Journal of Convex Analysis
Volume 19 (2012), No. 1, 23–48

Generic Fréchet Differentiability on Asplund Spaces via A.E. Strict Differentiability on Many Lines*

Luděk Zajíček
Charles University, Faculty of Mathematics and Physics,
Sokolovská 83, 186 75 Praha 8-Karlín, Czech Republic
zajicek@karlin.mff.cuni.cz

Received: July 11, 2010
Revised manuscript received: November 11, 2010

We prove that a locally Lipschitz function on an open subset G of an Asplund space X, whose restrictions to "many lines" are essentially smooth (i.e., almost everywhere strictly differentiable), is generically Fréchet differentiable on X. In this way we obtain new proofs of known Fréchet differentiability properties of approximately convex functions, Lipschitz regular functions, saddle (or biconvex) Lipschitz functions, and essentially smooth functions (in the sense of Borwein and Moors), and also some new differentiability results (e.g., for partially DC functions). We show that classes of functions $\mathcal{S}_e^g(G)$ and $\mathcal{R}_e^g(G)$ (defined via linear essential smoothness) are respectively larger than classes $\mathcal{S}_e(G)$ (of essentially smooth functions) and $\mathcal{R}_e(G)$ studied by Borwein and Moors, and have also nice properties. In particular, we prove that members of $\mathcal{S}_e^g(G)$ are uniquely determined by their Clarke subdifferentials. We also show the inclusion $\mathcal{S}_e(G) \subset \mathcal{R}_e(G)$ for Borwein-Moors classes.

Keywords: Generic Fréchet differentiability, essentially smooth functions, separable reduction

2010 Mathematics Subject Classification: Primary 46G05; Secondary 46T20

1. Introduction

It is well-known that some kinds of (non-convex) locally Lipschitz functions which naturally arise in applications have similar differentiability properties as convex functions. In this article we are interested in *generic Fréchet differentiability*; i.e., in Fréchet differentiability except a first category (meager) set. Recall that a Banach space X is called an *Asplund space* if each continuous convex function on X is generically Fréchet differentiable.

For the sake of brevity, we will say (*in this section only*) that a property (P) (which a function can have) is *a (GFD) property* (i.e. a generic Fréchet differentiability property), if each function f which is locally Lipschitz on an open subset G of an Asplund space and has the property (P) is generically Fréchet differentiable on

*The research was partially supported by the grant MSM 0021620839 from the Czech Ministry of Education and partially supported by the grant GAČR 201/09/0067 from the Grant Agency of Czech Republic.

ISSN 0944-6532 / $ 2.50

G. Using this abbreviation, we will now present a brief survey of known relevant differentiability results. The definitions of involved notions are recalled in Section 2.

Now it is well known that *approximate convexity* in the sense of [25] (and therefore also *local semiconvexity* and *strong paraconvexity*) are (GFD) properties in the above sense (see [36] or [26]). It seems that the first (GFD) property (weaker than convexity) appears in [15], where it is considered the following property (P1) of f:

(P1) $f(x) = \sup_{\alpha \in A} f_\alpha(x),\ x \in G$, where each f_α is Fréchet differentiable on G and the derivatives $f'_\alpha,\ \alpha \in A$, are equally uniformly continuous on G.

Ekeland and Lebourg [15] proved that "(P1) is a (GFD) property" if X admits a Fréchet smooth bump function, and it was proved that (P1) is a (GFD) property in [34]. The proof in [34] is based on the fact that the property (P1) implies that f is *Fréchet subdifferentiable* at all points of G. Moreover, the following property (P2) is a (GFD) property (see [35, Theorem 10]).

(P2) f is generically Fréchet subdifferentiable on G.

Note that if f has the property (P1), then f is semiconvex (=strongly paraconvex), and if X is superreflexive, then also the converse implication holds (see [14]). Further note that generic approximate convexity is also a sufficient property (see [26]), and that this fact immediately follows (via [25, Theorem 3.6]) from the fact that (P2) is a (GFD) property.

Independently, by different methods, and almost in the same time, further (GFD) properties were found. Georgiev [17] (see also [18] and [26] for another proofs) proved that *regularity* (in Clarke's sense) is a a (GFD) property, and de Barra, Giles and Fitzpatrick [12] proved that even *pseudoregularity* is a (GFD) property. Moreover, they proved (using the deep Preiss theorem on Fréchet differentiability of general Lipschitz functions) that the following weaker property is a (GFD) property (cf. also [5]).

(P3) If f is Fréchet differentiable at a point $x \in G$, then f is strictly Gâteaux differentiable at x.

Notice that generic strict Gâteaux differentiability *is not a (GFD) property*. Indeed, in ℓ^2 there exists a Lipschitz everywhere Gâteaux differentiable function f which is generically Fréchet non-differentiable (see [28]). Since f is generically pseudoregular (see (11) below), it is generically strictly Gâteaux differentiable. Two other (GFD) properties weaker than pseudoregularity (cf. Proposition BM above Remark 6.1) are considered in [2], [6] and [8]:

(P4) f is strictly Gâteaux differentiable at all points $x \in G$ except a Haar null set.

