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On Surjectivity Results for Maximal Monotone Operators of Type (D)

M. Rocco*
Dipartimento di Matematica, Statistica, Informatica e Applicazioni, Università degli Studi di Bergamo, Via dei Caniana 2, Bergamo 24127, Italy
marco.rocco@unibg.it

J. E. Martínez-Legaz†
Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain
juanenrique.martinez.legaz@uab.cat

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A generalization of Rockafellar's surjectivity theorem was provided in [14], replacing the duality mapping by any maximal monotone operator having finite-valued Fitzpatrick function. The present paper extends this result to the nonreflexive setting for maximal monotone operators of type (D) and refines the finite-valuedness condition on the Fitzpatrick function. Moreover, a characterization of surjectivity properties for the sum of two maximal monotone operators of type (D) in terms of Fenchel duality is given.

Keywords: Monotone operator, type (D), convex representation, bidual, surjectivity, Fenchel duality

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1. Introduction

The use of convex functions for the study of maximal monotone operators can be traced back to Krauss [8] and Fitzpatrick [3]. The approach of the latter, in particular, has generated intense research since it was independently rediscovered in [17] and [2]. Given a Banach space X and a maximal monotone operator $T : X \rightrightarrows X^*$ (we will come back to definitions and notations in Section 2), we define the Fitzpatrick family, or the family of convex representations of T as the set

$$\begin{aligned}\mathcal{H}_T \;=\; &\{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} : h \text{ is lower semicontinuous} \\ &\text{and convex, } h \geq \pi \text{ and } h = \pi \text{ on } \mathcal{G}(T)\},\end{aligned}$$

where $\pi : X \times X^* \to \mathbb{R}$, $(x, x^*) \mapsto \langle x, x^* \rangle$, denotes the duality product and $\mathcal{G}(T)$ the graph of T. In fact, it is well known that, since T is maximal monotone, one has

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$h = \pi$ exactly on $\mathcal{G}(T)$, i.e.

$$\forall (x, x^*) \in X \times X^* : \quad h(x, x^*) = \langle x, x^* \rangle \quad \Leftrightarrow \quad (x, x^*) \in \mathcal{G}(T).$$

This family has a minimum (the Fitzpatrick function φ_T) and a maximum (see [2] for the proof), namely $\varphi_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ and $\sigma_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, respectively defined by

$$\begin{aligned} \varphi_T(x, x^*) &= \langle x, x^* \rangle - \inf_{(y,y^*) \in \mathcal{G}(T)} \langle x - y, x^* - y^* \rangle \\ &= \sup_{(y,y^*) \in \mathcal{G}(T)} \{ \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle \} \end{aligned}$$

and

$$\sigma_T(x, x^*) = \operatorname{cl} \operatorname{conv}(\pi + \delta_{\mathcal{G}(T)})(x, x^*)$$

for all $(x, x^*) \in X \times X^*$.

Nice convex analytical arguments, applied to elements of $\mathcal{H}_T$, were employed by Simons and Zălinescu [25] to obtain a new proof of Rockafellar's characterization of maximal monotone operators [20] in the setting of a reflexive Banach space.

By means of a proof technique based on Fenchel duality and inspired by [25], [14] provided several generalizations of Rockafellar's surjectivity result and of its version with the sum of the graphs introduced by Simons [22], replacing the duality mapping involved in those results by an arbitrary maximal monotone operator having finite-valued Fitzpatrick function.

However, the results in [14] are mainly restricted to reflexive Banach spaces. Dealing with nonreflexive spaces introduces difficulties that can be overcome by imposing some conditions on the behavior of the operator with respect to the elements of the bidual. Surjectivity results for maximal monotone operators of type (D) and of type (NI) (we will recall the definitions in Section 2) were provided by Gossez [4], [5] and by Marques Alves and Svaiter [13], respectively. Finally, in a very recent work, Marques Alves and Svaiter [12] proved that the classes (D) and (NI) actually coincide.

The aim of the present paper is to further investigate in the domain of the convex analytical proofs contained in [25] and [14], especially with respect to their relevance for surjectivity results and applications of them. In this sense, we generalize [14] mainly along two directions.

First, by considering, in the main theorems, the case of a (possibly) nonreflexive Banach space with maximal monotone operators of type (D) defined on it. We mainly provide surjectivity properties that are stated in a natural way in terms of the (unique) extensions of the operators to the bidual, but we also consider a couple of results concerning density properties for the operators themselves, on the lines of [13].

Second, even for those results that hinge upon the hypothesis of reflexivity, we provide some generalizations with respect to [14] by refining the constraint qualifications and analyzing in full detail the structure and the scope of the proof techniques involved there. Namely, we weaken the requirement of finite-valued Fitzpatrick functions typically used in [14], replacing it by conditions on the sum of the domains of convex representers, and characterize surjectivity properties in terms of the existence of Fenchel functionals (we recall in Section 2 the meaning of this terminology borrowed

from [24]). This characterization, moreover, makes explicit the equivalent role played in our duality based proofs by any member of the Fitzpatrick family (not necessarily the most commmon ones, like the Fitzpatrick function or, in the case of subdifferentials, their Fenchel representation). The symmetry is such that they essentially have the same Fenchel functionals, if any.
The paper is organized as follows. In the second section we set notation and recall basic definitions. Moreover, we collect some important results from [24] and [10], which we will need later on and we prove some simple preliminary lemmas. In the third section, we prove the surjectivity theorems in their form related to the sum of the graphs. In the fourth section, we prove the surjectivity result for the range of the sum of two maximal monotone operators of type (D) (satisfying appropriate conditions) and derive some corollaries (in particular an existence theorem for variational inequalities on reflexive Banach spaces) that refine the corresponding results in [14]. Finally, the last section provides, as an application of the previous results, a new convex analytical proof of the relations between the range of a maximal monotone operator of type (D) and the projections of the domains of its convex representations on the dual space, yielding as a consequence the convexity of the closure of the range.

2. Notation and preliminary results

Given two normed spaces Y and Z, the functions

$$p_1 : Y \times Z \to Y, \quad (y,z) \mapsto p_1(y,z) = y,$$
$$p_2 : Y \times Z \to Z, \quad (y,z) \mapsto p_2(y,z) = z$$

will denote the canonical projections onto Y and Z, respectively.
According to a useful notation introduced in [18], given a function $f : Y \times Z \to \mathbb{R} \cup \{+\infty\}$ and a set $A \subseteq Y \times Z$, we will write $f^\top$ and $A^\top$ to denote respectively the function

$$f^\top : Z \times Y \to \mathbb{R} \cup \{+\infty\}, \quad (z,y) \mapsto f^\top(z,y) = f(y,z)$$

and the set

$$A^\top = \{(z,y) \in Z \times Y : (y,z) \in A\}.$$

Moreover, for any $f : Y \to \mathbb{R} \cup \{+\infty\}$, $g : Z \to \mathbb{R} \cup \{+\infty\}$, we define

$$f \oplus g : Y \times Z \to \mathbb{R} \cup \{+\infty\}, \quad (y,z) \mapsto (f \oplus g)(y,z) = f(y) + g(z).$$

Note that $\operatorname{dom}(f \oplus g) = \operatorname{dom} f \times \operatorname{dom} g$.
Given $(y,z) \in Y \times Z$, we will denote by $\tau_y : Y \to Y$ the translation in Y of vector y (i.e. $\tau_y(y') = y' + y$ for all $y' \in Y$), by $\tau_z : Z \to Z$ the translation in Z of vector z and by $\tau_{(y,z)} : Y \times Z \to Y \times Z$ the translation in $Y \times Z$ of vector (y,z).
Other isometries we will be dealing with are the reflections

$$\varrho_1 : Y \times Z \to Y \times Z, \quad (y,z) \mapsto (-y,z)$$
$$\varrho_2 : Y \times Z \to Y \times Z, \quad (y,z) \mapsto (y,-z).$$

In the following, we will consider a Banach space X and denote by X^* and X^{**} its (topological) dual and bidual, respectively. In particular, we will identify X with its canonical injection in X^{**}, whenever necessary, without using a specific notation for the image of elements of X in X^{**}. Analogously, we will denote the norm in X, X^* and X^{**} with the same standard symbol $\|\,.\,\|$ and the duality product in $X \times X^*$ and in $X^{**} \times X^*$ with $\langle .,. \rangle$, or, in function notation, as

$$\pi : X \times X^* \to \mathbb{R}, \quad (x, x^*) \mapsto \pi(x, x^*) = \langle x, x^* \rangle$$

and analogously for $\pi : X^{**} \times X^* \to \mathbb{R}$.
For any subset A of X (or X^*, X^{**}), the notation $\operatorname{cl} A$ will stand for the norm closure of A in X (X^*, X^{**}, respectively).
Recall that an operator, or multifunction, $T : X \rightrightarrows X^*$ is a point-to-set function and it is univocally determined by its graph

$$\mathcal{G}(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}.$$

The inverse operator of T is then defined as the multifunction $T^{-1} : X^* \rightrightarrows X^{**}$ having graph

$$\mathcal{G}(T^{-1}) = \{(x^*, x) \in X^* \times X^{**} : (x, x^*) \in \mathcal{G}(T)\} = \mathcal{G}(T)^{\top}.$$

A monotone operator is a multifunction such that

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \mathcal{G}(T).$$

A monotone operator T is maximal if it doesn't exist any monotone operator S whose graph strictly contains the graph of T. A point $(x, x^*) \in X \times X^*$ is monotonically related to the points of a given set $A \subseteq X \times X^*$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in A.$$

Thus a monotone operator $T : X \rightrightarrows X^*$ is maximal monotone if and only if any point in $X \times X^*$ which is monotonically related to $\mathcal{G}(T)$ belongs to $\mathcal{G}(T)$.
Since we will not assume the Banach space X to be reflexive (unless otherwise specified), it is natural to consider Gossez's extension (defined in [4]) of a maximal monotone operator $T : X \rightrightarrows X^*$ to the bidual, i.e. the operator $\widetilde{T} : X^{**} \rightrightarrows X^*$ whose graph contains all the points in $X^{**} \times X^*$ that are monotonically related to $\mathcal{G}(T)$

$$\mathcal{G}(\widetilde{T}) = \{(x^{**}, x^*) \in X^{**} \times X^* : \ \langle x^{**} - y, x^* - y^* \rangle \geq 0, \ \forall (y, y^*) \in \mathcal{G}(T)\}.$$

Remark 2.1. Note that, according to our previous definition of a (maximal) monotone operator, saying that an extension $S : X^{**} \rightrightarrows X^*$ of $T : X \rightrightarrows X^*$ is (maximal) monotone would imply, strictly speaking, that we consider it as an operator taking its values in X^{***}. However, since in this paper we are only interested in extending the space in which the domain of T is defined, when we say that S is (maximal) monotone, we mean (with an abuse of language meant to keep notation as simple as possible) that $S^{-1} : X^* \rightrightarrows X^{**}$ is (maximal) monotone.

In general a maximal monotone operator can have several different maximal monotone extensions to the bidual. While $\widetilde{T}$ is not necessarily monotone, if this is the case, then it is actually maximal monotone. It is worthwile considering the cases in which the maximal monotone extension to the bidual is unique. More specifically, we are interested in those operators whose unique maximal monotone extension coincides with $\widetilde{T}$.

Gossez [6], refining [4], introduced an important class of operators satisfying this property, that of maximal monotone operators of type (D) (see [19] for an exposition of the main results concerning this class). We give the definition introducing more notation, since we will need it in the following. Namely, given any operator $T : X \rightrightarrows X^*$, we will denote by $\overline{T} : X^{**} \rightrightarrows X^*$ the operator whose graph is the set of points $(x^{**}, x^*) \in X^{**} \times X^*$ for which there exists a bounded net (x_α, x^*_α) in $\mathcal{G}(T)$ such that $x_\alpha \to x^{**}$ in the $\sigma(X^{**}, X^*)$ topology of X^{**} and $x^*_\alpha \to x^*$ in the norm topology of X^*. It is easy to verify that, for any monotone operator T, $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widetilde{T})$.

Definition 2.2. Let X be a Banach space. An operator $T : X \rightrightarrows X^*$ is of type (D) if $\overline{T} = \widetilde{T}$.

Simons [21], in an attempt at defining a broader class of operators still keeping the nice properties of those of type (D), introduced maximal monotone operators of type (NI).

Definition 2.3. Let X be a Banach space. An operator $T : X \rightrightarrows X^*$ is of type (NI) if, for all $(x^{**}, x^*) \in X^{**} \times X^*$,

$$\inf_{(y,y^*)\in\mathcal{G}(T)} \langle x^{**} - y, x^* - y^* \rangle \le 0.$$

Marques Alves and Svaiter [9, 10, 11, 13] proved several useful properties of maximal monotone operators of type (NI).

