Maximal Monotonicity, Conjugation and the Duality Product in Non-Reflexive Banach Spaces

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Received: September 23, 2008

In this work we study some conditions which guarantee that a convex function represents a maximal monotone operator in non-reflexive Banach spaces.

Keywords: Fitzpatrick function, maximal monotone operator, non-reflexive Banach spaces

2000 Mathematics Subject Classification: 47H05, 49J52, 47N10

1. Introduction

Let $X$ be a real Banach space and $X^*$ its topological dual, both with norms denoted by $\| \cdot \|$. The duality product in $X \times X^*$ will be denoted by:

$$\pi : X \times X^* \to \mathbb{R}, \quad \pi(x, x^*) := \langle x, x^* \rangle = x^*(x).$$

(1)

A point to set operator $T : X \rightrightarrows X^*$ is a relation on $X \times X^*$:

$$T \subseteq X \times X^*$$

and $T(x) = \{ x^* \in X^* \mid (x, x^*) \in T \}$. An operator $T : X \rightrightarrows X^*$ is monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T$$

and it is maximal monotone if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of $X$ into $X^*$. The domain of $T : X \rightrightarrows X^*$ is defined by $D(T) := \{ x \in X \mid T(x) \neq \emptyset \}$.

Fitzpatrick proved constructively that maximal monotone operators are representable by convex functions. Before discussing his findings, let us establish some notation. We

*Partially supported by Brazilian CNPq scholarship 140525/2005-0.
†Partially supported by CNPq grants 300755/2005-8, 475647/2006-8 and by PRONEX-Optimization.
denote the set of extended-real valued functions on \( X \) by \( \mathbb{R}^X \). The epigraph of \( f \in \mathbb{R}^X \) is defined by
\[
E(f) := \{(x, \mu) \in X \times \mathbb{R} \mid f(x) \leq \mu \}.
\]
We say that \( f \in \mathbb{R}^X \) is lower semicontinuous (l.s.c. from now on) if \( E(f) \) is closed in the strong topology of \( X \times \mathbb{R} \).

Let \( T : X \rightrightarrows X^* \) be maximal monotone. The Fitzpatrick function of \( T \) is \[ \varphi_T \in \mathbb{R}^{X \times X^*}, \quad \varphi_T(x, x^*) := \sup_{(y, y^*) \in T} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \] (2)
and the Fitzpatrick family associated with \( T \) is
\[
\mathcal{F}_T := \left\{ h \in \mathbb{R}^{X \times X^*} \mid \begin{array}{l}
h \text{ is convex and l.s.c.} \\
h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^* \\
(x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle
\end{array} \right\}.
\]

In the next theorem we summarize the Fitzpatrick’s results:

**Theorem 1.1 ([4, Theorem 3.10]).** Let \( X \) be a real Banach space and \( T : X \rightrightarrows X^* \) be maximal monotone. Then for any \( h \in \mathcal{F}_T \)
\[
(x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle
\]
and \( \varphi_T \) is the smallest element of the family \( \mathcal{F}_T \).

Fitzpatrick’s results described above were rediscovered by Martínez-Legaz and Théra [9], and Burachik and Svaiter [2].

It seems interesting to study conditions under which a convex function \( h \in \mathbb{R}^X \) represents a maximal monotone operator, that is, \( h \in \mathcal{F}_T \) for some maximal monotone operator \( T \). Our aim is to extend previous results on this direction. We will need some auxiliary results and additional notation for this aim.

The Fenchel-Legendre conjugate of \( f \in \mathbb{R}^X \) is
\[
f^* \in \mathbb{R}^{X^*}, \quad f^*(x^*) := \sup_{x \in X} \langle x, x^* \rangle - f(x).
\]
Whenever necessary, we will identify \( X \) with its image under the canonical injection of \( X \) into \( X^{**} \). Burachik and Svaiter proved that the family \( \mathcal{F}_T \) is invariant under the mapping
\[
\mathcal{J} : \mathbb{R}^{X \times X^*} \rightarrow \mathbb{R}^{X \times X^*}, \quad \mathcal{J} h(x, x^*) := h^*(x^*, x).
\]
This means that if \( T : X \rightrightarrows X^* \) is maximal monotone, then [2]
\[
\mathcal{J}(\mathcal{F}_T) \subset \mathcal{F}_T.
\]
In particular, for any \( h \in \mathcal{F}_T \) it holds that \( h \geq \pi \), \( \mathcal{J}h \geq \pi \), that is,
\[
h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*.
\]
So, the above conditions are necessary for a convex function \( h \) on \( X \times X^* \) to represent a maximal monotone operator. Burachik and Svaiter proved that these conditions are also sufficient, in a reflexive Banach space, for \( h \) to represent a maximal monotone operator [3]:

**Theorem 1.2 ([3, Theorem 3.1])**. Let \( h \in \mathbb{R}^{X \times X^*} \) be proper, convex, l.s.c. and

\[
h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*. \tag{5}
\]

If \( X \) is reflexive, then

\[
T := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}
\]

is maximal monotone and \( h, Jh \in \mathcal{F}_T \).

Marques Alves and Svaiter generalized Theorem 1.2 to non-reflexive Banach spaces as follows:

**Theorem 1.3 ([5, Corollary 4.4 ])**. If \( h \in \mathbb{R}^{X \times X^*} \) is convex and

\[
h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*,
\]

\[
h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**} \tag{6}
\]

then

\[
T := \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}
\]

is maximal monotone and \( Jh \in \mathcal{F}_T \). Moreover, if \( h \) is l.s.c. then \( h \in \mathcal{F}_T \).

Condition (6) of Theorem 1.3 enforces the operator \( T \) to be of type (NI) [6] and is not necessary for maximal monotonicity of \( T \) in a non-reflexive Banach space. Note that the weaker condition (5) of Theorem 1.2 is still necessary in non-reflexive Banach spaces for the inclusion \( h \in \mathcal{F}_T \), where \( T \) is a maximal monotone operator. The main result of this paper is another generalization of Theorem 1.2 to non-reflexive Banach spaces which uses condition (5) instead of (6). To obtain this generalization, we add a regularity assumption on the domain of \( h \).

If \( T : X \rightrightarrows X^* \) is maximal monotone, it is easy to prove that \( \varphi_T \) is minimal in the family of all convex functions in \( X \times X^* \) which majorizes the duality product. So, it is natural to ask whether the converse also holds, that is:

Is any minimal element of this family (convex functions which majorizes the duality product) a Fitzpatrick function of some maximal monotone operator?

To give a partial answer to this question, Martínez-Legaz and Svaiter proved the following results, which we will use latter on:

**Theorem 1.4 ([8, Theorem 5])**. Let \( \mathcal{H} \) be the family of convex functions in \( X \times X^* \) which majorizes the duality product:

\[
\mathcal{H} := \left\{ h \in \mathbb{R}^{X \times X^*} \mid h \text{ is proper, convex and } h \geq \pi \right\}. \tag{7}
\]

The following statements hold true:
1. The family $\mathcal{H}$ is (downward) inductively ordered;
2. For any $h \in \mathcal{H}$ there exists a minimal $h_0 \in \mathcal{H}$ such that $h \geq h_0$;
3. Any minimal element $g$ of $\mathcal{H}$ is l.s.c. and satisfies $Jg \geq g$.

Note that item 2. is a direct consequence of item 1. Combining item 3. with Theorem 1.2, Martínez-Legaz and Svaiter concluded that in a reflexive Banach space, any minimal element of $\mathcal{H}$ is the Fitzpatrick function of some maximal monotone operator [8, Theorem 5]. We will also present a partial extension of this result for non-reflexive Banach spaces.

2. Basic results and notation

The weak-star topology of $X^*$ will be denoted by $\omega^*$ and by $s$ we denote the strong topology of $X$. A function $h \in \mathbb{R}^{X \times X^*}$ is lower semicontinuous in the strong $\times$ weak-star topology if $E(h)$ is a closed subset of $X \times X^* \times \mathbb{R}$ in the $s \times \omega^* \times | \cdot |$ topology.