(P5) f is *essentially smooth* on G (i.e., f is locally Lipschitz on G and, for each $v \in X$, f is strictly differentiable in the direction v at all points $x \in G$ except a Haar null set).

Clearly, (P4) is stronger than (P5), but (see [8, Theorem 3.3]) for locally Lipschitz functions,

$$\text{if } X \text{ is separable, then (P4) and (P5) are equivalent.} \tag{1}$$

The properties (P4) and (P5) are (GFD) properties. This fact is proved in [2, Theorem 4.1] in the separable case. The non-separable result is not mentioned in [8], but it follows from [8, Theorem 3.4] as in the proof of [2, Theorem 4.1]. Namely, [13, Lemma 1.6] easily implies (cf., e.g., [2, Proposition 3.1(d)]) that the property

(P6) the Clarke subdifferential mapping $\partial^C f : (G, \|\cdot\|) \to (X^*, w^*)$ is a minimal usco mapping

is a (GFD) property and [8, Theorem 3.4] asserts that (P5) implies (P6).

Recall that a multi-valued mapping $F : (G, \|\cdot\|) \to (X^*, w^*)$ is *usco* (where G is an open subset of X and (X^*, w^*) is the dual space endowed with w^*-topology), if $F(x)$ is a non-empty compact in (X^*, w^*) for each $x \in G$ and F is upper semicontinuous. (In this situation, it is frequently said that F is a w^*-usco from G to X^*.) The mapping F is called a *minimal usco* if there exists no usco $\tilde{F} : (G, \|\cdot\|) \to (X^*, w^*)$ whose graph is a proper subset of graph of F. The important Christensen-Kenderov result [13, Lemma 1.6] implies that if $F : (G, \|\cdot\|) \to (X^*, w^*)$ is a minimal usco, then there exists a first category set $N \subset G$ such that F is single-valued and semicontinuous as a mapping from $(G, \|\cdot\|)$ to $(X^*, \|\cdot\|)$ at all points of $G \setminus N$.

To formulate our results, which give new (GFD) properties, we need the following definition.

Definition 1.1. Let f be a real function defined on an open subset G of a Banach space X.

(i) We say that f is *essentially smooth on the line* $L = a + \mathbb{R}v$ (where $a \in X$, $0 \neq v \in X$) if the function $\varphi(t) := f(a + tv)$ of one variable is strictly differentiable at a.e. points of its (possibly empty) domain. (Obviously, the definition is correct; it does not depend on the choice of a and v).

(ii) We say that f is *linearly essentially smooth*, and write $f \in \mathcal{S}_e^l(G)$, if f is essentially smooth on all lines.

(iii) We say that f is *essentially smooth on a generic line parallel to* $0 \neq v \in X$, if f is essentially smooth on all lines parallel to v, except a first category set of lines in the factor space $X/\operatorname{span}\{v\}$.

Remark 1.2. The system of all essentially smooth functions on G is denoted by $\mathcal{S}_e(G)$ in [8]. So, f is essentially smooth on the line $L = a + \mathbb{R}v$ if and only if $H := \{t \in \mathbb{R} : a + tv \in G\} = \emptyset$ or $H \neq \emptyset$ and $\varphi \in \mathcal{S}_e(H)$, where $\varphi(t) := f(a + tv)$, $t \in H$. Further, if f is strictly differentiable in the direction v at a.e. points of L (i.e., except a set of null Hausdorff one-dimensional measure), then f is clearly essentially smooth on L (but the opposite implication does not hold).

We will show that also the following property (P7) is a (GFD) property:

(P7) f is linearly essentially smooth.

In fact, we prove that the following weaker properties are (GFD) properties.

(P8) There exist closed subspaces $X_1, \dots, X_n$ of X such that $X = X_1 \oplus \cdots \oplus X_n$ and f is essentially smooth on each line, which is parallel to some $X_i, 1 \leq i \leq n$.

(P9) There exists a dense subset D of the unit sphere S_X such that, for each $v \in D$,

f is essentially smooth on a generic line parallel to v.

The fact that (P8) is a (GFD) property generalizes the well-known fact (see [1]) that all locally Lipschitz saddle and biconvex functions on Asplund spaces are generically Fréchet differentiable. Moreover (cf. Section 8), it shows that also locally Lipschitz partially DC (or partially approximately convex) functions are generically Fréchet differentiable on Asplund spaces.

The condition (P9) is weaker than (P5) (see Proposition 6.4), and so also weaker than (P1) and psudoregularity. (On the other hand, it seems that (P9) is neither weaker nor stronger than (P2), (P3) or (P6).)

Recall that the class of all essentially smooth functions on G is denoted by $\mathcal{S}_e(G)$ in [8]. We will consider several related classes:

Definition 1.3. Let G be an open subset of a Banach space X.

(i) We will denote by $\mathcal{S}_e^g(G)$ (resp. $\mathcal{S}_e^{gg}(G)$, resp. $\mathcal{S}_e^{gd}(G)$) the class of all locally Lipschitz functions on G such that f is essentially smooth on a generic line parallel to an arbitrary $0 \neq v \in X$ (resp. to an arbitrary v from a set which is residual in S_X, resp. to an arbitrary v from a dense subset of S_X).

