ < \infty,$$

then the continuous seminorm $p : E^* \longrightarrow \mathbb{R}$ defined by

$$p := \left(\sup_{v \in C,\ a(\cdot,v) \neq 0} \frac{y_0^*(v)}{\|a(\cdot, v)\|} \right)_+ \|\cdot\|$$

can be used in Theorem 1.2 and such result guarantees the existence of $x_0^{**} \in \mathcal{R}(a, y_0^*)$ such that $x_0^{**} \le p$, so

$$\|x_0^{**}\| \le \left(\sup_{y \in C,\ a(\cdot,y) \neq 0} \frac{y_0^*(y)}{\|a(\cdot, y)\|} \right)_+ .$$

Therefore we have proved the equalities

$$\|x_0^{**}\| = \min\{\|x^{**}\| : x^{**} \in \mathcal{R}(a, y_0^*)\} = \left(\sup_{y\in C,\ a(\cdot,y)\neq 0} \frac{y_0^*(y)}{\|a(\cdot, y)\|} \right)_+$$

and thus the proof is complete. □

Observe that if $C = F$ and a is nondegenerate in the second variable ($a(\cdot, y) = 0 \Leftrightarrow y = 0$), then

$$|y| := \|a(\cdot, y)\|, \quad (y \in F)$$

defines a norm on F. Under the assumptions of the preceding result, if one of the equivalent assertions above is satisfied, then

$$\min\{\|x_0^{**}\| : x_0^{**} \in E^{**} \text{ and for all } y \in F,\ y_0^*(y) \leq x_0^{**}(a(\cdot, y))\} = |y_0^*|.$$

Obviously, for reflexive Banach spaces we can say something more concrete: if E is a nontrivial real reflexive Banach space, F is a nontrivial real normed space, $y_0^* \in F^*$, C is a nonempty convex subset of F and $a : E \times F \longrightarrow \mathbb{R}$ is a bilinear form such that for all $y \in C$, $a(\cdot, y) \in E^*$, then the variational inequality

$$\text{there exists } x_0 \in E \text{ such that for all } y \in C, \quad y_0^*(y) \leq a(x_0, y)$$

is equivalent to

$$\text{there exists } \alpha \geq 0 \text{ such that for all } y \in C, \quad y_0^*(y) \leq \alpha\|a(\cdot, y)\|.$$

Moreover, if one of these conditions is satisfied and there exists $y \in C$ such that $a(\cdot, y) \neq 0$, then

$$\min\{\|x_0\| : x_0 \in E \text{ and for all } y \in C,\ y_0^*(y) \leq a(x_0, y)\} = \left(\sup_{y\in C,\ a(\cdot,y)\neq 0} \frac{y_0^*(y)}{\|a(\cdot, y)\|} \right)_+.$$

When E is not reflexive x_0^{**} is not necessarily in E:

Corollary 1.4. *Let E be a nontrivial real Banach space, let F be a nontrivial real normed space and let $y_0^* \in F^*\backslash\{0\}$. Suppose that for all continuous bilinear form $a : E \times F \longrightarrow \mathbb{R}$ satisfying that there exists $x_0^{**} \in E^{**}$ such that*

$$\textit{for all } y \in F, \quad y_0^*(y) = x_0^{**}(a(\cdot, y)),$$

it holds that $x_0^{**} \in E$. Then E is reflexive.

Proof. By James's sup theorem (see [2] or [7] for a more general result), to show that E is reflexive we must prove that every continuous linear functional on E attains its norm, that is, whenever $x^* \in E^*$ then there exists $x \in E$ with $\|x\| = 1$ and $x^*(x) = \|x^*\|$. Let $x_0^* \in E^*$. We assume without loss of generality that $\|x_0^*\| = 1$. We define $a : E\times F \longrightarrow \mathbb{R}$ by

$$a(x, y) := x_0^*(x)y_0^*(y), \quad (x \in E,\ y \in F).$$

Then a is a continuous bilinear form and for all $y \in F$, $a(\cdot,y) = y_0^*(y)x_0^*$, so $y_0^*(y) \leq \|a(\cdot,y)\|$, hence Corollary 1.3 applied with $C = F$ guarantees that there exists $x_0^{**} \in E^{**}$ such that

$$\|x_0^{**}\| = \left(\sup_{y\in F,\ a(\cdot,y)\neq 0} \frac{y_0^*(y)}{\|a(\cdot,y)\|}\right)_+ = 1$$

and

$$y \in F \Rightarrow y_0^*(y) = x_0^{**}(x_0^*)y_0^*(y).$$

By hypothesis $x_0^{**} = x_0 \in E$ and since $y_0^* \neq 0$, the above condition implies that $x_0^*(x_0) = 1$, so we have shown that x_0^* attains the norm (at the norm one element $x_0 \in E$) as we wished to prove. $\square$