Theorem 2.4 ([10, Theorem 1.1]). *Let X be a Banach space and $T : X \rightrightarrows X^*$ a maximal monotone operator of type* (NI)*, which is equivalent to*

$$(\sigma_T)^*(x^*, x^{**}) \ge \langle x^{**}, x^* \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Then

(a) $\widetilde{T} : X^{**} \rightrightarrows X^*$ *is the unique maximal monotone extension of T to the bidual;*
(b) $(\sigma_T)^{*\top} = \varphi_{\widetilde{T}}$*;*
(c) *for all $h \in \mathcal{H}_T$,*

$$h^*(x^*, x^{**}) \ge \langle x^{**}, x^* \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**},$$
$$h^{*\top} \in \mathcal{H}_{\widetilde{T}};$$

(d) *T satisfies the strict Brønsted-Rockafellar property.*

Note that, as is well known, when X is a reflexive Banach space, for every $h \in \mathcal{H}_T$ we have $h^{*\top} \in \mathcal{H}_T$ and $(h^{*\top})^{*\top} = h$.

An immediate consequence of statement (b) above is that any maximal monotone operator T of type (D) satisfies

$$\varphi_{\widetilde{T}}|_{X\times X^*} = \varphi_T; \tag{1}$$

indeed, from the definition of φ_T it follows that $((\sigma_T)^{*\top})|_{X\times X^*} = \varphi_T$.
Moreover, recall that the strict Brønsted-Rockafellar property is defined in [9] as follows.

Definition 2.5. Let X be a Banach space and $T : X \rightrightarrows X^*$ an operator. We say that T satisfies the strict Brønsted-Rockafellar property if, for all η, ε such that $0 < \varepsilon < \eta$ and for all $(x, x^*) \in X \times X^*$, if

$$\inf_{(y,y^*)\in\mathcal{G}(T)} \langle x - y, x^* - y^*\rangle \geq -\varepsilon,$$

then, for any $\lambda > 0$ there exists $(x_\lambda, x^*_\lambda) \in \mathcal{G}(T)$ such that

$$\|x - x_\lambda\| < \lambda, \qquad \|x^* - x^*_\lambda\| < \frac{\eta}{\lambda}.$$

Obviously an operator T satisfies the strict Brønsted-Rockafellar property if and only if it is of type (BR), in the sense of Simons [24, Definition 36.13].
As already observed in [21], any maximal monotone operator of type (D) is of type (NI).
The fact that the converse holds as well has been proved recently by Marques Alves and Svaiter [12]. Thus, taking into account Theorem 2.4, the following implications between classes of maximal monotone operators hold:

$$(D) \Longleftrightarrow (NI) \Longrightarrow (BR).$$

If X is reflexive, every maximal monotone operator $T : X \rightrightarrows X^*$ is of type (D). Even when X is not reflexive, a fundamental example of maximal monotone operators of type (D) is given by subdifferentials of lower semicontinuous proper convex functions (see [4]).
Recall that, given a lower semicontinuous proper convex function $f : X \to \mathbb{R}\cup\{+\infty\}$, the subdifferential of f is the operator $\partial f : X \rightrightarrows X^*$ with domain $D(\partial f) \subseteq \operatorname{dom} f$ and such that, for all $x \in D(\partial f)$, it holds $x^* \in \partial f(x)$ if and only if

$$f(y) \geq f(x) + \langle y - x, x^*\rangle \ \ \forall y \in X.$$

By Fenchel-Young inequality, which states that

$$f(z) + f^*(z^*) \geq \langle z, z^*\rangle$$

for all $(z, z^*) \in X \times X^*$, the following characterization of $\mathcal{G}(\partial f)$ is obtained as well:

$$(x, x^*) \in \mathcal{G}(\partial f) \quad \Leftrightarrow \quad f(x) + f^*(x^*) = \langle x, x^*\rangle.$$

If $f = \frac{1}{2}\|\cdot\|^2$, where $\|\,.\,\|$ denotes the norm of X, then we call $J_X^{\|\cdot\|} := \partial f$ the duality mapping on X and we will denote it simply by J_X, or J, when no risk of confusion arises.
The case in which $f = \delta_K$, for some nonempty closed convex set $K \subseteq X$, is worthwhile considering as well. The maximal monotone operator $N_K = \partial\delta_K$ is called the normal cone operator to K and is given by

$$N_K(x) = \begin{cases} \{x^* \in X^* : \langle y - x, x^*\rangle \leq 0, \ \forall y \in K\}, & x \in K \\ \emptyset, & x \notin K. \end{cases}$$

Obviously, for any $x \in K$ its image $N_K(x)$ is a convex cone in X^*. Notice that, throughout the paper, we will consider cones as containing the origin, according to the following definition.

Definition 2.6. Let Y be a normed space and $K \subseteq Y$. We say that K is a cone if

$$\forall k \in K, \forall \lambda \geq 0 : \quad \lambda k \in K.$$

For any closed convex cone K we will denote by B_K its barrier cone, i.e. the domain of the support function δ_K^*.
It is useful to consider the approximate subdifferential, or $\varepsilon-$subdifferential, for some $\varepsilon > 0$, as well, i.e. the operator

$$\begin{aligned} \partial_\varepsilon f : \ & X \rightrightarrows X^* \\ & x \mapsto \{x^* \in X^* : f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \varepsilon\}, \end{aligned}$$

or, equivalently,

$$x \mapsto \{x^* \in X^* : f(y) \geq f(x) + \langle y - x, x^* \rangle - \varepsilon, \ \forall y \in X\}.$$

In the case $\varepsilon = 0$, one has $\partial_0 f = \partial f$, by definition.
Thus, the $\varepsilon-$subdifferential corresponding to the duality mapping will be the operator

$$\begin{aligned} J_\varepsilon : \ & X \rightrightarrows X^* \\ & x \mapsto \{x^* \in X^* : \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 \leq \langle x, x^* \rangle + \varepsilon\}. \end{aligned}$$

Similar enlargements can be considered for arbitrary maximal monotone operators, as in [2]. Given a maximal monotone operator $T : X \rightrightarrows X^*$, for any $\varepsilon > 0$, we will consider the $\varepsilon-$enlargement of T defined by the multifunction $T^\varepsilon : X \rightrightarrows X^*$ with graph

$$\begin{aligned} \mathcal{G}(T^\varepsilon) &= \{(x, x^*) \in X \times X^* : \varphi_T(x, x^*) \leq \langle x, x^* \rangle + \varepsilon\} \\ &= \{(x, x^*) \in X \times X^* : \langle x - y, x^* - y^* \rangle \geq -\varepsilon, \ \forall (y, y^*) \in \mathcal{G}(T)\}. \end{aligned}$$

To complete this brief review of the notation of Convex Analysis, we record two simple but useful results.

Lemma 2.7. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function. Then $\mathcal{G}(\widetilde{\partial f}) = \mathcal{G}(\partial f^*)^\top$.*

Proof. Let $(y^{**}, y^*) \in \mathcal{G}(\widetilde{\partial f})$. Since ∂f is a maximal monotone operator of type (D) and $f \oplus f^* \in \mathcal{H}_{\partial f}$, then, by Theorem 2.4 (c), $f^{**} \oplus f^* = (f \oplus f^*)^{*\top} \in \mathcal{H}_{\widetilde{\partial f}}$, yielding

$$f^{**}(y^{**}) + f^*(y^*) = \langle y^{**}, y^* \rangle,$$

which in turn is satisfied if and only if $(y^*, y^{**}) \in \mathcal{G}(\partial f^*)$, since $f^* \oplus f^{**} \in \mathcal{H}_{\partial f^*}$. $\square$

Lemma 2.8. *Let X be a Banach space, $\alpha > 0$ and $|\cdot| : X \to \mathbb{R}$ be the norm on X defined by $|\cdot| = \alpha \|\cdot\|$. Then, for all $\varepsilon \geq 0$,*

$$\left(J_X^{|\cdot|}\right)_\varepsilon = \alpha^2 \left(J_X^{\|\cdot\|}\right)_{\varepsilon/\alpha^2}.$$

Proof. Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be defined by $g(x) = 1/2|x|^2$ for all $x \in X$, so that $(J_X^{|\cdot|})_\varepsilon = \partial_\varepsilon g$.
For all $\varepsilon \geq 0$ and $(x, x^*) \in X \times X^*$, the inclusion $x^* \in (J_X^{|\cdot|})_\varepsilon(x)$ is equivalent to $g(y) \geq g(x) + \langle y - x, x^* \rangle - \varepsilon$ for all $y \in X$, i.e.

$$\frac{1}{2}\alpha^2\|y\|^2 \geq \frac{1}{2}\alpha^2\|x\|^2 + \langle y - x, x^* \rangle - \varepsilon$$

and, dividing both sides by α^2,

$$\frac{1}{2}\|y\|^2 \geq \frac{1}{2}\|x\|^2 + \left\langle y - x, \frac{1}{\alpha^2}x^* \right\rangle - \frac{\varepsilon}{\alpha^2},$$

which is in turn equivalent to $x^* \in \alpha^2 (J_X^{\|\cdot\|})_{\varepsilon/\alpha^2}(x)$.
Thus $(J_X^{|\cdot|})_\varepsilon = \alpha^2 (J_X^{\|\cdot\|})_{\varepsilon/\alpha^2}$. □

We collect now two important theorems of [24] that will be crucial to prove the results in the following sections. First, we adopt the terminology of [24], as specified in the definition below.

Definition 2.9. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions. We call $z^* \in X^*$ a Fenchel functional for f and g if

$$f^*(z^*) + g^*(-z^*) \leq 0.$$

Theorem 2.10 ([24, Theorem 7.4]). *Let E be a normed space with dual E^*, and $f, g : E \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions.*
Then:

(*a*) *f and g have a Fenchel functional if, and only if, there exists $M \geq 0$ such that, for all $x, y \in E$,*
$$f(x) + g(y) + M\|x - y\| \geq 0.$$

(*b*) *if $z^* \in E^*$ is a Fenchel functional for f and g, then*

$$\sup_{x,y \in E,\ x \neq y} \frac{-f(x) - g(y)}{\|x - y\|} \leq \|z^*\|.$$

(*c*) *if $f + g \geq 0$ on E and*

$$\sup_{x,y \in E,\ x \neq y} \frac{-f(x) - g(y)}{\|x - y\|} < +\infty$$

then

$$\min\{\|z^*\| : z^* \textit{ is a Fenchel functional for } f \textit{ and } g\}$$
$$= \sup_{x,y \in E,\ x \neq y} \frac{-f(x) - g(y)}{\|x - y\|} \vee 0.$$

If $E = \{0\}$, the conditions on the supremum in (b) and (c) hold trivially with the usual convention $\sup \emptyset = -\infty$.

Theorem 2.11 ([24, Theorem 15.1]). *Let E be a Banach space, $f, g : E \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous proper convex functions,*

$$F := \bigcup_{\lambda > 0} \lambda[\operatorname{dom} f - \operatorname{dom} g] \quad \textit{be a closed subspace of } E$$

and

$$f + g \geq 0 \quad \textit{on } E.$$

Then there exists a Fenchel functional for f and g.

As a consequence of the previous theorem, one can obtain Attouch-Brézis theorem:

Theorem 2.12 ([24, Remark 15.2]). *Let E be a Banach space, $f, g : E \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous proper convex functions and*

$$\bigcup_{\lambda > 0} \lambda[\operatorname{dom} f - \operatorname{dom} g] \quad \textit{be a closed subspace of } E.$$

Then, for all $x^* \in E^*$,

$$(f + g)^*(x^*) = \min_{z^* \in E^*} \{f^*(x^* - z^*) + g^*(z^*)\}.$$

Taking $x^* = 0_{X^*}$ in the previous theorem, one obtains

$$\inf_{x \in E} (f + g)(x) = \max_{z^* \in E^*} \{-f^*(-z^*) - g^*(z^*)\}.$$

Lemma 2.13 ([13, Lemma 3.5]). *Let X be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type* (D)*, $h \in \mathcal{H}_S$, $k \in \mathcal{H}_T$ and define*

$$\begin{aligned} &H : X \times X^* \to \overline{\mathbb{R}} \\ &H(x, x^*) = (h(x, \cdot) \Box k(x, \cdot))(x^*) = \inf_{y^* \in X^*} \{h(x, y^*) + k(x, x^* - y^*)\}. \end{aligned}$$

If

$$\bigcup_{\lambda > 0} \lambda[p_1 \operatorname{dom} h - p_1 \operatorname{dom} k]$$

is a closed subspace of X, then $S + T$ is maximal monotone of type (D) *and* $\operatorname{cl} H \in \mathcal{H}_{S+T}$.

Remark 2.14. In the following sections, we will frequently deal with translations of maximal monotone operators. To this respect, it can be useful to note that, given a maximal monotone operator $T : X \rightrightarrows X^*$, for all $(w, w^*) \in X \times X^*$,

$$\mathcal{G}(\tau_{-w^*} \circ T \circ \tau_w) = \mathcal{G}(T) - (w, w^*)$$

and an order preserving bijection between $\mathcal{H}_T$ and $\mathcal{H}_{\tau_{-w^*}\circ T\circ\tau_w}$ can be established as in [15], by means of the operator $\mathcal{T}_{(w,w^*)} : \mathcal{H}_T \to \mathcal{H}_{\tau_{-w^*}\circ T\circ\tau_w}$, such that $(\mathcal{T}_{(w,w^*)}h)(x,x^*) = h(x+w,x^*+w^*) - (\langle x,w^*\rangle + \langle w,x^*\rangle + \langle w,w^*\rangle)$ for any $h \in \mathcal{H}_T$, $(x,x^*) \in X \times X^*$. Therefore, it is equivalent to consider a convex representation of $\tau_{-w^*}\circ T\circ\tau_w$, or a convex representation of T to which apply the bijection $\mathcal{T}_{(w,w^*)}$. Though we will usually work with the first representation, the equivalence of the two will sometimes be used.
Note that the translation of a maximal monotone operator of type (D) is still maximal monotone of type (D) and that Gossez's extension of $\tau_{-w^*}\circ T\circ\tau_w$ coincides with $\tau_{-w^*}\circ \widetilde{T}\circ\tau_w$.