The indicator function of $V \subset X$ is $\delta_V$, $\delta_V(x) := 0$, $x \in V$ and $\delta_V(x) := \infty$, otherwise. The closed convex closure of $f \in \mathbb{R}^X$ is defined by

$$
\text{cl conv } f \in \mathbb{R}^X, \quad \text{cl conv } f(x) := \inf\{\mu \in \mathbb{R} | (x, \mu) \in \text{cl conv } E(f)\}
$$

where for $U \subset X$, $\text{cl conv } U$ is the closed convex hull (in the $s$ topology) of $U$. The effective domain of a function $f \in \mathbb{R}^X$ is

$$
D(f) := \{x \in X | f(x) < \infty\},
$$

and $f$ is proper if $D(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. If $f$ is proper, convex and l.s.c., then $f^*$ is proper. For $h \in \mathbb{R}^{X \times X^*}$, we also define

$$
\text{Pr}_X D(h) := \{x \in X | \exists x^* \in X^* | (x, x^*) \in D(h)\}.
$$

Let $T : X \rightrightarrows X^*$ be maximal monotone. In [2] Burachik and Svaiter defined and studied the biggest element of $\mathcal{F}_T$, namely, the $\mathcal{S}$-function, $\mathcal{S}_T \in \mathcal{F}_T$ defined by

$$
\mathcal{S}_T \in \mathbb{R}^{X \times X^*}, \quad \mathcal{S}_T := \sup_{h \in \mathcal{F}_T} \{h\},
$$

or, equivalently

$$
\mathcal{S}_T = \text{cl conv}(\pi + \delta_T).
$$

Recall that $\mathcal{J}(\mathcal{F}_T) \subset \mathcal{F}_T$. Additionally [2]

$$
\mathcal{J} \mathcal{S}_T = \varphi_T
$$

and, in a reflexive Banach space, $\mathcal{J} \varphi_T = \mathcal{S}_T$.

In what follows we present the Attouch-Brezis’s version of the Fenchel-Rockafellar duality theorem:
Theorem 2.1 ([1, Theorem 1.1]). Let $Z$ be a Banach space and $\varphi, \psi \in \mathbb{R}^Z$ be proper, convex and l.s.c. functions. If

$$\bigcup_{\lambda > 0} \lambda [D(\varphi) - D(\psi)],$$

is a closed subspace of $Z$, then

$$\inf_{z \in Z} \varphi(z) + \psi(z) = \max_{z^* \in Z^*} -\varphi^*(z^*) - \psi^*(-z^*).$$

Given $X, Y$ Banach spaces, $\mathcal{L}(Y, X)$ denotes the set of continuous linear operators of $Y$ into $X$. The range of $A \in \mathcal{L}(Y, X)$ is denoted by $R(A)$ and the adjoint by $A^* \in \mathcal{L}(X^*, Y^*)$:

$$\langle Ay, x^* \rangle = \langle y, A^* x^* \rangle \ \forall y \in Y, x^* \in X^*,$$

where $X^*, Y^*$ are the topological duals of $X$ and $Y$, respectively. The next proposition is a particular case of Theorem 3 of [10]. For the sake of completeness, we give the proof in the Appendix A.

Proposition 2.2. Let $X, Y$ Banach spaces and $A \in \mathcal{L}(Y, X)$. For $h \in \mathbb{R}^{X \times X^*}$, proper convex and l.s.c., define $f \in \mathbb{R}^{Y \times Y^*}$

$$f(y, y^*) := \inf_{x^* \in X^*} h(Ay, x^*) + \delta_{\{0\}}(y^* - A^* x^*).$$

If

$$\bigcup_{\lambda > 0} \lambda [\text{Pr}_X D(h) - R(A)],$$

is a closed subspace of $X$, then

$$f^*(z^*, z) = \min_{u^* \in X^*} h^*(u^*, Az) + \delta_{\{0\}}(z^* - A^* u^*).$$

Martínez-Legaz and Svaiter [7] defined, for $h \in \mathbb{R}^{X \times X^*}$ and $(x_0, x_0^*) \in X \times X^*$, $h_{(x_0, x_0^*)} \in \mathbb{R}^{X \times X^*}$

$$h_{(x_0, x_0^*)}(x, x^*) := h(x + x_0, x^* + x_0^*) - [\langle x, x_0^* \rangle + \langle x_0, x^* \rangle + \langle x_0, x_0^* \rangle]$$

$$= h(x + x_0, x^* + x_0^*) - \langle x + x_0, x^* + x_0^* \rangle + \langle x, x^* \rangle.$$  (12)

The operation $h \mapsto h_{(x_0, x_0^*)}$ preserves many properties of $h$, as convexity and lower semicontinuity. Moreover, one can easily prove the following Proposition:

Proposition 2.3. Let $h \in \mathbb{R}^{X \times X^*}$. Then it holds that

1. $h \geq \pi \iff h_{(x_0, x_0^*)} \geq \pi, \ \forall (x_0, x_0^*) \in X \times X^*$;
2. $Jh_{(x_0, x_0^*)} = (Jh)_{(x_0, x_0^*)}, \ \forall (x_0, x_0^*) \in X \times X^*.$
3. Main results

In the next theorem we generalize Theorem 1.2 to non-reflexive Banach spaces under condition (5) instead of condition (6) used in Theorem 1.3. To obtain this generalization, we add a regularity assumption (14) on the domain of $h$.