(ii) If $X = X_1 \oplus \cdots \oplus X_n$ (and this decomposition is fixed), then we denote by $\mathcal{S}_e^p(G)$ the class of all locally Lipschitz functions on G such that f is essentially smooth on each line parallel to some X_i $(1 \leq i \leq n)$.

Remark 1.4.

(i) $\mathcal{S}_e^{gd}(G)$ (resp. $\mathcal{S}_e^p(G)$) is the system of all locally Lipschitz functions fulfilling (P9) (resp. (P8)) on G.

(ii) In the above notation, "g" is for "generic", "d" for "dense", and "p" for "partial".

(iii) We can define by the obvious way also the class $\mathcal{S}_e^{dd}(G)$. But the members of this class need not be generically differentiable even for $X = \mathbb{R}^2$. (In this time, I know only a rather technical example, so I do not present it here.)

Obviously, $\mathcal{S}_e^l(G) \subset \mathcal{S}_e^g(G) \subset \mathcal{S}_e^{gg}(G) \subset \mathcal{S}_e^{gd}(G)$ and we will show (see Proposition 6.4) that, if $\dim X > 1$, then

$$\mathcal{S}_e(G) \subset \mathcal{S}_e^g(G) \text{ and this inclusion is strict.} \tag{2}$$

It seems that the class $\mathcal{S}_e^{gd}(G)$ is rather unstable, but the classes $\mathcal{S}_e^g(G)$ and $\mathcal{S}_e^{gg}(G)$ have the same stability properties as $\mathcal{S}_e(G)$: they are linear lattices closed under multiplication and division (when it is defined on G), see Proposition 7.1. Further, on Asplund spaces, the members of these classes are generically Fréchet differentiable and are uniquely determined by their Clarke subdifferentials (see Proposition 7.5). So, they can be, similarly as the smaller class $\mathcal{S}_e(G)$, of some interest.

The structure of the article is the following. In Section 2 we recall some notions and known facts. In Section 3 we prove a general differentiability result (Proposition 3.3) in a separable Asplund space; its proof contains the main idea of the article. The results in nonseparable Asplund spaces are proved by the well-known method of separable reduction (see, e.g., [35]). However, for the proof of some more subtle results (e.g. Theorem 5.2), the improvement (from [9] and [22]) of the separable reduction method, which is based on the notion of a "rich family" of separable subspaces, was very useful. The method of rich families is used in Section 4 to prove, among others, Theorem 4.7 on separable reduction of generic Fréchet differentiability, which improves [35, Theorem 8]. Differentiability results in nonseparable spaces (which imply that (P7), (P8) and (P9) are (GFD) properties) are proved in Section 5. In Section 6, we discuss the relation between Borwein-Moors classes $\mathcal{S}_e(G)$ and $\mathcal{R}_e(G)$ and the corresponding classes $\mathcal{S}_e^g(G)$, $\mathcal{R}_e^g(G)$. We prove (2) and also the inclusion $\mathcal{S}_e(X) \subset \mathcal{R}_e(X)$ which is not mentioned in [9], where the class $\mathcal{R}_e(X)$ is defined. In Section 7 we show stability properties of our classes and also prove that the members of the classes $\mathcal{S}_e^{gg}(G)$ and $\mathcal{S}_e^p(G)$ are uniquely determined by their Clarke subdifferentials. In Section 8 we present some consequences of our results.

2. Preliminaries

In the following, if it is not said otherwise, X will be a real Banach space. We set $S_X := \{x \in X : \|x\| = 1\}$. If $a, b \in X$, then $\overline{a, b}$ denotes the closed segment. By $\operatorname{span} M$ we denote the linear span of $M \subset X$. The equality $X = X_1 \oplus \cdots \oplus X_n$ means that X is the direct sum of non-trivial closed linear subspaces $X_1, \ldots, X_n$ and the corresponding projections $\pi_i : X \to X_i$ are continuous. The symbol $B(x, r)$ will denote the open ball with center x and radius r. The word *"generically"* has the usual sense; it means "at all points except a first category set".

Recall that X is called an Asplund space if each continuous convex function on X is generically Fréchet differentiable and that

$$X \text{ is Asplund if and only if } Y^* \text{ is separable for each separable subspace } Y \subset X. \tag{3}$$

We will need several times the following easy well-known fact.

Lemma 2.1. *Let X be a Banach space, $0 \neq u \in X$, and let $X = W \oplus \operatorname{span}\{u\}$. Then the mapping $w \in W \mapsto w + \mathbb{R}u \in X/\operatorname{span}\{u\}$ is a linear homeomorphism.*

In the following, f is a real function defined on an open subset G of X.

We say that f has a property generically on G, if f has this property at each point of G except a first category set. We allow also the case $G = \emptyset$ (then f is the empty function and has generically each property on G).