Observe that in the proof of the preceding result we have applied Corollary 1.3 instead of the Hahn–Banach theorem. As we announced in the introduction, in Section 3 we will show that both results are equivalent. Now we give a consequence of Theorem 1.2 for normed spaces, different from Corollary 1.3, that will be useful in order to state such equivalence:

Corollary 1.5. *If E and F are nontrivial real normed spaces, $y_0^* \in F^*$, $a : E^* \times F \longrightarrow \mathbb{R}$ is bilinear, and C is a nonempty convex subset of F satisfying that for all $y \in C$ we have that $a(\cdot,y) \in E$, then*

$$\text{there exists } x_0^* \in E^* \text{ such that for all } y \in C,\quad y_0^*(y) \leq a(x_0^*,y)$$
$$\Updownarrow$$
$$\text{there exists } \alpha \geq 0 \text{ such that for all } y \in C,\quad y_0^*(y) \leq \alpha\|a(\cdot,y)\|.$$

In addition, if one of these equivalent statements is satisfied and there exists $y \in C$ such that $a(\cdot,y) \neq 0$, then

$$\min\{\|x_0^*\| : x_0^* \in E^* \text{ and for all } y \in C,\ y_0^*(y) \leq a(x_0^*,y)\} = \left(\sup_{y\in C,\ a(\cdot,y)\neq 0} \frac{y_0^*(y)}{\|a(\cdot,y)\|}\right)_+ .$$

Proof. The equivalence follows from Theorem 1.2, just by considering in E^* its weak–star topology, i.e, $\tau_{E^*} = w(E^*,E)$ (see the statement of Theorem 1.2). The second part has a proof analogous to that of Corollary 1.3. $\square$

The versions of the Lax–Milgram theorem [1], [4], [5] and [6] (and therefore the classical theorem for Hilbert spaces [3]) are a direct consequence of Corollary 1.3. Moreover, the results in those papers are only stated for reflexive Banach spaces, not giving an explicit control of the norm of those x_0 that represent the functional y_0^*, taking just the convex subset $C = F$ and assuming the bilinear form a be continuous. Indeed, it is possible to find a real reflexive Banach space E, a normed space F and a bilinear form $a : E \times F \longrightarrow \mathbb{R}$ for which we can apply Corollary 1.3 but not the results of [1], [4], [5] or [6], either because a is not continuous or F is not complete:

Example 1.6. Let E be the real separable Hilbert space of sequences ℓ_2, let F be the linear subspace of E generated by its usual basis $\{e_n\}_{n\geq 1}$ and let $T : F \longrightarrow E^* \equiv E$ be the linear operator defined for each $n \geq 1$ as

$$Te_n := < ne_n | \cdot >,$$

where $< \cdot|\cdot >$ denotes the inner product, and extended by linearity. It is clear that T is not continuous and that for all $y \in F$ it holds that

$$\|y\| \leq \|Ty\|.$$

Then the bilinear form $a : E \times F \longrightarrow \mathbb{R}$ given by

$$a(\cdot, y) :=< Ty|\cdot >, \quad (y \in F)$$

is not continuous, but for all $y \in F$, $a(\cdot, y) \in E^*$ and thanks to the preceding inequality

$$y_0^* \in F^*, \ y \in F \Rightarrow y_0^*(y) \leq \|y_0^*\|\|a(\cdot, y)\|.$$

Therefore Corollary 1.3 (but none of the result in [1], [4], [5] or [6]) guarantees that if $y_0^* \in F^*$ then there exists $x_0 \in E$ with $a(x_0, \cdot) = y_0^*$.

As we have just seen, the hypotheses of Corollary 1.3 do not necessarily imply that a is continuous. However, imposing some additional restrictions we can automatically obtain that a is continuous. For instance, in [8] Saint Raymond states a result implying continuity of a bilinear form on a Hilbert space satisfying a certain coercivity hypothesis.

Let us also point out that if E is a nontrivial real reflexive Banach space, F is a nontrivial normed space and $a : E \times F \longrightarrow \mathbb{R}$ is a continuous bilinear form such that for all $y^* \in F^*$ there exists a unique $x \in E$ satisfying

$$y \in F \Rightarrow y_0^*(y) = a(x, y),$$

then the norm on F

$$|y| := \|a(\cdot, y)\|, \quad (y \in F)$$

is equivalent to the original one and, in view of the control of the norm in Corollary 1.3, the operator

$$\begin{aligned} T : \ & E \longrightarrow F^* \\ & x \mapsto a(x, \cdot) \end{aligned}$$

is a linear isometry from E onto $(F^*, |\cdot|)$.