**Theorem 3.1.** Let $h \in \mathbb{R}^{X \times X^*}$ be proper, convex and

$$
\begin{align*}
  h(x, x^*) &\geq \langle x, x^* \rangle, \\
  h^*(x^*, x) &\geq \langle x, x^* \rangle, \\
  \forall (x, x^*) &\in X \times X^*.
\end{align*}
$$

If

$$
\bigcup_{\lambda > 0} \lambda \text{Pr}_X D(h),
$$

is a closed subspace of $X$, then

$$
T := \{ (x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle \}
$$

is maximal monotone and $Jh \in \mathcal{F}_T$.

**Proof.** First, define $\tilde{h} := \text{cl} \, h$ and note that $\tilde{h}$ is proper, convex, l.s.c., satisfies (13), (14) and $\tilde{J}h = Jh$. So, it suffices to prove the theorem for the case where $h$ is l.s.c., and we assume it from now on in this proof. Monotonicity of $T$ follows from Theorem 5 of [7]. Note that for any $x \in X$

$$
T(x) = \{ x^* \in X^* \mid h^*(x^*, x) - \langle x, x^* \rangle \leq 0 \}.
$$

Therefore, $T(x)$ is convex and $\omega^*$-closed.

To prove maximality of $T$, take $(x_0, x^*_0) \in X \times X^*$ such that

$$
\langle x - x_0, x^* - x^*_0 \rangle \geq 0, \quad \forall (x, x^*) \in T
$$

and suppose $x^*_0 \notin T(x_0)$. As $T(x_0)$ is convex and $\omega^*$-closed, using the geometric version of the Hahn-Banach theorem in $X^*$ endowed with the $\omega^*$ topology we conclude that (even if $T(x_0)$ is empty) there exists $z_0 \in X$ such that

$$
\langle z_0, x^*_0 \rangle < \langle z_0, x^* \rangle, \quad \forall x^* \in T(x_0).
$$

Let $Y := \text{span} \{ x_0, z_0 \}$. Define $A \in \mathcal{L}(Y, X), A \, y := y, \forall y \in Y$ and the convex function $f \in \mathbb{R}^{Y \times Y^*}$,

$$
\begin{align*}
  f(y, y^*) := \inf_{x^* \in X^*} h(Ay, x^*) + \delta_{\{0\}}(y^* - A^* x^*).
\end{align*}
$$

Using Proposition 2.2 we obtain

$$
\begin{align*}
  f^*(y^*, y) &= \min_{x^* \in X^*} h^*(x^*, Ay) + \delta_{\{0\}}(y^* - A^* x^*).
\end{align*}
$$

Using (13), (17) and (18) it is easy to see that

$$
\begin{align*}
  f(y, y^*) &\geq \langle y, y^* \rangle, \\
  f^*(y^*, y) &\geq \langle y, y^* \rangle, \\
  \forall (y, y^*) &\in Y \times Y^*.
\end{align*}
$$
Define \( g := \mathcal{J}f \). As \( Y \) is reflexive we have \( \mathcal{J}g = \text{cl} \ f \). Therefore, using (19) we also have
\[
g(y, y^*) \geq \langle y, y^* \rangle, \quad g^*(y^*, y) \geq \langle y, y^* \rangle, \quad \forall (y, y^*) \in Y \times Y^*.
\tag{20}
\]

Now, using (20) and item 1. of Proposition 2.3 we obtain
\[
g_{(x_0, A^*x_0^*)}(y, y^*) + \frac{1}{2} \|y\|^2 + \frac{1}{2} \|y^*\|^2 \\
\geq \langle y, y^* \rangle + \frac{1}{2} \|y\|^2 + \frac{1}{2} \|y^*\|^2 \geq 0, \quad \forall (y, y^*) \in Y \times Y^*.
\tag{21}
\]

and
\[
(\mathcal{J}g)_{(x_0, A^*x_0^*)}(y, y^*) + \frac{1}{2} \|y\|^2 + \frac{1}{2} \|y^*\|^2 \\
\geq \langle y, y^* \rangle + \frac{1}{2} \|y\|^2 + \frac{1}{2} \|y^*\|^2 \geq 0, \quad \forall (y, y^*) \in Y \times Y^*.
\tag{22}
\]