Recall (see [24]) that $x^* \in X^*$ is called a strict derivative of f at $a \in G$ if

$$\lim_{(x,y)\to(a,a),\ x\neq y} \frac{f(y) - f(x) - x^*(y - x)}{\|y - x\|} = 0.$$

By [35, Theorem 3],

$$\text{the set of all points of strict differentiability of } f \text{ is a } G_\delta \text{ set.} \tag{4}$$

Strict differentiability is a stronger condition than Fréchet differentiability, but (see e.g. [35, Theorem B, p. 476])

$$f \text{ is generically strictly differentiable iff it is generically Fréchet differentiable.} \tag{5}$$

The directional and one-sided directional derivatives of f at x in the direction v are defined respectively by

$$f'(x,v) := \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \quad \text{and} \quad f'_+(x,v) := \lim_{t\to 0+} \frac{f(x+tv)-f(x)}{t}.$$

The upper (Dini) one-sided directional derivative of f at x in the direction v is defined by

$$d^+ f(x,v) := \limsup_{t\to 0+} (f(x+tv)-f(x))t^{-1}.$$

We will use the obvious fact that, if $a, v \in X$ and $\varphi(t) := f(a+tv)$, $t \in \mathbb{R}$, then

$$d^+ f(a+tv, v) = D^+\varphi(t), \quad \text{whenever } a+tv \in G, \tag{6}$$

where $D^+\varphi$ is the classical right upper Dini derivative.

If f is locally Lipschitz on G then (see [21, Proposition 3.1]), for each $x \in G$,

$$\text{the function } v \mapsto d^+ f(x,v) \text{ is continuous on } X \tag{7}$$

and (see the proof of [21, Lemma 3.5]), for each $v \in X$,

$$\text{the function } x \mapsto d^+ f(x,v) \text{ is Borel measurable on } G. \tag{8}$$

The Fréchet subdifferential of f at a is defined by

$$\partial^F f(a) := \left\{ x^* \in X^* : \ \liminf_{h\to 0} \frac{f(a+h)-f(a)-x^*(h)}{\|h\|} \geq 0 \right\}.$$

We say that f is *Fréchet subdifferentiable* at a, if $\partial^F f(a) \neq \emptyset$.

We will say that f is (ε)-Fréchet differentiable at $a \in G$ for some $\varepsilon > 0$ if there exists $p \in X^*$ such that

$$\limsup_{h\to 0} \frac{|f(a+h)-f(a)-p(h)|}{\|h\|} \leq \varepsilon.$$

We will need the well-known fact that

$$f \text{ is Fréchet differentiable at } a \text{ iff it is } (\varepsilon)\text{-Fréchet differentiable at } a \text{ for each } \varepsilon > 0. \tag{9}$$

This fact follows from the note mentioned in [20] after Definition 1.1 (for an easy proof see [22]).

Further we suppose that f *is locally Lipschitz on* G. Then

$$f^0(a,v) := \limsup_{z\to a, t\to 0+} \frac{f(z+tv)-f(z)}{t}$$

is the Clarke derivative of f at a in the direction v and

$$\partial^C f(a) := \{x^* \in X^* : \ x^*(v) \leq f^0(a,v) \text{ for all } v \in X\}$$

is the Clarke subdifferential of f at a (which is always non-empty). We say that f is (Clarke) *regular* at $x \in G$ if $f^0(x,v) = f'_+(x,v)$ for each $v \in X$. We say that f is *pseudoregular* at $x \in G$ if $f^0(x,v) = d^+f(x,v)$ for each $v \in X$. We say that f is regular (psudoregular) on G, if f is regular (psudoregular) at each point of G.

Let f be a locally Lipschitz function on an open $G \subset X$, $a \in X$, $0 \neq v \in X$, $t \in \mathbb{R}$, and $x := a + tv \in G$. Set $\varphi(\tau) := f(a + \tau v)$, $\tau \in \mathbb{R}$. It is easy to see, that

$$\text{if } f \text{ is pseudoregular (regular) at } x, \text{ then } \varphi \text{ is pseudoregular (regular) at } t. \tag{10}$$

We will need the well-known fact (see [21, Lemma (3.6)] or [19]) that if G is an open subset of a separable Banach space, then

$$\text{each locally Lipschitz function on } G \text{ is generically pseudoregular on } G. \tag{11}$$

We say that f is strictly differentiable at x in a direction v if

$$\lim_{z \to x, t \to 0+} \frac{f(z+tv) - f(z)}{t} = f'(x,v).$$

We say that f is strictly Gâteaux differentiable at x if it is strictly differentiable at x in all directions $v \in X$.

Recall (see [3, Proposition 1], cf. [8, p. 316]) that f is strictly differentiable at x in the direction v, if and only if, $f^0(x,v) = -f^0(x,-v)$, and in this case $f^0(x,v) = -f^0(x,-v) = f'(x,v)$.

Further observe that, for each $v \in X$, the function $x \mapsto f^0(x,v)$ is upper semicontinuous. Consequently, for each $a \in G$,

$$f^0(a,v) \geq \limsup_{x \to a} f^0(x,v) \geq \limsup_{x \to a} d^+f(x,v). \tag{12}$$

Moreover, we obtain that the "oscillation function"

$$h(x) := \omega(f,x,v) := f^0(x,v) + f^0(x,-v) \text{ is upper semicontinuous on } G. \tag{13}$$

Note that clearly $\omega(f,x,v) \geq 0$, and f is strictly differentiable at $x \in G$ in the direction v if and only if $\omega(f,x,v) = 0$.