2. The multilinear problem

When one tries to represent by means of an $n-$linear form another $k-$linear form with $k < n$, that is, to generalize the version of the Lax–Milgram theorem given in Theorem 1.2 for locally convex spaces to the multilinear context, it becomes clear that such a generalization is not possible:

Example 2.1. Suppose that E is a nontrivial real reflexive Banach space and that $a : E^* \times E^* \times E \times E \longrightarrow \mathbb{R}$ is the continuous $4-$linear form defined by

$$a(x_1^*, x_2^*, x_3, x_4) = x_1^*(x_3)x_2^*(x_4), \quad (x_1^*, \ x_2^* \in E^*, \ x_3, \ x_4 \in E).$$

Then for all $x_3, x_4 \in E$, $a(\cdot, \cdot, x_3, x_4) \in \mathcal{L}^2(E^*, E^*)$ and

$$\|a(\cdot, \cdot, x_3, x_4)\| = \|x_3\|\|x_4\|,$$

so given a continuous bilinear form $b : E \times E \longrightarrow \mathbb{R}$ we have that

$$\text{for all } x_3,\ x_4 \in E, \quad b(x_3, x_4) \le \beta \|a(\cdot, \cdot, x_3, x_4)\|,$$

with $\beta = \|b\|$, but we can not assure that there exist $x_1^*,\ x_2^* \in E^*$ such that $a(x_1^*, x_2^*, \cdot, \cdot) = b$, since this would imply that b is the product of two linear functionals. Note that, even in the case that $\dim(E) = 2$, we can not guarantee the existence of such x_1^* and x_2^*, despite the preceding inequality.

We can, however, obtain a positive partial result, that also generalizes Theorem 1.2:

Theorem 2.2. *Let $n \ge 1$ and let $E_1, \ldots, E_n, F$ be nontrivial real Hausdorff locally convex spaces and assume that $\mathcal{L}^n(E_1, \ldots, E_n)$ is also a nontrivial real Hausdorff locally convex space. Let $a : E_1 \times \cdots \times E_n \times F \longrightarrow \mathbb{R}$ be a $(n+1)-$linear form, let $y_0^* \in F^*$ and let C be a convex subset of F such that for all $y \in F$, $a(\cdot, \ldots, \cdot, y) \in \mathcal{L}^n(E_1, \ldots, E_n)$. Then*

$$\begin{array}{c} \text{there exists } \Phi \in (\mathcal{L}^n(E_1, \ldots, E_n))^* \\ \text{such that for all } y \in C, \quad y_0^*(y) \le \Phi(a(\cdot, \ldots, \cdot, y)) \\ \Updownarrow \\ \text{there exists a continuous seminorm } p : \mathcal{L}^n(E_1, \ldots, E_n) \longrightarrow \mathbb{R} \\ \text{such that for all } y \in C, \quad y_0^*(y) \le p(a(\cdot, \ldots, \cdot, y)). \end{array}$$

Moreover, if one of these equivalent conditions is satisfied, then we can take Φ and p with $\Phi \le p$.

Proof. The proof is very similar to that of Theorem 1.2. Let us first suppose that there exists a continuous seminorm $p : \mathcal{L}^n(E_1, \ldots, E_n) \longrightarrow \mathbb{R}$ such that for all $y \in C$ we have the inequality $y_0^*(y) \le p(a(\cdot, \ldots, \cdot, y))$. If we apply the Hahn–Banach–Lagrange theorem with the sublinear function $S = p$, the proper convex function

$$\begin{aligned} k :\ & C \longrightarrow \mathbb{R} \\ & y \mapsto -y_0^*(y) \end{aligned}$$

and the $S-$convex mapping

$$\begin{aligned} j :\ & C \longrightarrow \mathcal{L}^n(E_1, \ldots, E_n) \\ & y \mapsto a(\cdot, \ldots, \cdot, y), \end{aligned}$$

it follows that there exists a linear form $\Phi : \mathcal{L}^n(E_1, \ldots, E_n) \longrightarrow \mathbb{R}$ such that $\Phi \le p$ (hence $\Phi \in (\mathcal{L}^n(E_1, \ldots, E_n))^*$) and

$$\inf_{y \in C} \left(\Phi(a(\cdot, \ldots, \cdot, y)) - y_0^*(y)\right) = \inf_{y \in C} \left(p(a(\cdot, \ldots, \cdot, y)) - y_0^*(y)\right).$$

But, since we are assuming that

$$\inf_{y \in C} \left(p(a(\cdot, \ldots, \cdot, y)) - y_0^*(y)\right) \ge 0,$$

then

$$\inf_{y \in C} \left(\Phi(a(\cdot, \ldots, \cdot, y)) - y_0^*(y)\right) \ge 0,$$

that is,

$$\text{for all } y \in C, \quad y_0^*(y) \le \Phi(a(\cdot, \ldots, \cdot, y))$$

and thus we conclude that implication $\Uparrow$ is true.