We will also be interested in the effects of the composition of elements of $\mathcal{H}_T$ with reflections in the first or in the second component of points of $X \times X^*$. Such compositions will be essential for the duality proofs to work with elements of the Fitzpatrick family. The following lemma will then be useful.

Lemma 2.15. *Let $f : X \times X^* \to \mathbb{R}\cup\{+\infty\}$ be a proper convex function. Then, for all $(x^{**},x^*) \in X^{**}\times X^*$, it holds $(f\circ\varrho_1)^*(x^*,x^{**}) = (f^*\circ\varrho_1)(x^*,x^{**}) = f^*(-x^*,x^{**})$ and $(f\circ\varrho_2)^*(x^*,x^{**}) = (f^*\circ\varrho_2)(x^*,x^{**}) = f^*(x^*,-x^{**})$.*

Proof. We will prove the first relation. The other follows similarly.

$$\begin{aligned}(f\circ\varrho_1)^*(x^*,x^{**}) &= \sup_{(y,y^*)\in X\times X^*}\{\langle x^{**},y^*\rangle + \langle y,x^*\rangle - (f\circ\varrho_1)(y,y^*)\}\\ &= \sup_{(y,y^*)\in X\times X^*}\{\langle x^{**},y^*\rangle + \langle y,x^*\rangle - f(-y,y^*)\}\\ &= \sup_{(y,y^*)\in X\times X^*}\{\langle x^{**},y^*\rangle + \langle -y,-x^*\rangle - f(-y,y^*)\}\\ &= \sup_{(y,y^*)\in X\times X^*}\{\langle x^{**},y^*\rangle + \langle y,-x^*\rangle - f(y,y^*)\}\\ &= f^*(-x^*,x^{**}).\end{aligned}$$

□

The following three simple algebraic lemmas will help us to manipulate Attouch-Brézis type conditions.

Lemma 2.16. *Let Y be a normed space, $A, B \subseteq Y$ and*

$$\bigcup_{\lambda>0}\lambda[A-B]$$

be closed in Y.
Then

$$\bigcup_{\lambda>0}\lambda[A-B] = \bigcup_{\lambda>0}\lambda[\operatorname{cl} A - B].$$

Proof. The inclusion $\subseteq$ follows from $A \subseteq \operatorname{cl} A$, implying $A - B \subseteq \operatorname{cl} A - B$.
In order to prove the opposite inclusion, let $p \in \bigcup_{\lambda>0}\lambda[\operatorname{cl} A - B]$ (the inclusion is

obvious if either A or B are empty). Then there exist $\mu > 0$, $x \in \operatorname{cl} A$ and $y \in B$ such that $p = \mu(x - y)$. Moreover, since $x \in \operatorname{cl} A$, there exists a sequence (x_n) in A such that $x_n \to x$. Therefore

$$p = \mu(x - y) = \mu\left(\lim_n x_n - y\right) = \lim_n [\mu(x_n - y)] \in \bigcup_{\lambda > 0} \lambda[A - B]$$

because, for every $n \in \mathbb{N}$, $\mu(x_n - y) \in \bigcup_{\lambda>0} \lambda[A - B]$, which is a closed set by hypothesis. □

Lemma 2.17. *Let Y and Z be normed spaces, $A \subseteq Y \times Z$ and $B \subseteq Y$. Let S be any subspace of Z containing $p_2 A$. Then*

$$A - (B \times S) = (p_1 A - B) \times S.$$

Proof. If either A or B are empty, the result is trivial.
If $A, B \neq \emptyset$, obviously, by definition of S, $A - (B \times S) \subseteq (p_1 A - B) \times S$.
Let now $w \in (p_1 A - B) \times S$. Then there exist $(a_1, a_2) \in A$, $b \in B$ and $c \in S$ such that $w = (a_1 - b, c)$. Therefore, letting $d := a_2 - c \in S$,

$$w = (a_1 - b, c) = (a_1 - b, a_2 - d) = (a_1, a_2) - (b, d) \in A - (B \times S).$$

□

Lemma 2.18. *Let Y and Z be normed spaces, $B \subseteq Y$, $C \subseteq Z$ and*

$$L := \bigcup_{\lambda>0} \lambda(B \times C), \qquad M := \bigcup_{\lambda>0} \lambda B, \qquad N := \bigcup_{\lambda>0} \lambda C.$$

Then:

(a) *if L is a closed subspace of $Y \times Z$, then M and N are closed subspaces of Y and Z, respectively;*

(b) *if C is a cone, then $L = M \times C$; in particular, if M and C are closed subspaces of Y and Z, respectively, then L is a closed subspace of $Y \times Z$. Analogously, if B is a cone, then $L = B \times N$; if B and N are closed subspaces of Y and Z, respectively, then L is a closed subspace of $Y \times Z$.*

Proof. (a) Since L is a subspace of $Y \times Z$, then $(0_Y, 0_Z) \in L$, so that $0_Y \in B$ and $0_Z \in C$. Let $\lambda y_1, \mu y_2 \in M$, where $\lambda, \mu > 0$ and $y_1, y_2 \in B$. Then, for all $\alpha, \beta \in \mathbb{R}$, since $0_Z \in C$ and L is a subspace, there exist $\tau > 0$ and $y \in B$ such that

$$\begin{aligned}\alpha(\lambda y_1) + \beta(\mu y_2) &= \alpha p_1(\lambda y_1, 0_Z) + \beta p_1(\mu y_2, 0_Z) \\ &= p_1(\alpha\lambda(y_1, 0_Z) + \beta\mu(y_2, 0_Z)) = p_1(\tau y, 0_Z) = \tau y.\end{aligned}$$

Therefore M is a subspace of Y. Moreover, if $(\lambda_n y_n)$ is a sequence in M ($\lambda_n > 0$ and $y_n \in B$) converging to a given $x \in Y$, being L closed, there exist $\varrho > 0$, $y \in B$ such that

$$(x, 0_Z) = \lim_n (\lambda_n y_n, 0_Z) = \varrho(y, 0_Z) \in L,$$

which yields $x = \varrho y \in M$. Thus M is closed in Y.
In a similar way it can be proved that N is a closed subspace of Z.

(b) If B or C are empty, the result is trivial. Thus, suppose that $B, C \neq \emptyset$ and, for instance, that C is a cone (if B is a cone, the proof is similar). Obviously $L \subseteq M \times N = M \times C$. On the other hand, given $x \in M$, $z \in C$, there exist $\lambda > 0$, $y \in B$ such that $x = \lambda y$, so that

$$(x, z) = (\lambda y, z) = \lambda \left(y, \frac{1}{\lambda} z \right) \in L,$$

since C is a cone. In particular, if M is a closed subspace of Y and C is a closed subspace of Z, then L is a closed subspace of $Y \times Z$, since it is the cartesian product of two closed subspaces. □

3. The sum of the graphs

We begin with a lemma which takes on much of the burden needed to prove Theorem 3.2. Though, the purpose of stating a lemma on its own doesn't restrict to issues of ease, but it also enables us to underline the fact that the points $(x^{**}, x^*) \in X^{**} \times X^*$ satisfying the properties that are listed below are the same for any couple of representations $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, a fact which is not stressed in the statement of Theorem 3.2.

Lemma 3.1. *Let X be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type* (D)*, $(x^{**}, x^*) \in X^{**} \times X^*$ and $(u, u^*), (v, v^*) \in X \times X^*$. Then the following facts are equivalent:*

(a) $(u + x^{**}, u^* + x^*) \in \mathcal{G}(\widetilde{S})$ *and* $(v - x^{**}, x^* - v^*) \in \mathcal{G}(\widetilde{T})$;

(b) *for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, the point (x^*, x^{**}) is a Fenchel functional for h and $k \circ \varrho_1$;*

(c) *there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that (x^*, x^{**}) is a Fenchel functional for h and $k \circ \varrho_1$.*

If X is reflexive, the previous statements are also equivalent to:

(d) *for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$,*

$$(h + k \circ \varrho_1)(x^{**}, x^*) = 0; \tag{2}$$

(e) *there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that* (2) *holds.*

Proof. (a) $\Longrightarrow$ (b) By hypothesis,

$$(x^{**} + u, x^* + u^*) \in \mathcal{G}(\widetilde{S}) \quad \text{and} \quad (v - x^{**}, x^* - v^*) \in \mathcal{G}(\widetilde{T}),$$

that is to say

$$(x^{**}, x^*) \in \mathcal{G}(\tau_{-u^*} \circ \widetilde{S} \circ \tau_u) \quad \text{and} \quad (-x^{**}, x^*) \in \mathcal{G}(\tau_{v^*} \circ \widetilde{T} \circ \tau_v). \tag{3}$$

Let $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$. By Theorem 2.4,

$$h^{*\top} \in \mathcal{H}_{\tau_{-u^*} \circ \widetilde{S} \circ \tau_u} \quad \text{and} \quad k^{*\top} \in \mathcal{H}_{\tau_{v^*} \circ \widetilde{T} \circ \tau_v}.$$

Thus, by (3), we have

$$h^{*\top}(x^{**}, x^*) = \langle x^{**}, x^* \rangle \quad \text{and} \quad k^{*\top}(-x^{**}, x^*) = \langle -x^{**}, x^* \rangle,$$

which entails, by Lemma 2.15,

$$\begin{aligned} h^*(x^*, x^{**}) + (k \circ \varrho_1)^*(-x^*, -x^{**}) &= h^*(x^*, x^{**}) + k^*(x^*, -x^{**}) \\ &= h^{*\top}(x^{**}, x^*) + k^{*\top}(-x^{**}, x^*) \\ &= \langle x^{**}, x^* \rangle + \langle -x^{**}, x^* \rangle = 0, \end{aligned}$$

i.e. (x^*, x^{**}) is a Fenchel functional for h and $k \circ \varrho_1$.

$(b) \Longrightarrow (c)$ Obvious.

$(c) \Longrightarrow (a)$ Suppose we are given $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that (x^*, x^{**}) is a Fenchel functional for h and $k \circ \varrho_1$. Therefore

$$h^*(x^*, x^{**}) + (k \circ \varrho_1)^*(-x^*, -x^{**}) \leq 0.$$

On the other hand, by Lemma 2.15 and Theorem 2.4, we obtain the opposite inequality as well, i.e.

$$\begin{aligned} h^*(x^*, x^{**}) + (k \circ \varrho_1)^*(-x^*, -x^{**}) &= h^{*\top}(x^{**}, x^*) + k^{*\top}(-x^{**}, x^*) \\ &\geq \langle x^{**}, x^* \rangle + \langle -x^{**}, x^* \rangle = 0. \end{aligned}$$

Thus, since

$$h^{*\top}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle \quad \text{and} \quad k^{*\top}(-x^{**}, x^*) \geq \langle -x^{**}, x^* \rangle,$$

then

$$h^{*\top}(x^{**}, x^*) = \langle x^{**}, x^* \rangle \quad \text{and} \quad k^{*\top}(-x^{**}, x^*) = \langle -x^{**}, x^* \rangle.$$

Hence, by the maximality of the operators $\tau_{-u^*} \circ \widetilde{S} \circ \tau_u$ and $\tau_{v^*} \circ \widetilde{T} \circ \tau_v$,

$$(x^{**}, x^*) \in \mathcal{G}(\tau_{-u^*} \circ \widetilde{S} \circ \tau_u) = \mathcal{G}(\widetilde{S}) - (u, u^*)$$

and

$$(-x^{**}, x^*) \in \mathcal{G}(\tau_{v^*} \circ \widetilde{T} \circ \tau_v) = \mathcal{G}(\widetilde{T}) - (v, -v^*),$$

yielding (a).

Suppose now that X is reflexive.

$(b) \Longrightarrow (d)$ Let $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$. We also have $h^{*\top} \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k^{*\top} \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$. Since (b) holds, the point (x^*, x^{**}) is a Fenchel functional for $h^{*\top}$ and $k^{*\top} \circ \varrho_1$, so that

$$\begin{aligned} 0 = \langle x^{**}, x^* \rangle + \langle -x^{**}, x^* \rangle &\leq (h + k \circ \varrho_1)(x^{**}, x^*) \\ &= (h^{*\top})^*(x^*, x^{**}) + (k^{*\top} \circ \varrho_1)^*(-x^*, -x^{**}) \leq 0, \end{aligned}$$

implying (2).

$(d) \Longrightarrow (e)$ Obvious.