Using Theorem 2.1 and item 2. of Proposition 2.3 we conclude that there exists \((\tilde{z}, \tilde{z}^*) \in Y \times Y^*\) such that
\[
\inf_{(y, y^*) \in Y \times Y^*} g_{(x_0, A^*x_0^*)}(y, y^*) + \frac{1}{2} \|y\|^2 + \frac{1}{2} \|y^*\|^2 = 0.
\tag{23}
\]

From (21), (22) and (23) we have
\[
\inf_{(y, y^*) \in Y \times Y^*} g_{(x_0, A^*x_0^*)}(y, y^*) + \frac{1}{2} \|y\|^2 + \frac{1}{2} \|y^*\|^2 = 0.
\tag{24}
\]

As \( Y \) is reflexive, from (12), (24) we conclude that there exists \((\hat{y}, \hat{y}^*) \in Y \times Y^*\) such that
\[
g(\hat{y} + x_0, \hat{y}^* + A^*x_0^*) - \langle \hat{y} + x_0, \hat{y}^* + A^*x_0^* \rangle + \langle \hat{y}, \hat{y}^* \rangle + \frac{1}{2} \|\hat{y}\|^2 + \frac{1}{2} \|\hat{y}^*\|^2 = 0.
\tag{25}
\]

Using (25) and the first inequality of (20) (and the definition of \( g \)) we have
\[
f^*(\hat{y}^* + A^*x_0^*, \hat{y} + x_0) = \langle \hat{y} + x_0, \hat{y}^* + A^*x_0^* \rangle
\tag{26}
\]
and
\[
\langle \hat{y}, \hat{y}^* \rangle + \frac{1}{2} \|\hat{y}\|^2 + \frac{1}{2} \|\hat{y}^*\|^2 = 0.
\tag{27}
\]

Using (18) we have that there exists \( w_0^* \in X^* \) such that
\[
f^*(\hat{y}^* + A^*x_0^*, \hat{y} + x_0) = h^*(w_0^*, A(\hat{y} + x_0)), \quad \hat{y}^* + A^*x_0^* = A^*w_0^*.
\tag{28}
\]

So, combining (26) and (28) we have
\[
h^*(w_0^*, A(\hat{y} + x_0)) = \langle \hat{y} + x_0, A^*w_0^* \rangle = \langle A(\hat{y} + x_0), w_0^* \rangle.
\]

In particular, \( w_0^* \in T(A(\hat{y} + x_0)) \). As \( x_0 \in Y \), we can use (15) and the second equality of (28) to conclude that
\[
\langle A(\hat{y} + x_0) - x_0, w_0^* - x_0^* \rangle = \langle \hat{y}, A^*(w_0^* - x_0^*) \rangle = \langle \hat{y}, \hat{y}^* \rangle \geq 0.
\tag{29}
\]
Using (27) and (29) we conclude that $\hat{y} = 0$ and $\hat{y}^* = 0$. Therefore,

$$w_0^* \in T(x_0), \quad A^* x_0 = A^* w_0^*.$$ 

As $z_0 \in Y$, we have $z_0 = A z_0$ and so

$$\langle z_0, x_0^* \rangle = \langle A z_0, x_0^* \rangle = \langle z_0, A^* x_0^* \rangle = \langle z_0, A^* w_0^* \rangle = \langle z_0, w_0^* \rangle,$$

that is,

$$\langle z_0, x_0^* \rangle = \langle z_0, w_0^* \rangle, \quad w_0^* \in T(x_0)$$

which contradicts (16). Therefore, $(x_0, x_0^*) \in T$ and so $T$ is maximal monotone and $J h \in F_T$. 

Observe that if $h$ is convex, proper and l.s.c. in the strong $\times$ weak-star topology, then $J^2 h = h$. Therefore, using this observation we have the following corollary of Theorem 3.1:

**Corollary 3.2.** Let $h \in \mathbb{R}^{X \times X^*}$ be proper, convex, l.s.c. in the strong $\times$ weak-star topology and

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*.$$ 

If

$$\bigcup_{\lambda > 0} \lambda \Pr_X D(h),$$

is a closed subspace of $X$, then

$$T := \{ (x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle \}$$

is maximal monotone and $h, J h \in F_T$. 