It is easy to see that f is strictly Gâteaux differentiable at $x \in G$ if and only if f is both Gâteaux differentiable at x and is regular (or pseudoregular) at x.

Recall also that (see e.g. [6, Proposition 1.1]), if f is locally Lipschitz on G, then

$$f \text{ is strictly Gâteaux differentiable at } a \in G \text{ if and only if } \partial^C f(a) \text{ is a singleton.} \tag{14}$$

(Note that, in [6], strict differentiability is called "Fréchet strict differentiability", and Gâteaux strict differentiability is called "strict differentiability".) If X is finite-dimensional and f is Lipschitz on a neighbourhood of $a \in X$, then it is easy to show (see also the note after Ex. 3.64 of [24]) that

$$f \text{ is strictly differentiable at } a \text{ if and only if it is Gâteaux strictly differentiable at } a. \tag{15}$$

We will work also with approximately convex functions in the sense of [25] (which are different from ε-convex functions in the sense of Hyers and Ulam) and with semiconvex (and semiconcave; see [10]) functions with general modulus.

Definition 2.2 ([25]). A real valued function f on an open subset Ω of a Banach space X is called *approximately convex at* $x_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \varepsilon\lambda(1-\lambda)\|x-y\| \tag{16}$$

whenever $\lambda \in [0,1]$ and $x, y \in B(x_0, \delta)$. We say that f is *approximately convex* on Ω if it is approximately convex at each $x_0 \in \Omega$.

Definition 2.3. Denote by $\mathcal{M}$ the set of all functions $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ which are non-decreasing and right continuous at 0. A continuous real valued function f on an open convex subset Ω of a Banach space X is called *semiconvex with modulus* $\omega \in \mathcal{M}$ if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \lambda(1-\lambda)\omega(\|x-y\|)\|x-y\|, \tag{17}$$

whenever $\lambda \in [0,1]$ and $x, y \in \Omega$.

A function is called *semiconvex on* Ω if it is semiconvex on Ω with some modulus $\omega \in \mathcal{M}$.

A function g on Ω is called *semiconcave* (with modulus ω) if the function $-g$ is semiconvex (with modulus ω).

Note that semiconvex functions coincide with Rolewicz's strongly paraconvex functions (cf. [36]).

We will use the following facts about these notions.

If f is a locally Lipschitz function on an open subset of a Banach space, then:

$$\text{if } f \text{ is locally convex, then } f \text{ is locally semiconvex (=strongly paraconvex);} \tag{18}$$

$$\text{if } f \text{ is locally semiconvex, then } f \text{ is approximately convex;} \tag{19}$$

$$\text{if } f \text{ is is approximately convex, then } f \text{ is regular.} \tag{20}$$

Note that (18) and (19) follow easily from definitions, and (20) is proved in [25, Corollary 3.5 and Theorem 3.6]).

Since Borwein-Moors class $S_e(G)$ is defined using Haar null sets in Banach spaces, we recall basic relevant facts about these sets (cf. [8]). Let X be a Banach space. A

Borel set is called Haar null, if there exists a Borel Radon probability measure μ on X such that $\mu(B+x) = 0$ for each $x \in X$. The system of all Borel Haar null sets in X is stable with respect to countable unions and contains no nonempty open set. If $X = \mathbb{R}^n$, then Borel Haar null sets coincide with Borel Lebesgue null sets. We will need the following fact which is an easy consequence of [8, Theorem 2.3].

Lemma 2.4. *Let X be a Banach space, $0 \neq v \in X$, and let W be a topological complement of* span$\{v\}$. *Let $B \subset X$ be a Borel Haar null set. Then there exists a set $D \subset W$ dense in W such that the set $\{t \in \mathbb{R} : \ d + tv \in B\}$ is Lebesgue null for each $d \in D$.*

3. Differentiability results in separable Asplund spaces

Lemma 3.1. *Let X be a Banach space and g a locally Lipschitz function on an open set $B \subset X$. Let $a \in B$, $s \geq 0$, $u \in S_X$, and let $\overline{a, b} \subset B$, where $b := a + su$. Let g be essentially smooth on a generic line parallel to u. Let $\alpha \in \mathbb{R}$, and let there exist a first category set $M \subset X$ such that $d^+g(x, u) \geq \alpha$ for each $x \in B \setminus M$. Then $g(b) - g(a) \geq \alpha s$.*