$(e) \Longrightarrow (c)$ By (e), there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that

$$0 = (h + k \circ \varrho_1)(x^{**}, x^*) = (h^{*\top})^*(x^*, x^{**}) + (k^{*\top} \circ \varrho_1)^*(-x^*, -x^{**}).$$

Thus (x^*, x^{**}) is a Fenchel functional for $h^{*\top}$ and $k^{*\top} \circ \varrho_1$, where $h^{*\top} \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k^{*\top} \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$. □

As already announced, Lemma 3.1 makes the proof of Theorem 3.2 immediate. This theorem, along with its version for the range (Theorem 4.2), can be regarded as the basis of the paper, since it provides a punctual characterization of the set $\mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$ in terms of Fenchel functionals of arbitrary convex representations of $\widetilde{S}$ and $\widetilde{T}$ or, equivalently, of their translations. Upon this duality characterization eventually hinge all the results that follow.

Note that condition (4) in Theorem 3.2 below and the analogous conditions in the results that follow are simply the necessary and suffcient condition for the existence of Fenchel functionals given by Simons (see Theorem 2.10 above). Although they are not new results, we include them in our statements for the sake of completeness, in order to give a thorough understanding of the correspondences involved.

Theorem 3.2. *Let X be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type* (D) *and $(u, u^*), (v, v^*), (w, w^*) \in X \times X^*$ such that $u + v = w$ and $u^* + v^* = w^*$. The following statements are equivalent:*

- (a) $(w, w^*) \in \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$;
- (b) *there exists $(x^{**}, x^*) \in X^{**} \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, the point (x^*, x^{**}) is a Fenchel functional for h and $k \circ \varrho_1$;*
- (c) *there exist $(x^{**}, x^*) \in X^{**} \times X^*$, $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that the point (x^*, x^{**}) is a Fenchel functional for h and $k \circ \varrho_1$;*
- (d)

$$\inf_{\substack{(y,y^*) \in \operatorname{dom} \varphi_{\tau_{-u^*} \circ S \circ \tau_u} \\ (z,z^*) \in \varrho_1(\operatorname{dom} \varphi_{\tau_{v^*} \circ T \circ \tau_v}) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_{\tau_{-u^*} \circ S \circ \tau_u}(y, y^*) + (\varphi_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(z, z^*)}{\|(y, y^*) - (z, z^*)\|} > -\infty; \tag{4}$$

- (e) *relation (4) holds with $\varphi_{\tau_{-u^*} \circ S \circ \tau_u}$ and $\varphi_{\tau_{v^*} \circ T \circ \tau_v}$ replaced by $\sigma_{\tau_{-u^*} \circ S \circ \tau_u}$ and $\sigma_{\tau_{v^*} \circ T \circ \tau_v}$, respectively.*

Moreover, if X is reflexive, the previous items are also equivalent to:

- (f) *there exists $(x, x^*) \in X \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$,*

$$(h + k \circ \varrho_1)(x, x^*) = 0; \tag{5}$$

- (g) *there exist $(x, x^*) \in X \times X^*$, $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that (5) holds.*

A sufficient condition for (a)–(e) to hold is the existence of $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that

$$\bigcup_{\lambda > 0} \lambda[\operatorname{dom} h - \varrho_1(\operatorname{dom} k) - (w, w^*)] \quad \textit{is a closed subspace of } X \times X^*. \tag{6}$$

Proof. The equivalence $(a) \iff (b) \iff (c)$ ($\iff (f) \iff (g)$, if X is reflexive) is an immediate consequence of Lemma 3.1.

$(b) \implies (d)$ It follows from assertion (b) of Theorem 2.10, since

$$\begin{aligned}
&\inf_{\substack{(y,y^*)\in \operatorname{dom}\varphi_{\tau_{-u^*}\circ S\circ\tau_u}\\ (z,z^*)\in\varrho_1(\operatorname{dom}\varphi_{\tau_{v^*}\circ T\circ\tau_v})\\ (y,y^*)\neq(z,z^*)}} \frac{\varphi_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\varphi_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|}\\
&= -\sup_{\substack{(y,y^*),(z,z^*)\in X\times X^*\\ (y,y^*)\neq(z,z^*)}} \frac{-\varphi_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)-(\varphi_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|}\\
&\geq -\|(x^*,x^{**})\| > -\infty.
\end{aligned} \tag{7}$$

$(d) \implies (e)$ Obvious.

$(e) \implies (c)$ It follows from assertion (a) of Theorem 2.10. Indeed, by setting

$$M := \sup_{\substack{(y,y^*)\in \operatorname{dom}\sigma_{\tau_{-u^*}\circ S\circ\tau_u}\\ (z,z^*)\in\varrho_1(\operatorname{dom}\sigma_{\tau_{v^*}\circ T\circ\tau_v})\\ (y,y^*)\neq(z,z^*)}} \frac{-\sigma_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)-(\sigma_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|}\vee 0$$

and recalling (7), we have $0\leq M<+\infty$ and $\sigma_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\sigma_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)+M\|(y,y^*)-(z,z^*)\|\geq 0$ for any $(y,y^*),(z,z^*)\in X\times X^*$ with $(y,y^*)\neq(z,z^*)$. On the other hand, when $(y,y^*)=(z,z^*)$,

$$\begin{aligned}
&\sigma_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\sigma_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(y,y^*)+M\|(y,y^*)-(y,y^*)\|\\
&=\sigma_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\sigma_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(y,y^*)\geq\langle y,y^*\rangle+\langle -y,y^*\rangle\\
&=0,\quad \forall (y,y^*)\in X\times X^*.
\end{aligned}$$

Finally, given $h\in\mathcal{H}_S$ and $k\in\mathcal{H}_T$ satisfying (6), we have $\mathcal{T}_{(u,u^*)}h\in\mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$, $\mathcal{T}_{(v,-v^*)}k\in\mathcal{H}_{\tau_{v^*}\circ T\circ\tau_v}$ and

$$\bigcup_{\lambda>0}\lambda[\operatorname{dom}h-\varrho_1(\operatorname{dom}k)-(w,w^*)]=\bigcup_{\lambda>0}\lambda[\operatorname{dom}\mathcal{T}_{(u,u^*)}h-\operatorname{dom}((\mathcal{T}_{(v,-v^*)}k)\circ\varrho_1)].$$

Moreover, $(\mathcal{T}_{(u,u^*)}h)(x,x^*)+((\mathcal{T}_{(v,-v^*)}k)\circ\varrho_1)(x,x^*)\geq\langle x,x^*\rangle+\langle -x,x^*\rangle=0$ for all $(x,x^*)\in X\times X^*$. Therefore, by Theorem 2.11, (c) is satisfied. Consequently, (a)–(e) hold. $\square$

Remark 3.3. (a) Note that assertion (d) can be restated by expressing the set over which the infimum in condition (4) is taken by means of the graphs of S and T instead of the domains of $\varphi_{\tau_{-u^*}\circ S\circ\tau_u}$ and $\varphi_{\tau_{v^*}\circ T\circ\tau_v}$, i.e.

$$\inf_{\substack{(y,y^*)\in\operatorname{cl}\operatorname{conv}\mathcal{G}(S)-(u,u^*)\\ (z,z^*)\in\varrho_1(\operatorname{cl}\operatorname{conv}\mathcal{G}(T))+(v,v^*)\\ (y,y^*)\neq(z,z^*)}} \frac{\varphi_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\varphi_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|}>-\infty. \tag{8}$$

Denote by (d') this new statement. Obviously (d) implies (d'). Vice versa, if (d') holds, since for any maximal monotone operator $A : X \rightrightarrows X^*$ we have $\operatorname{conv}\mathcal{G}(A) \subseteq \operatorname{dom}\sigma_A \subseteq \operatorname{cl}\operatorname{conv}\mathcal{G}(A)$, then

$$\begin{aligned}
&\inf_{\substack{(y,y^*)\in\operatorname{dom}\sigma_{\tau_{-u^*}\circ S\circ\tau_u}\\ (z,z^*)\in\varrho_1(\operatorname{dom}\sigma_{\tau_{v^*}\circ T\circ\tau_v})\\ (y,y^*)\neq(z,z^*)}} \frac{\sigma_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\sigma_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|}\\
=&\inf_{\substack{(y,y^*)\in\operatorname{cl}\operatorname{conv}\mathcal{G}(S)-(u,u^*)\\ (z,z^*)\in\varrho_1(\operatorname{cl}\operatorname{conv}\mathcal{G}(T))+(v,v^*)\\ (y,y^*)\neq(z,z^*)}} \frac{\sigma_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\sigma_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|}\\
\geq&\inf_{\substack{(y,y^*)\in\operatorname{cl}\operatorname{conv}\mathcal{G}(S)-(u,u^*)\\ (z,z^*)\in\varrho_1(\operatorname{cl}\operatorname{conv}\mathcal{G}(T))+(v,v^*)\\ (y,y^*)\neq(z,z^*)}} \frac{\varphi_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\varphi_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|} > -\infty,
\end{aligned}$$

so that (e) is verified and, consequently, (d) holds as well.

(b) The sufficient condition (6) in the case $h = \sigma_S$ and $k = \sigma_T$ can be stated analogously in terms of the graphs of S and T as

$$\bigcup_{\lambda>0}\lambda[\operatorname{conv}\mathcal{G}(S)-\varrho_1(\operatorname{conv}\mathcal{G}(T))-(w,w^*)] \text{ is a closed subspace of } X\times X^*. \tag{9}$$

Indeed, since for any $A \subseteq X \times X^*$, $\operatorname{conv}(\varrho_1 A) = \varrho_1(\operatorname{conv} A)$, $\operatorname{cl}\operatorname{conv}(\varrho_1 A) = \varrho_1(\operatorname{cl}\operatorname{conv} A)$ and $\operatorname{conv}[A-(w,w^*)] = \operatorname{conv} A - (w,w^*)$, we have

$$\begin{aligned}
&\bigcup_{\lambda>0}\lambda[\operatorname{conv}\mathcal{G}(S)-\varrho_1(\operatorname{conv}\mathcal{G}(T))-(w,w^*)]\\
\subseteq&\bigcup_{\lambda>0}\lambda[\operatorname{dom}\sigma_S-\operatorname{dom}(\sigma_T\circ\varrho_1)-(w,w^*)]\\
\subseteq&\bigcup_{\lambda>0}\lambda[\operatorname{cl}\operatorname{conv}\mathcal{G}(S)-\varrho_1(\operatorname{cl}\operatorname{conv}\mathcal{G}(T))-(w,w^*)].
\end{aligned}$$

Therefore, by Lemma 2.16, the sufficient condition (6) is satisfied with $h = \sigma_S$ and $k = \sigma_T$.

(c) Conditions (8) and (9) simplify whenever $\mathcal{G}(S)$ or $\mathcal{G}(T)$ are convex. By [10, Lemma 1.2] and [1, Theorem 4.2], this is the case if and only if S or T are translates of monotone linear relations.

(d) Since we will need it to prove Theorem 4.2, we observe that statement (a) of Theorem 3.2 could be formulated in a less concise way by saying that there exists $(x^{**}, x^*) \in X^{**}\times X^*$ such that $(u+x^{**}, u^*+x^*) \in \mathcal{G}(\widetilde{S})$ and $(v-x^{**}, x^*-v^*) \in \mathcal{G}(\widetilde{T})$.

The following corollary extends to the nonreflexive setting, for maximal monotone operators of type (D), the surjectivity property in its version related to the sum of the graphs introduced in [22] (note that, strictly speaking, this version could not be called a surjectivity property, for it deals with the graphs, not with the ranges; anyway, as we will see in the next section, it is in some sense equivalent to surjectivity). On the basis of this corollary we will provide two possible reformulations of [14, Theorem 2.1] (the main result of that paper) in the nonreflexive setting for maximal monotone operators of type (D) (see Remark 3.5 and Corollary 3.6 below).

Corollary 3.4. *Let X be a Banach space and $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type* (D)*. Then the following statements are equivalent:*

(*a*) $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$;

(*b*) *for all $(u, u^*), (v, v^*) \in X \times X^*$, there exists $(x^{**}, x^*) \in X^{**} \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, (x^*, x^{**}) is a Fenchel functional for h and $k \circ \varrho_1$;*

(*c*) *for all $(u, u^*), (v, v^*) \in X \times X^*$, there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that h and $k \circ \varrho_1$ have a Fenchel functional;*

(*d*) *for all $(u, u^*), (v, v^*) \in X \times X^*$,*

$$\inf_{\substack{(y,y^*) \in \operatorname{cl} \operatorname{conv} \mathcal{G}(S) - (u,u^*) \\ (z,z^*) \in \varrho_1(\operatorname{cl} \operatorname{conv} \mathcal{G}(T)) + (v,v^*) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_{\tau_{-u^*} \circ S \circ \tau_u}(y, y^*) + (\varphi_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(z, z^*)}{\|(y, y^*) - (z, z^*)\|} > -\infty; \quad (10)$$

(*e*) *for all $(u, u^*), (v, v^*) \in X \times X^*$, relation* (10) *holds with $\varphi_{\tau_{-u^*} \circ S \circ \tau_u}$ and $\varphi_{\tau_{v^*} \circ T \circ \tau_v}$ replaced by $\sigma_{\tau_{-u^*} \circ S \circ \tau_u}$ and $\sigma_{\tau_{v^*} \circ T \circ \tau_v}$, respectively.*

If X is reflexive, they are also equivalent to:

(*f*) *for all $(u, u^*), (v, v^*) \in X \times X^*$, there exists $(x, x^*) \in X \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, $(h + k \circ \varrho_1)(x, x^*) = 0$;*

(*g*) *for all $(u, u^*), (v, v^*) \in X \times X^*$, there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that $0 \in \operatorname{Im}(h + k \circ \varrho_1)$.*

Thus, if for all $(w, w^*) \in X \times X^*$ there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that

$$\bigcup_{\lambda > 0} \lambda[\operatorname{dom} h - \varrho_1(\operatorname{dom} k) - (w, w^*)] \quad \textit{is a closed subspace of } X \times X^*,$$

then $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$. In particular, this is true whenever

$$\operatorname{dom} \varphi_S - \varrho_1(\operatorname{dom} \varphi_T) = X \times X^*.$$

Proof. It follows from Theorem 3.2. □

Remark 3.5. As we anticipated, the previous corollary provides an immediate generalization of [14, Theorem 2.1], which reads as follows:
Let X be a Banach space and $S : X \rightrightarrows X^*$ be a monotone operator.