**Proof.** Using Theorem 3.1 we conclude that the set

$$S := \{ (x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle \}$$

is maximal monotone. Take $(x, x^*) \in S$. As $\pi$ is Gateaux differentiable, $h \geq \pi$ and $\pi(x, x^*) = h(x, x^*)$, we have (see Lemma 4.1 of [5])

$$D\pi(x, x^*) \in \partial J h(x, x^*),$$

where $D\pi$ stands for the Gateaux derivative of $\pi$. As $D\pi(x, x^*) = (x^*, x)$, we conclude that

$$J h(x, x^*) + J^2 h(x, x^*) = \langle (x, x^*), (x^*, x) \rangle.$$ 

Substituting $J h(x, x^*)$ by $(x, x^*)$ in the above equation we conclude that $J^2 h(x, x^*) = \langle x, x^* \rangle$. Therefore, as $J^2 h(x, x^*) = h(x, x^*)$,

$$S \subset T.$$ 

To end the proof use the maximal monotonicity of $S$ (Theorem 3.1) and the monotonicity of $T$ (see Theorem 5 of [7]) to conclude that $S = T$. 

It is natural to ask whether we can drop lower semicontinuity assumptions. In the context of non-reflexive Banach spaces, we should use the l.s.c. closure in the strong $\times$ weak-star topology. Unfortunately, as the duality product is not continuous in this topology, it is not clear whether the below implication holds:

$$ h \geq \pi \Rightarrow \text{cl}_{s\times\omega^*} h \geq \pi. $$

**Corollary 3.3.** Let $h \in \mathbb{R}^{X \times X^*}$ be proper, convex and

$$ h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*. $$

If

$$ \bigcup_{\lambda > 0} \lambda \text{Pr}_X D(h) $$

is a closed subspace of $X$, then

$$ \text{cl}_{s\times\omega^*} h \in F_T, $$

where $\text{cl}_{s\times\omega^*}$ denotes the l.s.c. closure in the strong $\times$ weak-star topology and $T$ is the maximal monotone operator defined as in Theorem 3.1:

$$ T := \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}. $$

In particular, $\text{cl}_{s\times\omega^*} h \geq \pi$.

**Proof.** First use Theorem 3.1 to conclude that $T$ is maximal monotone and $\mathcal{J} h \in F_T$. In particular,

$$ \mathcal{S}_T \geq \mathcal{J} h \geq \varphi_T. $$

Therefore,

$$ \mathcal{J} \varphi_T \geq \mathcal{J}^2 h \geq \mathcal{J} \mathcal{S}_T. $$

As $\mathcal{J} \mathcal{S}_T = \varphi_T \in F_T$ and $\mathcal{J} \varphi_T \in F_T$, we conclude that $\text{cl}_{s\times\omega^*} h = \mathcal{J}^2 h \in F_T$. \hfill $\square$

In the next corollary we give a partial answer for an open question proposed by Martínez-Legaz and Svaiter in [8], in the context of non-reflexive Banach spaces.

**Corollary 3.4.** Let $\mathcal{H}$ be the family of convex functions on $X \times X^*$ bounded below by the duality product, as defined in (7). If $g$ is a minimal element of $\mathcal{H}$ and

$$ \bigcup_{\lambda > 0} \lambda \text{Pr}_X D(g) $$

is a closed subspace of $X$, then there exists a maximal monotone operator $T$ such that $g = \varphi_T$, where $\varphi_T$ is the Fitzpatrick function of $T$.

**Proof.** Using item 3. of Theorem 1.4 and Theorem 3.1 we have that

$$ T := \{(x, x^*) \in X \times X^* \mid g^*(x^*, x) = \langle x, x^* \rangle\} $$

is maximal monotone, $\mathcal{J} g \in F_T$ and

$$ T \subset \{(x, x^*) \in X \times X^* \mid g(x, x^*) = \langle x, x^* \rangle\}. $$

As $g$ is convex and bounded below by the duality product, using Theorem 5 of [7], we conclude that the rightmost set on the above inclusion is monotone. Since $T$ is maximal monotone, the above inclusion holds as an equality and, being l.s.c., $g \in F_T$.