Proof. Choose a topological complement W of span$\{u\}$. Using the canonical isomorphism between $X = W \oplus \text{span}\{u\}$ and $W \times \mathbb{R}$, and using the Kuratowski-Ulam theorem ("Fubini theorem for category", see [27, p. 56]) in $W \times \mathbb{R}$, we obtain that there exists a residual set $S_1 \subset W$ such that $M \cap (w + \mathbb{R}u)$ is a first category set in the line $w + \mathbb{R}u$ for each $w \in S_1$. Further, using Lemma 2.1, we obtain a residual set $S_2 \subset W$ such that g is essentially smooth on the line $w + \mathbb{R}u$ for each $w \in S_2$. Write $a = w_a + \lambda u$, $w_a \in W$, $\lambda \in \mathbb{R}$, and, for each $n \in \mathbb{N}$, choose $w_n \in S_1 \cap S_2$ with $\|w - w_n\| < 1/n$. Setting $a_n := w_n + \lambda u$, we have $\|a - a_n\| < 1/n$ and $L_n := a_n + \mathbb{R}u = w_n + \mathbb{R}u$. Consequently, $M \cap L_n$ is a first category set in the line L_n and g is essentially smooth on L_n. Obviously, for each sufficiently large n, we have $\overline{a_n, b_n} \subset B$, where $b_n := a_n + su$. Set, for any such n, $h_n(t) = g(a_n + tu)$, $t \in [0, s]$. Then h_n is a.e. strictly differentiable on $(0, s)$, and since $D^+h_n(t) \geq \alpha$ generically on $(0, s)$ by (6), we easily infer from (12) that $h_n'(t) \geq \alpha$ for a.e. $t \in (0, s)$. Consequently

$$g(b_n) - g(a_n) = h_n(s) - h_n(0) = \int_0^s h_n'(t) \, dt \geq \alpha s.$$

Since $g(b_n) \to g(b)$ and $g(a_n) \to g(a)$, we obtain $g(b) - g(a) \geq \alpha s$. □

Lemma 3.2. *Let X be a separable Banach space, f a locally Lipschitz function on an open set $G \subset X$, and $p \in X^*$. Then the function*

$$g(x) := \limsup_{h \to 0} \frac{|f(x+h) - f(x) - p(h)|}{\|h\|}$$

is Borel measurable on G.

Proof. Since g is clearly Borel measurable if and only if it is locally Borel measurable in G, and each Lipschitz function on $B(a, r) \subset G$ can be extended to a Lipschitz function on X, we can suppose without any loss of generality that $G = X$. Then, for each $0 \neq h \in X$, the function $x \mapsto g(x,h) := |f(x+h) - f(x) - p(h)|/\|h\|$ is clearly continuous on X. So, for each $n \in \mathbb{N}$, the function $g_n(x) := \sup\{g(x,h) : 0 < \|h\| < 1/n\}$ is lower semicontinuous on X. Therefore, $g(x) = \lim_{n\to\infty} g_n(x)$ is Borel measurable on X. □

Our main results follow from the following proposition.

Proposition 3.3. *Let $X = X_1 \oplus \cdots \oplus X_n$ be a Banach space with a separable dual X^*. Let $G \subset X$ be an open set and $f : G \to \mathbb{R}$ a locally Lipschitz function. Let, for each $1 \leq i \leq n$, there exists a dense set $D_i \subset S_{X_i}$ such that, for each $v \in D_i$, f is essentially smooth on a generic line parallel to v. Then f is generically Fréchet differentiable on G.*

Proof. Choose $K > 0$ such that $K\|v\| \geq \sum_{i=1}^n \|v_i\|$, whenever $v = \sum_{i=1}^n v_i$ and $v_i \in X_i$ for each $1 \leq i \leq n$.

Suppose on the contrary that the set N of all points $x \in G$ at which f is not Fréchet differentiable is of the second category (i.e., is not of the first category).

Choose a sequence $(p_j)_{j=1}^\infty$ which is dense in X^*, and define, for $m, j \in \mathbb{N}$, the set N_m^j as the set of all $x \in G$ for which

$$\limsup_{h\to 0} \frac{|f(x+h) - f(x) - p_j(h)|}{\|h\|} > \frac{1}{m}, \quad \text{and} \tag{21}$$

$$d^+ f(x,v) \geq p_j(v) - \frac{1}{2Km} \quad \text{for each } v \in S_X. \tag{22}$$

First we will show that each set N_m^j is Borel. To this end observe that each set A_m^j of all $x \in G$ for which (21) holds is Borel by Lemma 3.2. Second, denote by C_m^j the set of all $x \in G$ for which (22) holds. Let E be a countable dense subset of S_X. By (7), we can write equivalently "$v \in E$" instead of "$v \in S_X$" in (22). Thus (8) easily implies that C_m^j is Borel. So each $N_m^j = A_m^j \cap C_m^j$ is Borel.

Further we will show that

$$N_m^j \text{ is of the second category for some } m, j \in \mathbb{N}. \tag{23}$$

To this end denote by A the set of points $x \in G$ at which f is not pseudoregular. By (11), A is a first category set. So, to prove (23), it is sufficient to show that

$$N \setminus A \subset \bigcup_{m,j=1}^{\infty} N_m^j. \tag{24}$$

For each $x \in G$, choose a functional $\varphi^x \in \partial^C f(x)$. To prove (24), choose an arbitrary $x \in N \setminus A$. By (9), we can choose $m \in \mathbb{N}$ such that (21) holds for each $j \in \mathbb{N}$. Now choose $j \in \mathbb{N}$ such that $\|\varphi^x - p_j\| \leq \frac{1}{2Km}$. Since $x \notin A$, we obtain that, for each $v \in S_X$,

$$d^+ f(x,v) = f^0(x,v) \geq \varphi^x(v) \geq p_j(v) - \frac{1}{2Km},$$

and thus $x \in N^j_m$. So (24) is proved and we can choose m, j such that N^j_m is of the second category.