(a) If S is maximal monotone of type (D), then, for any maximal monotone operator $T : X \rightrightarrows X^*$ of type (D) such that $\operatorname{dom} \varphi_S - \varrho_1(\operatorname{dom} \varphi_T) = X \times X^*$, one has $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$.

(b) If there exist a multifunction $T : X \rightrightarrows X^*$, such that $\mathcal{G}(S) + \mathcal{G}(-T) = X \times X^*$, and a point $(p, p^*) \in X \times X^*$, such that $\langle p - y, p^* - y^* \rangle > 0$ for any $(y, y^*) \in \mathcal{G}(T) \backslash \{(p, p^*)\}$, then S is maximal monotone.

Assertion (a) is a consequence of Corollary 3.4 that extends implication $(a) \Longrightarrow (b)$ of [14, Theorem 2.1] to a nonreflexive setting, for operators of type (D), and substitutes a constraint on the sum of the domains of the Fitzpatrick functions for the original condition requiring the second of these domains to be the whole space $X \times X^*$. Statement (b) is instead a weakened version of implication $(c) \Longrightarrow (a)$ of the same theorem, taking into account that this implication already worked in a nonreflexive setting and that some hypotheses (namely, T being a maximal monotone operator having finite-valued Fitzpatrick function) can be dropped.

A closer similarity to the structure of [14, Theorem 2.1] can be obtained with a bit more involved version of statement (c) of that theorem. We prove this fact in the following corollary, where we denote by $\mathrm{cl}_{(w,n)}(A)$ the closure of a set $A \subseteq X \times X^*$ in the $\sigma(X, X^*) \otimes$ norm topology of $X \times X^*$.

Corollary 3.6. *Let X be a Banach space and $S : X \rightrightarrows X^*$ be a monotone operator of type* (D)*, whose graph is closed in the $\sigma(X, X^*) \otimes$ norm topology of $X \times X^*$. Then the following facts are equivalent:*

(a) *S is maximal;*

(b) *for every maximal monotone operator $T : X \rightrightarrows X^*$ of type* (D) *such that $\operatorname{dom}\varphi_S - \varrho_1(\operatorname{dom}\varphi_T) = X \times X^*$, one has $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$, so that, in particular, $\mathrm{cl}_{(w,n)}(\mathcal{G}(S) + \mathcal{G}(-T)) = X \times X^*$;*

(c) *there exist a monotone operator $T : X \rightrightarrows X^*$ of type* (D) *such that $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$ and a point $(p, p^*) \in \mathcal{G}(T)$ such that, for every net (x_α, x_α^*) in $\mathcal{G}(T)$, if $\lim_\alpha \langle p - x_\alpha, p^* - x_\alpha^* \rangle = 0$, then (x_α) converges to p in the $\sigma(X, X^*)$ topology of X and (x_α^*) converges to p^* in the norm topology of X^*.*

Proof. $(a) \Longrightarrow (b)$ It is a consequence of Corollary 3.4. To prove the density result, let $(w, w^*) \in X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$. Thus, there exist $(x^{**}, x^*) \in \mathcal{G}(\widetilde{S})$ and $(y^{**}, y^*) \in \mathcal{G}(\widetilde{T})$ such that $(x^{**}+y^{**}, x^*-y^*) = (w, w^*)$. Since S and T are both of type (D), there exist two nets $(x_\alpha, x_\alpha^*)_{\alpha \in A}$ in $\mathcal{G}(S)$ and $(y_\beta, y_\beta^*)_{\beta \in B}$ in $\mathcal{G}(T)$ converging to (x^{**}, x^*) and to (y^{**}, y^*), respectively, in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$. The net $(z_\gamma, z_\gamma^*)_{\gamma \in \Gamma}$, with $\Gamma = A \times B$ and $z_\gamma := x_\alpha + y_\beta$, $z_\gamma^* := x_\alpha^* - y_\beta^*$ for every $\gamma = (\alpha, \beta)$, converges then to (w, w^*) in the same topology of $X^{**} \times X^*$ and therefore in the $\sigma(X, X^*) \otimes$ norm topology of $X \times X^*$.

$(b) \Longrightarrow (c)$ The duality mapping $J : X \rightrightarrows X^*$ is maximal monotone of type (D) and $\operatorname{dom}\varphi_J = X \times X^*$. Therefore, by hypothesis, $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})$. Set then $(p, p^*) = (0_X, 0_{X^*})$ and consider a net (x_α, x_α^*) in $\mathcal{G}(J)$, such that $\lim_\alpha \langle x_\alpha, x_\alpha^* \rangle = 0$. By definition of J, this implies $\lim_\alpha \left(\frac{1}{2}\|x_\alpha\|^2 + \frac{1}{2}\|x_\alpha^*\|^2\right) = 0$. Thus (x_α, x_α^*) converges to $(0_X, 0_{X^*})$ in the norm topology of $X \times X^*$ and, consequently, in the $\sigma(X, X^*) \otimes$ norm topology.

$(c) \Longrightarrow (a)$ Let $(x, x^*) \in X \times X^*$ be monotonically related to every point in $\mathcal{G}(S)$. Since $(x + p, x^* - p^*) \in X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$, then there exist two bounded nets $(x_\alpha, x_\alpha^*) \in \mathcal{G}(S)$ and $(y_\beta, -y_\beta^*) \in \mathcal{G}(-T)$ converging in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$ and whose sum converges to $(x+p, x^*-p^*)$ in the same topology.

Hence $(x - x_\alpha, x^* - x^*_\alpha)$ and $(y_\beta - p, -y^*_\beta + p^*)$ have the same limit. Since (x, x^*) is monotonically related to $\mathcal{G}(S)$, then $\langle x - x_\alpha, x^* - x^*_\alpha\rangle \geq 0$ and, taking the limit, we obtain

$$\lim_\beta \langle y_\beta - p, -y^*_\beta + p^*\rangle = \lim_\alpha \langle x - x_\alpha, x^* - x^*_\alpha\rangle \geq 0,$$

which, taking into account the monotonicity of T, entails

$$\lim_\beta \langle y_\beta - p, y^*_\beta - p^*\rangle = 0.$$

Therefore, by the hypothesis on (p, p^*), we obtain that (y_β, y^*_β) converges to (p, p^*) in the $\sigma(X, X^*) \otimes$ norm topology of $X \times X^*$. As a consequence, (x_α, x^*_α) converges to (x, x^*) in the same topology. Thus, by the hypothesis that S has a closed graph in this topology, we conclude that (x, x^*) belongs to $\mathcal{G}(S)$. Therefore, being (x, x^*) an arbitrary point of $X \times X^*$ monotonically related to $\mathcal{G}(S)$, S is maximal monotone. □

Note that, in the previous proof, the hypothesis on the closure of the graph is needed only to prove the last implication.

A natural question to address at this point is whether the density property mentioned in statement (b) of the previous corollary can be strengthened, introducing the closure in the norm topology of the product space $X \times X^*$. The answer is in the positive, but to obtain it we have to use, along with Fenchel duality, the strict Brønsted-Rockafellar property of maximal monotone operators of type (D).

Theorem 3.7. *Let X be a Banach space and $S, T : X \rightrightarrows X^*$ be monotone operators of type* (D)*. If, for all $(w, w^*) \in X \times X^*$, there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_1(\operatorname{dom} k) - (w, w^*)] \quad \textit{is a closed subspace of } X \times X^*, \tag{11}$$

then:

(a) *for all $\varepsilon > 0$, $\mathcal{G}(S^\varepsilon) + \mathcal{G}(-T^\varepsilon) = X \times X^*$;*

(b) *if S and T are maximal monotone,* $\operatorname{cl}(\mathcal{G}(S) + \mathcal{G}(-T)) = X \times X^*$.

Proof. (a) Let $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ satisfy condition (11), $(w, w^*) \in X \times X^*$ and $\varepsilon > 0$. Then, by Remark 2.14, $\mathcal{T}_{(w,w^*)}h \in \mathcal{H}_{\tau_{-w^*} \circ S \circ \tau_w}$ and, by hypothesis,

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} \mathcal{T}_{(w,w^*)}h - \varrho_1(\operatorname{dom} k)]$$

is a closed subspace of $X \times X^*$. Therefore, as a consequence of Theorem 2.12, there exists $(z^*, z^{**}) \in X^* \times X^{**}$ such that

$$\begin{aligned} 0 \leq \inf_{X\times X^*}\{\mathcal{T}_{(w,w^*)}h + (k \circ \varrho_1)\} &= -(\mathcal{T}_{(w,w^*)}h)^*(z^*, z^{**}) - (k \circ \varrho_1)^*(-z^*, -z^{**}) \\ &= -\mathcal{T}_{(w^*,w)}h^*(z^*, z^{**}) - k^*(z^*, -z^{**}) \\ &\leq -\langle z^{**}, z^*\rangle - \langle -z^{**}, z^*\rangle = 0, \end{aligned} \tag{12}$$

where the property $(\mathcal{T}_{(w,w^*)}h)^* = \mathcal{T}_{(w^*,w)}h^*$ and Lemma 2.15 have been used.
Thus the infimum in (12) is equal to zero and, for all $\varepsilon > 0$, there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$\mathcal{T}_{(w,w^*)}h(x_\varepsilon, x_\varepsilon^*) + k(-x_\varepsilon, x_\varepsilon^*) \le \varepsilon,$$

yielding

$$\varphi_{\tau_{-w^*}\circ S\circ\tau_w}(x_\varepsilon, x_\varepsilon^*) \le \mathcal{T}_{(w,w^*)}h(x_\varepsilon, x_\varepsilon^*) \le -k(-x_\varepsilon, x_\varepsilon^*) + \varepsilon \le \langle x_\varepsilon, x_\varepsilon^*\rangle + \varepsilon,$$

$$\varphi_T(-x_\varepsilon, x_\varepsilon^*) \le k(-x_\varepsilon, x_\varepsilon^*) \le -\mathcal{T}_{(w,w^*)}h(x_\varepsilon, x_\varepsilon^*) + \varepsilon \le \langle -x_\varepsilon, x_\varepsilon^*\rangle + \varepsilon,$$

i.e. $(x_\varepsilon, x_\varepsilon^*) \in \mathcal{G}((\tau_{-w^*} \circ S \circ \tau_w)^\varepsilon) = \mathcal{G}(\tau_{-w^*} \circ S^\varepsilon \circ \tau_w)$ and $(-x_\varepsilon, x_\varepsilon^*) \in \mathcal{G}(T^\varepsilon)$. Hence

$$(w, w^*) = (w + x_\varepsilon, w^* + x_\varepsilon^*) + (-x_\varepsilon, -x_\varepsilon^*) \in \mathcal{G}(S^\varepsilon) + \mathcal{G}(-T^\varepsilon).$$

(b) Let $(w, w^*) \in X \times X^*$. It follows from (a) that, for all $n \in \mathbb{N}\backslash\{0\}$, there exists $(x_n, x_n^*) \in \mathcal{G}((\tau_{-w^*} \circ S \circ \tau_w)^{1/(9n^2)})$ such that $(-x_n, x_n^*) \in \mathcal{G}(T^{1/(9n^2)})$. By the strict Brønsted-Rockafellar property, there exist $(\overline{x}_n, \overline{x}_n^*) \in \mathcal{G}(\tau_{-w^*} \circ S \circ \tau_w)$ and $(-\overline{y}_n, \overline{y}_n^*) \in \mathcal{G}(T)$ such that

$$\|\overline{x}_n - x_n\| < \frac{1}{2\sqrt{2}n}, \quad \|\overline{x}_n^* - x_n^*\| < \frac{1}{2\sqrt{2}n}, \quad \|\overline{y}_n - x_n\| < \frac{1}{2\sqrt{2}n}, \quad \|\overline{y}_n^* - x_n^*\| < \frac{1}{2\sqrt{2}n},$$

implying

$$\|\overline{x}_n - \overline{y}_n\| < \frac{1}{\sqrt{2}n}, \quad \|\overline{x}_n^* - \overline{y}_n^*\| < \frac{1}{\sqrt{2}n}.$$

Therefore $(w + \overline{x}_n - \overline{y}_n, w^* + \overline{x}_n^* - \overline{y}_n^*)$ is a sequence in $\mathcal{G}(S) + \mathcal{G}(-T)$ such that

$$\|(w + \overline{x}_n - \overline{y}_n, w^* + \overline{x}_n^* - \overline{y}_n^*) - (w, w^*)\| = \left(\|\overline{x}_n - \overline{y}_n\|^2 + \|\overline{x}_n^* - \overline{y}_n^*\|^2\right)^{1/2} < \frac{1}{n},$$

i.e. $(w + \overline{x}_n - \overline{y}_n, w^* + \overline{x}_n^* - \overline{y}_n^*)$ converges in norm to (w, w^*). □

Corollary 3.4 yields a sort of "extended Brønsted-Rockafellar property" (in the sense that it involves the extension of the operator to the bidual), that we can state as follows.