To end the proof, note that $g \geq \varphi_T \in \mathcal{H}$. \hfill $\square$
A. Proof of Proposition 2.2

Proof of Proposition 2.2. Using the Fenchel-Young inequality we have, for any $(y, y^*), (z, z^*) \in Y \times Y^*$ and $x^*, u^* \in X^*$,
\[
    h(Ay, x^*) + \delta_{01}(y^* - A^*x^*) + h^*(u^*, Az) + \delta_{01}(z^* - A^*u^*) \geq \langle Ay, u^* \rangle + \langle Az, x^* \rangle.
\]
Taking the infimum over $x^*, u^* \in X^*$ on the above inequality we get
\[
f(y, y^*) + \inf_{u^* \in X^*} h^*(u^*, Az) + \delta_{01}(z^* - A^*u^*) \geq \langle y, z^* \rangle + \langle z, y^* \rangle = \langle (z^*, z), (y, y^*) \rangle,
\]
that is,
\[
\langle (z^*, z), (y, y^*) \rangle - f(y, y^*) \leq \inf_{u^* \in X^*} h^*(u^*, Az) + \delta_{01}(z^* - A^*u^*).
\]
Now, taking the supremum over $(y, y^*) \in Y \times Y^*$ on the left hand side of the above inequality we obtain
\[
f^*(z^*, z) \leq \inf_{u^* \in X^*} h^*(u^*, Az) + \delta_{01}(z^* - A^*u^*). \tag{30}
\]
For a fixed $(z, z^*) \in Y \times Y^*$ such that $f^*(z^*, z) < \infty$, define $\varphi, \psi \in \mathbb{R}^{Y \times X \times Y^* \times X^*}$,
\[
    \varphi(y, x, y^*, x^*) := f^*(z^*, z) - \langle y, z^* \rangle - \langle z, y^* + A^*x^* \rangle + \delta_{01}(y^*) + h(x, x^*),
\]
\[
    \psi(y, x, y^*, x^*) := \delta_{01}(x - Ay).
\]
Direct calculations yields
\[
    \bigcup_{\lambda > 0} \lambda[\varphi(\varphi) - D(\varphi)] = Y \times \bigcup_{\lambda > 0} \lambda[\Pr_X D(h)] - R(A)] \times Y^* \times X^*. \tag{31}
\]
Using (11), (31) and Theorem 2.1 for $\varphi$ and $\psi$, we conclude that there exists $(y^*, x^*, y^{**}, x^{**}) \in Y^* \times X^* \times Y^{**} \times X^{**}$ such that
\[
    \inf \varphi + \psi = -\varphi^*(y^*, x^*, y^{**}, x^{**}) - \psi^*(-y^*, -x^*, -y^{**}, -x^{**}). \tag{32}
\]
Now, notice that
\[
    (\varphi + \psi)(y, x, y^*, x^*) \geq f^*(z^*, z) + f(y, A^*x^*) - \langle (z^*, z), (y, A^*x^*) \rangle \geq 0. \tag{33}
\]
Using (32) and (33) we get
\[
    \varphi^*(y^*, x^*, y^{**}, x^{**}) + \psi^*(-y^*, -x^*, -y^{**}, -x^{**}) \leq 0. \tag{34}
\]
Direct calculations yields
\[
\psi^*(-y^*, -x^*, -y^{**}, -x^{**}) = \sup_{(y, z^*, w^*)} (y, -y^* - A^*x^*) + \langle z^*, -y^{**} \rangle + \langle w^*, -x^{**} \rangle
\]
\[
= \delta_{01}(y^* + A^*x^*) + \delta_{01}(y^{**}) + \delta_{01}(x^{**}). \tag{35}
\]
Now, using (34) and (35) we conclude that

\[ y^{**} = 0, \quad x^{**} = 0 \quad \text{and} \quad y^* = -A^*x^*. \]

Therefore, from (34) we have

\[
\varphi^*(-A^*x^*, x^*, 0, 0) = \sup_{(y, x, w^*)} \left( \langle y, z^* - A^*x^* \rangle + \langle x, x^* \rangle + \langle Az, w^* \rangle - h(x, w^*) \right) - f^*(z^*, z)
\]

\[
= h^*(x^*, Az) + \delta_{\{0\}}(z^* - A^*x^*) - f^*(z^*, z) \leq 0,
\]

that is, there exists \( x^* \in X^* \) such that

\[ f^*(z^*, z) \geq h^*(x^*, Az) + \delta_{\{0\}}(z^* - A^*x^*). \]

Finally, using (30) we conclude the proof. \( \square \)

References