Denoting $g := f - p_j$, we easily see that, for each $x \in N^j_m$, we have

$$\limsup_{h \to 0} \frac{|g(x+h) - g(x)|}{\|h\|} > \frac{1}{m}, \tag{25}$$

and

$$d^+ g(x,v) \geq -\frac{1}{2Km} \quad \text{for each } v \in S_X. \tag{26}$$

Further, obviously

$$g \text{ is essentially smooth on a line } L \text{ if and only if } f \text{ is essentially smooth on } L. \tag{27}$$

Since N^j_m is a second category set which is Borel (and so has the Baire property), we can choose a ball $B = B(x_0, r) \subset G$ and a first category set $M \subset X$, such that $B \setminus M \subset N^j_m$.

Since N^j_m is dense in B, by (25) we can find $x, y \in B(x_0, \frac{r}{8K})$ such that $|g(y)-g(x)| > (1/m)\|y-x\|$. We can and will suppose that

$$g(y) - g(x) < -\frac{1}{m}\|y - x\|. \tag{28}$$

Write $y - x = \sum_{i=1}^n v_i$ with $v_i \in X_i$. For each $\varepsilon > 0$ and $1 \leq i \leq n$ choose $u_i \in D_i$ and $s_i > 0$ such that $\|v_i - s_i u_i\| < \varepsilon$. Observe that

$$K\|y - x\| \geq \sum_{i=1}^n \|v_i\| \geq \sum_{i=1}^n s_i - n\varepsilon. \tag{29}$$

Set $c_0 := x$, $c_k := x + \sum_{i=1}^k s_i u_i$ for $1 \leq k \leq n$, and denote $y_\varepsilon := c_n$. Obviously, $\|y_\varepsilon - y\| \leq n\varepsilon$. Since $\|y - x\| \leq \frac{r}{4K}$, we easily obtain by (29) that, for all sufficiently small $\varepsilon > 0$, we have $\overline{c_{k-1}, c_k} \subset B$, $1 \leq k \leq n$. Observe that $c_k = c_{k-1} + s_k u_k$, $1 \leq k \leq n$. Since $u_k \in D_k$ (and (27) holds), $B \setminus M \subset N^j_m$ (and so (26) holds for each $x \in B \setminus M$), we can apply Lemma 3.1 with $a := c_{k-1}, b := c_k, s := s_k, u := u_k, \alpha := -\frac{1}{2Km}$, and obtain

$$g(c_k) - g(c_{k-1}) \geq -\frac{s_k}{2Km} \quad \text{for each } 1 \leq k \leq n.$$

Consequently, using also (29), we obtain

$$g(y_\varepsilon) - g(x) = \sum_{k=1}^n (g(c_k) - g(c_{k-1})) \geq -\frac{1}{2Km}\sum_{k=1}^n s_k \geq -\frac{1}{2Km}(K\|y - x\| + n\varepsilon).$$

Since $\lim_{\varepsilon \to 0+} y_\varepsilon = y$, we obtain

$$g(y) - g(x) = \lim_{\varepsilon \to 0+} (g(y_\varepsilon) - g(x)) \geq -\frac{1}{2m}\|y - x\|,$$

which contradicts (28). □

Corollary 3.4. *Let X be a separable Asplund space and $G \subset X$ an open set. Then:*

(*i*) *Each $f \in \mathcal{S}_e^{gd}(G)$ (resp. $f \in \mathcal{S}_e^{gg}(G)$, $f \in \mathcal{S}_e^{g}(G)$, $f \in \mathcal{S}_e^{l}(G)$) is generically Fréchet differentiable on G.*

(*ii*) *If $X = X_1 \oplus \cdots \oplus X_n$ and $f \in \mathcal{S}_e^{p}(G)$, then f is generically Fréchet differentiable on G.*

4. Separable reduction for generic Fréchet differentiability, rich families, and related lemmas

We will need the following [35, Lemma 1].

Lemma 4.1. *Let X be a normed linear space, S a Banach space, $G \subset X$ an open set, and let $f : G \to S$ be an arbitrary mapping. Then there exists a mapping t which assigns to each closed separable subspace V of X a separable closed space $V \subset t(V) \subset X$ such that the following assertion holds: If Y is a closed subspace of X such that the set $D(Y) := \bigcup\{V : t(V) \subset Y\}$ is dense in Y, then f is strictly differentiable at each point of Y at which $f|_Y$ is strictly differentiable.*

The following useful notion is taken from [9].

Definition 4.2. Let X be a normed linear space. A family $\mathcal{F}$ of closed separable subspaces of X is called a *rich family* if:

(R1) If $Y_i \in \mathcal{F}$ ($i \in \mathbb{N}$) and $Y_1 \subset Y_2 \subset \dots$, then $\overline{\bigcup\{Y_n : n \in \mathbb{N}\}} \in \mathcal{F}$.

(R2) For each closed separable subspace Y_0 of X there exists $Y \in \mathcal{F}$ such that $Y_0 \subset Y$.