Definition 3.8. Let $S : X \rightrightarrows X^*$ be an operator. We say that S satisfies the extended Brønsted-Rockafellar property if, for all $\lambda, \varepsilon > 0$ and $(x, x^*) \in S^\varepsilon$, there exists $(\overline{x}^{**}, \overline{x}^*) \in \overline{S}$ such that $\|x - \overline{x}^{**}\| \le \lambda$ and $\|x^* - \overline{x}^*\| \le \varepsilon/\lambda$.

Proposition 3.9. *Let X be a Banach space and $S : X \rightrightarrows X^*$ be a maximal monotone operator of type* (D). *Then S satisfies the extended Brønsted-Rockafellar property.*

Proof. Let $\lambda, \varepsilon > 0$ and define the norm $|\cdot| = \sqrt{\varepsilon}/\lambda\|\cdot\|$. By Lemma 2.8,

$$J' := J_X^{|\cdot|} = \frac{\varepsilon}{\lambda^2} J_X^{\|\cdot\|}.$$

In particular, $J' : X \rightrightarrows X^*$ is then a maximal monotone operator of type (D), with finite-valued Fitzpatrick function. Therefore, by Corollary 3.4, $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) +$

$\mathcal{G}(-\widetilde{J'})$. Hence, for any $(x, x^*) \in S^\varepsilon$, there exists $(\overline{x}^{**}, \overline{x}^*) \in \widetilde{S}$ such that $(x-\overline{x}^{**}, \overline{x}^* - x^*) \in \mathcal{G}(\widetilde{J'})$. Thus, by Lemma 2.7, item (b) of Theorem 2.4 and (1),

$$
\begin{aligned}
|x-\overline{x}^{**}|^2 = |\overline{x}^* - x^*|^2 &= -\langle x-\overline{x}^{**}, x^* - \overline{x}^*\rangle \\
&\leq \varphi_{\widetilde{S}}(x, x^*) - \langle x, x^*\rangle \\
&= \varphi_S(x, x^*) - \langle x, x^*\rangle \leq \varepsilon,
\end{aligned}
$$

i.e.

$$|x-\overline{x}^{**}| \leq \sqrt{\varepsilon} \quad \text{and} \quad |\overline{x}^* - x^*| \leq \sqrt{\varepsilon}.$$

Since the norm that makes X^* dual to $(X, |\cdot|)$ is $|\cdot| = \lambda/\sqrt{\varepsilon}\|\cdot\|$, this implies

$$\|x-\overline{x}^{**}\| \leq \lambda \quad \text{and} \quad \|\overline{x}^* - x^*\| \leq \frac{\varepsilon}{\lambda}.$$

□

Since in the reflexive case $\widetilde{S} = S$ and all maximal monotone operators are of type (D), then, in this setting, Proposition 3.9 yields Torralba's Theorem [26]. In order to recover the usual strict Brønsted-Rockafellar property, in the nonreflexive case one should invoke a different surjectivity result [13, Corollary 3.7], stating that, for a maximal monotone operator S of type (D), it holds $R(S(\cdot + w) + \mu J_\eta) = X^*$ for all $w \in X$, $\mu, \eta > 0$. This result, which for monotone operators with closed graph is equivalent to the property of S being maximal monotone of type (D), is obtained in [13] by means of the strict Brønsted-Rockafellar property. The opposite implication can be proved as well, as a consequence of the following statement (since it is easily verified that $R(S(\cdot + w) + \mu J_\eta) = X^*$ for all $w \in X$ and $\mu, \eta > 0$ is equivalent to $\mathcal{G}(S) + \mathcal{G}(-\mu J_\eta) = X \times X^*$ for all $\mu, \eta > 0$).

Proposition 3.10. *Let X be a Banach space and $S : X \rightrightarrows X^*$ be a monotone operator. If $\mathcal{G}(S) + \mathcal{G}(-\mu J_\eta) = X \times X^*$ for all $\mu, \eta > 0$, then S satisfies the strict Brønsted-Rockafellar property.*

Proof. Let $\lambda, \varepsilon, \widetilde{\varepsilon} > 0$, with $\varepsilon < \widetilde{\varepsilon}$, and $(w, w^*) \in \mathcal{G}(S^\varepsilon)$. Consider the norm

$$|\cdot| = \frac{\sqrt{\widetilde{\varepsilon}}}{\lambda}\|\cdot\|.$$

By Lemma 2.8, the hypothesis entails that, for any $\eta > 0$, there exists $(x_\eta, x^*_\eta) \in \mathcal{G}(S)$ such that $(w - x_\eta, x^*_\eta - w^*) \in \mathcal{G}(J^{|\cdot|}_\eta)$, that is

$$\frac{1}{2}|w - x_\eta|^2 + \frac{1}{2}|x^*_\eta - w^*|^2 \leq -\langle w - x_\eta, w^* - x^*_\eta\rangle + \eta \leq \varepsilon + \eta. \tag{13}$$

Recall that, for any $(z, z^*) \in \mathcal{G}(J^{|\cdot|}_\eta)$,

$$
\begin{aligned}
\frac{1}{2}(|z| - |z^*|)^2 &= \frac{1}{2}|z|^2 - |z|\,|z^*| + \frac{1}{2}|z^*|^2 \\
&\leq \frac{1}{2}|z|^2 + \frac{1}{2}|z^*|^2 - \langle z, z^*\rangle \leq \eta,
\end{aligned}
$$

implying $|z| \leq |z^*| + \sqrt{2\eta}$. Thus

$$|z|^2 = \frac{1}{2}|z|^2 + \frac{1}{2}|z|^2 \leq \frac{1}{2}|z|^2 + \frac{1}{2}|z^*|^2 + |z^*| \sqrt{2\eta} + \eta.$$

In our case, setting $z = w - x_\eta$ and $z^* = x^*_\eta - w^*$ and taking into account (13), we obtain that $|x^*_\eta - w^*|^2 \leq 2(\varepsilon + \eta)$ and

$$\begin{aligned} |w - x_\eta|^2 &\leq \frac{1}{2}|w - x_\eta|^2 + \frac{1}{2}|x^*_\eta - w^*|^2 + |x^*_\eta - w^*| \sqrt{2\eta} + \eta \\ &\leq \varepsilon + \eta + \sqrt{2(\varepsilon+\eta)} \sqrt{2\eta} + \eta = \varepsilon + 2\eta + 2\sqrt{(\varepsilon+\eta)\eta}. \end{aligned}$$

Since $\lim_{\eta \to 0^+}(2\eta + 2\sqrt{(\varepsilon+\eta)\eta}) = 0$, there exists $\eta_1 > 0$ such that, for all $\eta \in]0, \eta_1[$,

$$\varepsilon + 2\eta + 2\sqrt{(\varepsilon+\eta)\eta} < \widetilde{\varepsilon},$$

i.e. $|w - x_\eta| < \sqrt{\widetilde{\varepsilon}}$.
Analogously, one can show that, for all $0 < \eta < \eta_1$, $|x^*_\eta - w^*| < \sqrt{\widetilde{\varepsilon}}$.
Finally, by definition of $| \cdot |$, we obtain

$$\|w - x_\eta\| < \lambda \quad \text{and} \quad \|x^*_\eta - w^*\| < \frac{\widetilde{\varepsilon}}{\lambda}.$$

□

We close this section observing that Corollary 3.4 enables us to answer to a problem addressed by Simons and Zălinescu at the end of [25], where they wonder if it is possible to prove with a technique similar to that employed in their paper the fact that, given a maximal monotone operator $S : X \rightrightarrows X^*$ of type (D), there exists $(x^{**}, x^*) \in \mathcal{G}(\widetilde{S}) \cap \mathcal{G}(-J_{X^*})^\top$, a fact already proved in [22] by means of a more traditional approach. The answer is in the positive.

Corollary 3.11. *Let X be a Banach space and $S : X \rightrightarrows X^*$ be a maximal monotone operator of type* (D). *Then $\mathcal{G}(\widetilde{S}) \cap \mathcal{G}(-J_{X^*})^\top \neq \emptyset$.*

Proof. Since $\operatorname{dom} \varphi_J = X \times X^*$ and the duality mapping J is maximal monotone of type (D) (as the subdifferential of a lower semicontinuous proper convex function), then, by Corollary 3.4, $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})$. In particular, $(0_X, 0^*_X) \in \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})$, so that there exists $(x^{**}, x^*) \in \mathcal{G}(\widetilde{S})$ such that $(-x^{**}, -x^*) \in \mathcal{G}(-\widetilde{J})$, i.e., by Lemma 2.7, $(x^*, -x^{**}) \in \mathcal{G}(J_{X^*})$ and finally $(x^*, x^{**}) \in \mathcal{G}(-J_{X^*})$. □

4. The range of the sum

As in the previous section, we begin with a main theorem and then develop some of its consequences. In this case, the main result can be obtained directly from Theorem 3.2 by means of an appropriate transformation. Anyway, we state explicitly the analogous of Lemma 3.1, which is obtained exploiting the same transformation, since we will invoke it in the proof of Corollary 4.7.

Lemma 4.1. *Let X be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type* (D)*, $(x^{**}, x^*) \in X^{**} \times X^*$ and $(u, u^*), (v, v^*) \in X \times X^*$. Then the following facts are equivalent:*

(*a*) $(u + x^{**}, u^* + x^*) \in \mathcal{G}(\widetilde{S})$ *and* $(v + x^{**}, v^* - x^*) \in \mathcal{G}(\widetilde{T})$;

(*b*) *for all* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$, *the point* (x^*, x^{**}) *is a Fenchel functional for* h *and* $k \circ \varrho_2$;

(*c*) *there exist* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$ *such that* (x^*, x^{**}) *is a Fenchel functional for* h *and* $k \circ \varrho_2$.

If X is reflexive, the previous statements are also equivalent to:

(*d*) *for all* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$,

$$(h + k \circ \varrho_2)(x^{**}, x^*) = 0; \tag{14}$$

(*e*) *there exist* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$ *such that* (14) *holds.*

Proof. Define $T' : X \rightrightarrows X^*$ by

$$x^* \in T'(x) \iff x^* \in -T(-x)$$

for all $(x, x^*) \in X \times X^*$. Then T' is maximal monotone of type (D) and

$$(v + x^{**}, v^* - x^*) \in \mathcal{G}(\widetilde{T}) \iff (-v - x^{**}, -v^* + x^*) \in \mathcal{G}(\widetilde{T'}).$$

The result follows then from Lemma 3.1 with v replaced by $-v$, considering the bijection

$$\begin{array}{rccc} \mathcal{R}: & \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v} & \to & \mathcal{H}_{\tau_{v^*} \circ T' \circ \tau_{-v}} \\ & k & \mapsto & k \circ \varrho_2 \circ \varrho_1. \end{array}$$

$\square$

Theorem 4.2. *Let X be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type* (D) *and $w^* \in X^*$, $(u, u^*), (v, v^*) \in X \times X^*$ such that $u^* + v^* = w^*$. The following statements are equivalent:*

(*a*) $w^* \in R(\widetilde{S}(\,\cdot + u) + \widetilde{T}(\,\cdot + v))$;

(*b*) *there exists* $(x^{**}, x^*) \in X^{**} \times X^*$ *such that, for all* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$, *the point* (x^*, x^{**}) *is a Fenchel functional for* h *and* $k \circ \varrho_2$;

(*c*) *there exist* $(x^{**}, x^*) \in X^{**} \times X^*$, $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$ *such that* (x^*, x^{**}) *is a Fenchel functional for* h *and* $k \circ \varrho_2$;

(*d*)

$$\inf_{\substack{(y,y^*) \in \operatorname{dom} \varphi_{\tau_{-u^*} \circ S \circ \tau_u} \\ (z,z^*) \in \varrho_2(\operatorname{dom} \varphi_{\tau_{-v^*} \circ T \circ \tau_v}) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_{\tau_{-u^*} \circ S \circ \tau_u}(y, y^*) + (\varphi_{\tau_{-v^*} \circ T \circ \tau_v} \circ \varrho_2)(z, z^*)}{\|(y, y^*) - (z, z^*)\|} > -\infty; \tag{15}$$

(*e*) *relation* (15) *holds with* $\varphi_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $\varphi_{\tau_{-v^*} \circ T \circ \tau_v}$ *replaced by* $\sigma_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $\sigma_{\tau_{-v^*} \circ T \circ \tau_v}$, *respectively.*

If X is reflexive, the previous items are also equivalent to:

(f) *there exists $(x,x^*) \in X \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$, $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$,*

$$(h + k \circ \varrho_2)(x,x^*) = 0; \tag{16}$$

(g) *there exist $(x,x^*) \in X \times X^*$, $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ and $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$ such that* (16) *holds.*

A sufficient condition for (a)–(e) *to hold is the existence of $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u-v,w^*)] \quad \textit{is a closed subspace of } X \times X^*. \tag{17}$$

Proof. The theorem follows either from Theorem 3.2 by means of the bijection $\mathcal{R}$ used in the proof of Lemma 4.1 (taking into account observation (d) of Remark 3.3, replacing v by $-v$ and setting $w = u - v$), or directly from the same lemma with a proof similar to that of Theorem 3.2. □

Corollary 4.3. *Let X be a Banach space, $S,T : X \rightrightarrows X^*$ be maximal monotone operators of type* (D) *and $u,v \in X$. Then the following statements are equivalent:*

(a) $R(\widetilde{S}(\cdot+u) + \widetilde{T}(\cdot+v)) = X^*$;

(b) *for all $u^*,v^* \in X^*$, there exists $(x^{**},x^*) \in X^{**} \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$, $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$, (x^*,x^{**}) is a Fenchel functional for h and $k\circ\varrho_2$;*

(c) *for all $u^*,v^* \in X^*$, there exist $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ and $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$ such that h and $k\circ\varrho_2$ have a Fenchel functional;*

(d) *for all $u^*,v^* \in X^*$,*

$$\inf_{\substack{(y,y^*)\in \operatorname{cl}\operatorname{conv}\mathcal{G}(S)-(u,u^*)\\ (z,z^*)\in\varrho_2(\operatorname{cl}\operatorname{conv}\mathcal{G}(T))+(-v,v^*)\\ (y,y^*)\neq(z,z^*)}} \frac{\varphi_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*) + (\varphi_{\tau_{-v^*}\circ T\circ\tau_v}\circ\varrho_2)(z,z^*)}{\|(y,y^*)-(z,z^*)\|} > -\infty; \tag{18}$$

(e) *for all $u^*,v^* \in X^*$, relation* (18) *holds with $\varphi_{\tau_{-u^*}\circ S\circ\tau_u}$ and $\varphi_{\tau_{-v^*}\circ T\circ\tau_v}$ replaced by $\sigma_{\tau_{-u^*}\circ S\circ\tau_u}$ and $\sigma_{\tau_{-v^*}\circ T\circ\tau_v}$, respectively.*

If X is reflexive, they are also equivalent to:

(f) *for all $u^*,v^* \in X^*$, there exists $(x,x^*) \in X\times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ and $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$, $(h+k\circ\varrho_2)(x,x^*) = 0$;*

(g) *for all $u^*,v^* \in X^*$, there exist $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ and $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$ such that*

$$0 \in \operatorname{Im}(h + k\circ\varrho_2).$$

A sufficient condition for $\widetilde{S}(\cdot+u)+\widetilde{T}(\cdot+v)$ to be surjective is that, for all $w^* \in X^*$, there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u-v,w^*)] \quad \textit{is a closed subspace of } X \times X^*. \tag{19}$$

In particular, the previous condition is satisfied whenever there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that

$$\begin{aligned}&\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u-v,0) = A \times X^*,\\ &\textit{where } \bigcup_{\lambda>0} \lambda A \textit{ is a closed subspace of } X.\end{aligned} \tag{20}$$

Proof. The equivalence of (a)–(e) (and (f)–(g), when X is reflexive) is an immediate consequence of Theorem 4.2 (taking into account an observation similar to Remark 3.3, as (d) and (e) are concerned).
Condition (17) in the same theorem guarantees that the validity of (19) for any $w^* \in X^*$ is a sufficient condition for the surjectivity of $\widetilde{S}(\,\cdot+u) + \widetilde{T}(\,\cdot+v)$.
Finally, condition (20) yields

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u-v,w^*)] = \bigcup_{\lambda>0} \lambda[A \times X^* - (0,w^*)] = \bigcup_{\lambda>0} \lambda(A \times X^*),$$

so that, by Lemma 2.18, (19) is satisfied for any $w^* \in X^*$. □

Condition (20) slightly refines the analogous condition given in [24, Theorem 30.2]. As a consequence of the previous corollary, one can provide generalizations of Corollary 2.7, Theorem 2.8 and Proposition 2.9 of [14]. We state for instance a possible improvement of Proposition 2.9 of that paper.

Corollary 4.4. *Let X be a Banach space, $S : X \rightrightarrows X^*$ be a monotone operator and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function.*

(a) *If S is maximal monotone of type* (D) *and* $\operatorname{dom}\varphi_S + (-\operatorname{dom} f) \times \operatorname{dom} f^* = X \times X^*$, *then* $R(\widetilde{S}(\,\cdot+w) + \widetilde{\partial f}) = X^*$ *for all* $w \in X$.

(b) *If $R(S(\,\cdot+w) + \partial f) = X^*$ for all $w \in X$, f admits a unique global minimizer p and is Gâteaux differentiable at p, then S is maximal monotone.*

Proof. (a) It is a consequence of Corollary 4.3, setting $T = \partial f$, $u = w$, $v = 0$ and, in condition (20), $h = \varphi_S$ and $k = f \oplus f^*$.

(b) Let $(x, x^*) \in X \times X^*$ be monotonically related to every point in $\mathcal{G}(S)$. Since $x^* \in X^* = R(S(\,\cdot+x-p) + \partial f)$, we have $x^* \in S(a+x-p) + \partial f(a)$ for some $a \in X$. We can therefore write $x^* = a^* + s^*$ for some $a^* \in S(a+x-p)$ and $s^* \in \partial f(a)$. Using that $a^* - x^* = -s^*$, we obtain

$$\begin{aligned} 0 &\le \langle (a+x-p) - x, a^* - x^* \rangle \\ &= \langle a-p, a^*-x^* \rangle = -\langle a-p, s^* \rangle = -\langle a, s^* \rangle + \langle p, s^* \rangle \\ &= -f(a) - f^*(s^*) + \langle p, s^* \rangle \le -f(p) - f^*(s^*) + \langle p, s^* \rangle \le 0, \end{aligned}$$

hence $f(a) = f(p)$ and $-f(p) - f^*(s^*) + \langle p, s^* \rangle = 0$, that is, $s^* \in \partial f(p) = \{0\}$. We deduce that $s^* = 0$ and, by the assumption on f, that $a = p$. We then conclude $x^* = a^* \in S(x)$, thus proving the maximality of S. □

Remark 4.5. As a consequence of Lemma 2.17, when f is finite-valued the condition $\operatorname{dom}\varphi_S + (-\operatorname{dom} f) \times \operatorname{dom} f^* = X \times X^*$ is equivalent to $p_2(\operatorname{dom}\varphi_S) + \operatorname{dom} f^* = X^*$. Analogously, if f is cofinite, it is equivalent to $p_1(\operatorname{dom}\varphi_S) - \operatorname{dom} f = X$.

As in the previous section, one could be interested in obtaining a surjectivity property for the range of the sum of the operators themselves, instead of their extensions. As in Theorem 3.7, one can prove a density result, rather than one of surjectivity. Item (b) of the following theorem is a generalization of implication 4. $\Longrightarrow$ 1. in [13, Theorem 3.6]; our proof is along the same lines.

Theorem 4.6. *Let X be a Banach space, $S, T : X \rightrightarrows X^*$ be monotone operators of type* (D) *and $u, v \in X$. If, for all $w^* \in X^*$, there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u - v, w^*)] \quad \text{is a closed subspace of } X \times X^*, \tag{21}$$

then:

(a) *for all $\varepsilon > 0$, $R(S(\,\cdot + u)^\varepsilon + T(\,\cdot + v)^\varepsilon) = X^*$;*

(b) *if S and T are maximal monotone and*

$$\bigcup_{\lambda>0} \lambda[p_1 \operatorname{dom} \mathcal{T}_{(u,w^*)} h - p_1 \operatorname{dom} \mathcal{T}_{(v,0_{X^*})} k] \quad \text{is a closed subspace of } X, \tag{22}$$

then $\operatorname{cl}(R(S(\,\cdot + u) + T(\,\cdot + v))) = X^*$.

Proof. Let $w^* \in X^*$ and $\varepsilon > 0$ be given and $h \in \mathcal{H}_S$, $k \in \mathcal{H}_T$ satisfy condition (21).

(a) With a reasoning analogous to the proof of item (a) of Theorem 3.7, one can prove that

$$\inf_{(y,y^*) \in X \times X^*} \{\mathcal{T}_{(u,w^*)} h(y, y^*) + (\mathcal{T}_{(v,0_{X^*})} k \circ \varrho_2)(y, y^*)\} = 0. \tag{23}$$

Then there exists $(x_\varepsilon, x^*_\varepsilon) \in X \times X^*$ such that

$$\mathcal{T}_{(u,w^*)} h(x_\varepsilon, x^*_\varepsilon) + \mathcal{T}_{(v,0_{X^*})} k(x_\varepsilon, -x^*_\varepsilon) \le \varepsilon,$$

which entails

$$\varphi_{\tau_{-w^*} \circ S \circ \tau_u}(x_\varepsilon, x^*_\varepsilon) \le \mathcal{T}_{(u,w^*)} h(x_\varepsilon, x^*_\varepsilon) \le -\mathcal{T}_{(v,0_{X^*})} k(x_\varepsilon, -x^*_\varepsilon) + \varepsilon \le \langle x_\varepsilon, x^*_\varepsilon \rangle + \varepsilon$$

and

$$\varphi_{T \circ \tau_v}(x_\varepsilon, -x^*_\varepsilon) \le \mathcal{T}_{(v,0_{X^*})} k(x_\varepsilon, -x^*_\varepsilon) \le -\mathcal{T}_{(u,w^*)} h(x_\varepsilon, x^*_\varepsilon) + \varepsilon \le \langle x_\varepsilon, -x^*_\varepsilon \rangle + \varepsilon,$$

i.e. $(x_\varepsilon, x^*_\varepsilon) \in \mathcal{G}((\tau_{-w^*} \circ S \circ \tau_u)^\varepsilon)$ and $(x_\varepsilon, -x^*_\varepsilon) \in \mathcal{G}((T \circ \tau_v)^\varepsilon)$, yielding

$$w^* = (w^* + x^*_\varepsilon) - x^*_\varepsilon \in S(\,\cdot\, + u)^\varepsilon(x_\varepsilon) + T(\,\cdot\, + v)^\varepsilon(x_\varepsilon).$$

(b) Defining

$$H(x, x^*) := \inf_{y^* \in X^*} \{\mathcal{T}_{(u,w^*)} h(x, y^*) + \mathcal{T}_{(v,0_{X^*})} k(x, x^* - y^*)\},$$

by (22) and Lemma 2.13, one has that $\tau_{-w^*} \circ S \circ \tau_u + T \circ \tau_v$ is maximal monotone of type (D) and $\operatorname{cl} H \in \mathcal{H}_{\tau_{-w^*} \circ S \circ \tau_u + T \circ \tau_v}$.
By (23), there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$\begin{aligned}
\varphi_{\tau_{-w^*} \circ S \circ \tau_u + T \circ \tau_v}(x_\varepsilon, 0_{X^*}) \leq \; & \operatorname{cl} H(x_\varepsilon, 0_{X^*}) \\
& \leq \mathcal{T}_{(u,w^*)} h(x_\varepsilon, x_\varepsilon^*) + (\mathcal{T}_{(v,0_{X^*})} k \circ \varrho_2)(x_\varepsilon, x_\varepsilon^*) \\
& < \varepsilon^2 = \langle x_\varepsilon, 0_{X^*} \rangle + \varepsilon^2.
\end{aligned}$$

Then, since $\tau_{-w^*} \circ S \circ \tau_u + T \circ \tau_v$, being of type (D), is of type (BR), for all $\eta > \varepsilon$ there exists $(\overline{x}, \overline{x}^*) \in \mathcal{G}(\tau_{-w^*} \circ S \circ \tau_u + T \circ \tau_v)$ such that $\|\overline{x} - x_\varepsilon\| < \eta$ and $\|\overline{x}^* - 0_{X^*}\| < \eta$, i.e.

$$w^* + \overline{x}^* \in R(S(\,\cdot + u) + T(\,\cdot + v))$$

and $\|(w^* + \overline{x}^*) - w^*\| = \|\overline{x}^*\| < \eta$. The result follows from the arbitrariness of ε and η. □

We now present an application of the surjectivity results considered. A consequence of Theorem 4.2 is the possibility to provide a characterization of the solutions of variational inequalities concerning maximal monotone operators in reflexive Banach spaces. This is accomplished by the following corollary, which generalizes [14, Corollary 2.3].
Note that necessary and sufficient conditions for the existence of solutions to the variational inequality on T and K (nonempty closed convex subset of X) in principle do not require $T + N_K$ to be maximal monotone, unlike the standard sufficient conditions [20, 27].
Recall that, given a cone K in X, we denote by B_K the barrier cone of K.