And conversely, if for some $\Phi \in (\mathcal{L}^n(E_1, \ldots, E_n))^*$ it holds that

$$y \in C \Rightarrow y_0^*(y) \le \Phi(a(\cdot, \ldots, \cdot, y)),$$

then we just take $p = |\Phi|$ as continuous seminorm on $\mathcal{L}^n(E_1, \ldots, E_n)$.

Observe that in both directions we have also proven that $\Phi \le p$. □

With a more restrictive hypothesis we can obtain something better than in Theorem 2.2, as a consequence of this theorem or of Theorem 1.2:

Corollary 2.3. *Let E_1 and F be nontrivial real Hausdorff locally convex spaces, let $n \ge 1$ and let $E_2, \ldots, E_n$ be nontrivial vector spaces and let $a : E_1 \times \cdots \times E_n \times F \longrightarrow \mathbb{R}$ be an $(n+1)-$linear form. If $y_0^* \in F^*$ and C is a convex subset of F, $(x_2, \ldots, x_n) \in E_2 \times \cdots \times E_n$ and for all $y \in F$, $a(\cdot, x_2, \ldots, x_n, y) \in E_1^*$, then*

$$\begin{array}{l} \textit{there exists } x_1^{**} \in E_1^{**} \\ \qquad \textit{such that for all } y \in C, \quad y_0^*(y) \le x_1^{**}(a(\cdot, x_2, \ldots, x_n, y)) \\ \qquad\qquad\qquad \Updownarrow \\ \textit{there exists } p : E_1^* \longrightarrow \mathbb{R} \textit{ continuous seminorm} \\ \qquad \textit{such that for all } y \in C, \quad y_0^*(y) \le p(a(\cdot, x_2, \ldots, x_n, y)). \end{array}$$

In addition, when one of these equivalent conditions is satisfied we can take Φ and p in such a way that $\Phi \le p$.

Proof. The result follows from Theorem 1.2 or Theorem 2.2, with the bilinear form $b : E_1 \times F \longrightarrow \mathbb{R}$ defined by

$$b(x, y) := a(x, x_2, \ldots, x_n, y), \quad (x \in E_1, \ y \in F).$$

□

In a similar way to Section 1, we can state for normed spaces concrete versions of the multilinear Lax–Milgram Theorem 2.2, analogous to Corollaries 1.3 and 1.5.

3. Lax–Milgram and Hahn–Banach theorems are equivalent

As we asserted when we evoked the Hahn–Banach–Lagrange theorem, such a result is a generalization of the Hahn–Banach theorem, although it is actually an equivalent statement, deduced from the former. Theorem 1.2 has been obtained as a consequence of the Hahn–Banach–Lagrange theorem. Now we close the circle, by showing that Theorem 1.2 provides us with a proof of the Hahn–Banach theorem, in one of its equivalent versions (see for instance [9] for its classical forms).

Theorem 3.1. *Let E be a nontrivial real normed space, let M be a nontrivial vector subspace of E and let $y_0^* : M \longrightarrow \mathbb{R}$ be a continuous linear functional. Then there exists a norm–preserving continuous linear extension $x_0^* : E \longrightarrow \mathbb{R}$ of y_0^*.*

Proof. We apply Corollary 1.5 (a consequence of Theorem 1.2) with $F := M$, $C := M$ and the bilinear form

$$\begin{aligned} a : \quad & E^* \times M \longrightarrow \mathbb{R} \\ & (x^*, y) \mapsto x^*(y). \end{aligned}$$

Since for all $y \in M$ we have that $a(\cdot, y) = y$, and by hypothesis $y_0^* \in M^*$, Corollary 1.5 guarantees the existence of $x_0^* \in E^*$ such that

$$\text{for all } y \in M, \quad y_0^*(y) \leq a(x_0^*, y),$$

in other words, x_0^* is a continuous linear extension of y_0^*. In addition Corollary 1.5 allows us to choose x_0^* in such a way that

$$\|x_0^*\| = \left(\sup_{y \in M,\ a(\cdot,y) \neq 0} \frac{y_0^*(y)}{\|a(\cdot, y)\|} \right)_+ = \|y_0^*\|$$

and thus the proof is complete. □

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