Corollary 4.7. *Let X be a reflexive Banach space, $S : X \rightrightarrows X^*$ be a maximal monotone operator, K be a nonempty closed convex subset of X and $(x, x^*) \in X \times X^*$. Consider the following statements:*

(a) *(x, x^*) is a solution to the variational inequality on S and K, i.e. $x \in K \cap D(S)$, $x^* \in S(x)$ and*

$$\forall y \in K : \quad \langle y - x, x^* \rangle \geq 0; \tag{24}$$

(b) *for all $h \in \mathcal{H}_S$, the point (x^*, x) is a Fenchel functional for h and $(\delta_K \oplus \delta_K^*) \circ \varrho_2$;*

(c) *there exists $h \in \mathcal{H}_S$ such that (x^*, x) is a Fenchel functional for h and $(\delta_K \oplus \delta_K^*) \circ \varrho_2$;*

(d) *for all $h \in \mathcal{H}_S$,*

$$\left(h + (\delta_K \oplus \delta_K^*) \circ \varrho_2\right)(x, x^*) = 0; \tag{25}$$

(e) *there exists $h \in \mathcal{H}_S$ satisfying relation* (25)*;*

(f)

$$\inf_{\substack{(y,y^*) \in \operatorname{cl}\operatorname{conv} \mathcal{G}(S) \\ (z,z^*) \in K \times (-B_K) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_S(y, y^*) + \delta_K^*(-z^*)}{\|(y, y^*) - (z, z^*)\|} > -\infty; \tag{26}$$

(g) *relation* (26) *holds with φ_S replaced by σ_S;*

(*h*) *there exists* $h \in \mathcal{H}_S$ *such that*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - (K \times (-B_K))] \ \text{ is a closed subspace of } X \times X^*. \tag{27}$$

Statements (*a*)–(*e*) *are equivalent;* (*f*)–(*g*) *are necessary and sufficient conditions for the existence of solutions to the variational inequality* (24), *while* (*h*) *provides a sufficient condition.*

Proof. Note that (a) is equivalent to the inclusions $(x, x^*) \in \mathcal{G}(S)$ and $(x, -x^*) \in \mathcal{G}(N_K)$. Since $N_K = \partial\delta_K$ and $\delta_K \oplus \delta_K^* = \varphi_{N_K}$, the equivalence of (a)–(e) is a consequence of Lemma 4.1, with $x^{**} = x$, $u = v = 0_X$ and $u^* = v^* = 0_{X^*}$.
Moreover, since the existence of a solution to the variational inequality on S and K is equivalent to the inclusion $0 \in R(S+N_K)$, then, by Theorem 4.2 with $u = v = 0_X$ and $w^* = 0_{X^*}$ (taking into account an observation similar to Remark 3.3), the relations between (a)–(e) and (f)–(h) follow as well. In particular, condition (27) is an instance of (17) with $k = \delta_K \oplus \delta_K^*$, because

$$\varrho_2(\operatorname{dom}(\delta_K \oplus \delta_K^*)) = \varrho_2(\operatorname{dom}\delta_K \times \operatorname{dom}\delta_K^*) = K \times (-B_K).$$

□

Remark 4.8. (a) Similarly to Remark 3.3, a particular case of condition (27), namely when $h = \sigma_S$, can be stated as

$$\bigcup_{\lambda>0} \lambda[\operatorname{conv}\mathcal{G}(S) - (K \times (-B_K))] \ \text{ is a closed subspace of } X \times X^*. \tag{28}$$

(b) If B_K is a closed subspace of X^* containing $p_2 \operatorname{dom} h$, then, by Lemma 2.17 and Lemma 2.18, condition (27) simplifies to

$$\bigcup_{\lambda>0} \lambda(p_1 \operatorname{dom} h - K) \ \text{ is a closed subspace of } X.$$

Analogously, if B_K is a closed subspace of X^* containing $p_2\operatorname{conv}\mathcal{G}(S)$, then condition (28) can be reduced to

$$\bigcup_{\lambda>0} \lambda(p_1 \operatorname{conv}\mathcal{G}(S) - K) \ \text{ is a closed subspace of } X.$$

In particular, this is the case whenever K is bounded, since then $B_K = X^*$.

The previous corollary can be restated, with obvious changes, for the more general variational inequality (considered in [27, Proposition 32.36])

$$\forall y \in X: \ \ \langle y - x, x^* \rangle + \vartheta(y) \geq \vartheta(x),$$

where $\vartheta : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous proper convex function, replacing $\delta_K \oplus \delta_K^*$ by $\vartheta \oplus \vartheta^*$ in the proof.
On the other hand, it's worth stating explicitly how Corollary 4.7 specializes in the particular case of convex constrained optimization problems.

Corollary 4.9. *Let X be a reflexive Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function and K be a nonempty closed convex subset of X. If*

$$\bigcup_{\lambda>0} \lambda[(\operatorname{dom} f - K) \times (\operatorname{dom} f^* + B_K)] \quad \text{is a closed subspace of } X \times X^*, \tag{29}$$

then f has a global minimum on K.

Proof. Condition (29) is derived from condition (27) of Corollary 4.7, taking into account that

$$\begin{aligned} \operatorname{dom}(f \oplus f^*) - K \times (-B_K) &= \operatorname{dom} f \times \operatorname{dom} f^* - K \times (-B_K) \\ &= (\operatorname{dom} f - K) \times (\operatorname{dom} f^* + B_K). \end{aligned}$$

□

Remark 4.10. By Lemma 2.18, condition (29) is entailed by any of the two following conditions:

(a) $\bigcup_{\lambda>0} \lambda(\operatorname{dom} f - K)$ is a closed subspace of X and $\operatorname{dom} f^* + B_K$ is a closed subspace of X^*;

(b) $\operatorname{dom} f - K$ is a closed subspace of X and $\bigcup_{\lambda>0} \lambda(\operatorname{dom} f^* + B_K)$ is a closed subspace of X^*.

5. The closure of the range of a maximal monotone operator

The duality methods used in the previous sections can provide geometrical insight in the theoretical framework of maximal monotone operators. An example of this fact is given by the following proposition, which provides a convex analytical proof of the well known relations [24, Theorem 43.1 and Lemma 31.1] between the range of a maximal monotone operator of type (D) and the projection of the domains of its convex representations on X^*, implying as an immediate consequence the convexity of the closure of the former.

Lemma 5.1. *Let X be a Banach space and $K \subseteq X^*$ be a weak*-closed convex set. Then $((\delta_K^*)_{|X})^* = \delta_K$.*

Proof. It is easy to check that $((\delta_K^*)_{|X})^* \leq \delta_K$, given that, for all $x^* \in X^*$,

$$\begin{aligned} ((\delta_K^*)_{|X})^*(x^*) &= \sup_{x\in X}\{\langle x, x^*\rangle - \delta_K^*(x)\} \\ &\leq \sup_{x^{**}\in X^{**}}\{\langle x^{**}, x^*\rangle - \delta_K^*(x^{**})\} = \delta_K^{**}(x^*) = \delta_K(x^*). \end{aligned}$$

If $x^* \in K$, then both sides are equal to zero and equality follows. If $x^* \notin K$, the equality holds again, since K is convex and weak*-closed and therefore we can separate x^* from K by means of an hyperplane of X^* corresponding to a point $x \in X$ (replacing then x by tx and letting $t \to +\infty$, one concludes that the supremum on the left-hand side of the preceding inequality has to be infinite). □

Proposition 5.2. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator of type* (D)*. Then, for any $h \in \mathcal{H}_T$:*

(a) $\operatorname{cl}(p_2 \operatorname{dom} h) = \operatorname{cl} R(T)$*;*

(b) $\operatorname{int}(p_2 \operatorname{dom} h) \subseteq \operatorname{int} R(\widetilde{T})$*.*

Proof. (a) Since $\mathcal{G}(T) \subseteq \operatorname{dom} h \subseteq \operatorname{dom} \varphi_T$, it suffices to prove that $p_2 \operatorname{dom} \varphi_T \subseteq \operatorname{cl} R(T)$. Actually, since T is a maximal monotone operator of type (D), $\operatorname{cl} R(\widetilde{T}) = \operatorname{cl} R(T)$ (see e.g. [19, proof of Theorem 3.8]) and we will prove that $p_2 \operatorname{dom} \varphi_T \subseteq \operatorname{cl} R(\widetilde{T})$.

Suppose, by contradiction, that there exists $(x_0, x_0^*) \in \operatorname{dom} \varphi_T$ such that $x_0^* \notin \operatorname{cl} R(\widetilde{T})$. Without loss of generality, we can suppose that $(x_0, x_0^*) = (0_X, 0_{X^*})$.

Take $\eta \in]0, d_{X^*}(0, \operatorname{cl} R(\widetilde{T}))[$ and set $K := \operatorname{cl} B_{X^*}(0_{X^*}, \eta)$ (the closed ball of X^* centered at 0_{X^*} and with radius η) and $g := (\delta_K^*)_{|X}$. Then we have $g^* = \delta_K$ (by Lemma 5.1) and $(0_X, 0_{X^*}) \in X \times B_{X^*}(0_{X^*}, \eta) = \operatorname{int} \operatorname{dom}(g \oplus g^*)$. Hence

$$(0_X, 0_{X^*}) \in \operatorname{dom} \varphi_T \cap \operatorname{int} \operatorname{dom}((g \oplus g^*) \circ \varrho_1).$$

Clearly this condition implies that

$$\bigcup_{\lambda > 0} \lambda[\operatorname{dom} \varphi_T - \varrho_1 \operatorname{dom}(g \oplus g^*)] = X \times X^*.$$

Therefore, by Theorem 2.11, there exists a Fenchel functional $(y^*, y^{**}) \in X^* \times X^{**}$ for φ_T and $(g \oplus g^*) \circ \varrho_1$. By Lemma 3.1, this entails $y^* \in R(\widetilde{T}) \cap R(\widetilde{\partial g})$, which is absurd, since, by Lemma 2.7, $R(\widetilde{\partial g}) = D(\partial g^*) = K$ and $R(\widetilde{T}) \cap K = \emptyset$ by construction. Then $p_2 \operatorname{dom} \varphi_T \subseteq \operatorname{cl} R(\widetilde{T})$.

(b) Similarly to item (a), we only have to prove that $\operatorname{int}(p_2 \operatorname{dom} \varphi_T) \subseteq \operatorname{int} R(\widetilde{T})$.

The result is obvious when $\operatorname{int}(p_2 \operatorname{dom} \varphi_T) = \emptyset$. Suppose then that there exists $(x_0, x_0^*) \in \operatorname{dom} \varphi_T$, such that $x_0^* \in \operatorname{int}(p_2 \operatorname{dom} \varphi_T)$. This means that there exists $\varrho > 0$ such that $B_{X^*}(x_0^*, \varrho) \subseteq p_2 \operatorname{dom} \varphi_T$. Let $y^* \in B_{X^*}(x_0^*, \varrho)$ and define $A : X \rightrightarrows X^*$ by $\mathcal{G}(A) = X \times \{y^*\}$. It is easy to check that A is maximal monotone of type (D) and that $\operatorname{dom} \varphi_A = \mathcal{G}(A)$.

Since $y^* \in \operatorname{int}(p_2 \operatorname{dom} \varphi_T)$, we have $0_{X^*} \in \operatorname{int}(p_2 \operatorname{dom} \varphi_T) - \{y^*\} = \operatorname{int}(p_2 \operatorname{dom} \varphi_T - \{y^*\})$, which entails that the set $p_2 \operatorname{dom} \varphi_T - \{y^*\}$ is absorbing in X^*, i.e.

$$\bigcup_{\lambda > 0} \lambda(p_2 \operatorname{dom} \varphi_T - \{y^*\}) = X^*.$$

Therefore, by Lemma 2.17 (with the roles of Y and Z interchanged) and Lemma 2.18 (b), we obtain

$$\begin{aligned}\bigcup_{\lambda > 0} \lambda[\operatorname{dom} \varphi_T - \varrho_1(\operatorname{dom} \varphi_A)] &= \bigcup_{\lambda > 0} \lambda(\operatorname{dom} \varphi_T - X \times \{y^*\}) \\ &= \bigcup_{\lambda > 0} \lambda[X \times (p_2 \operatorname{dom} \varphi_T - \{y^*\})] = X \times X^*.\end{aligned}$$

Thus, by Theorem 3.2, $(0_X, 0_{X^*}) \in \mathcal{G}(\widetilde{T}) + \mathcal{G}(-\widetilde{A})$. Therefore, there exists $(x, x^*) \in X \times X^*$ such that $(x, x^*) \in \mathcal{G}(\widetilde{T})$ and $(-x, x^*) \in \mathcal{G}(\widetilde{A})$, so that $\emptyset \neq R(\widetilde{T}) \cap R(\widetilde{A}) = R(\widetilde{T}) \cap R(A) \subseteq \{y^*\}$. Thus $y^* \in R(\widetilde{T})$. Since y^* was arbitrarily chosen in $B_{X^*}(x_0^*, \varrho)$, we have $B_{X^*}(x_0^*, \varrho) \subseteq R(\widetilde{T})$, i.e. $x_0^* \in \operatorname{int} R(\widetilde{T})$. □

Remark 5.3. When X is a reflexive space, Proposition 5.2 (b) yields the relation

$$\operatorname{int}(p_2 \operatorname{dom} h) = \operatorname{int} R(T)$$

as a particular case, which is part of [24, Lemma 31.1].

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