A basic fact ([9, Proposition 1.1]) concerning rich families is the following.

Lemma 4.3. *Let X be a normed linear space and let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be rich families of closed separable subspaces of X. Then $\mathcal{F} := \bigcap\{\mathcal{F}_n : n \in \mathbb{N}\}$ is also a rich family of closed separable subspaces of X.*

We will need also the following simple facts.

Lemma 4.4. *Let X be a normed linear space and let $X = X_1 \oplus \cdots \oplus X_n$. Let $\mathcal{F}_k$ be a rich family of closed separable subspaces of X_k, $1 \le k \le n$. Then*

$$\mathcal{F} := \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_n := \{Y^1 \oplus \cdots \oplus Y^n : \ Y^k \in \mathcal{F}_k, \ 1 \le k \le n\}$$

is a rich family in X.

Proof. Suppose that $Z_1 \subset Z_2 \subset \cdots$ are spaces belonging to $\mathcal{F}$ and let $Z_i = Y_i^1 \oplus \cdots \oplus Y_i^n$. Then clearly $Y_1^k \subset Y_2^k \subset \cdots$, $1 \le k \le n$. So, since $\mathcal{F}_k$ is a rich family, we have $\overline{\bigcup_{i=1}^\infty Y_i^k} \in \mathcal{F}_k$, $1 \le k \le n$. It is easy to check that

$$\overline{\bigcup_{i=1}^\infty Z_i} = \overline{\bigcup_{i=1}^\infty Y_i^1 \oplus \cdots \oplus Y_i^n} = \overline{\bigcup_{i=1}^\infty Y_i^1} \oplus \cdots \oplus \overline{\bigcup_{i=1}^\infty Y_i^n}.$$

Thus $\mathcal{F}$ satisfies the property (R1) from Definition 4.2.

To prove (R2), consider an arbitrary closed separable space $Z_0 \subset X$. Then clearly

$$Z_0 \subset \overline{\pi_1(Z_0)} \oplus \cdots \oplus \overline{\pi_n(Z_0)},$$

where $\pi_k : X \to X_k$ are natural projections. Since $\overline{\pi_k(Z_0)}$, $1 \leq k \leq n$, is clearly a closed separable subspace of X_k, we can find $Y_k \in \mathcal{F}_k$ containing $\overline{\pi_k(Z_0)}$. So $Z_0 \subset Y_1 \oplus \cdots \oplus Y_n \in \mathcal{F}$, and (R2) is proved. □

Lemma 4.5. *Let X be a normed linear space and let t be a mapping which assigns to each closed separable subspace V of X a separable closed space $V \subset t(V) \subset X$. Denote by $\mathcal{F}_t$ the system of all closed separable subspaces Y of X such that $D(Y) := \bigcup\{V : t(V) \subset Y\}$ is dense in Y. Then $\mathcal{F}_t$ is a rich family.*

Proof. Let $Y_i \in \mathcal{F}_t$ $(i \in \mathbb{N})$ and $Y_1 \subset Y_2 \subset \ldots$. Denote $Y := \overline{\bigcup\{Y_i : i \in \mathbb{N}\}}$. Since clearly $D(Y_i) \subset D(Y)$, $i \in \mathbb{N}$, and $D(Y_i)$ is dense in Y_i, it is easy to see that $D(Y)$ is dense in Y. So condition (R1) of Definition 4.2 holds.

To prove (R2), let Y_0 be a closed separable subspace of X. Define $Y_1 := t(Y_0)$, $Y_2 := t(Y_1)$, and so on. Setting $Y := \overline{\bigcup\{Y_n : n \in \mathbb{N}\}}$, we have clearly $Y_0 \subset Y$, and also $Y \in \mathcal{F}_t$, since each Y_n, $n \in \mathbb{N}$, is obviously contained in $D(Y)$. □

The following fact is contained (with another proof) already in [22], where the method of rich families in the differentiability theory is developed.

Lemma 4.6. *Let X be a Banach space and $M \subset X$ a residual set. Then there exists a rich family $\mathcal{F}_M$ of closed separable subspaces of X such that $M \cap Y$ is residual in Y for each $Y \in \mathcal{F}$.*

Proof. By [35, Lemma 2] there exists a mapping s which assigns to each closed separable subspace V of X a separable closed space $V \subset s(V) \subset X$ such that the following assertion holds: If Y is a closed subspace of E such that the set $C(Y) := \bigcup\{V : s(V) \subset Y\}$ is dense in Y, then $M \cap Y$ is residual in Y. So, denoting by $\mathcal{F}_M$ the system of all closed separable subspaces Y of X such that $C(Y)$ is dense in Y, we have that $\mathcal{F}_M$ has the desired property. Moreover, using Lemma 4.5 with $t := s$, we obtain that $\mathcal{F}_M$ is a rich family. □

The following "separable reduction theorem for generic Fréchet differentiability" improves [35, Theorem 8] (which shows that $(ii) \Rightarrow (i)$ holds if $\mathcal{F}$ from (ii) is the family of all separable closed subspaces of X). Note that [22] contains a separable reduction statement (proved by a quite different method) which rather easily implies Theorem 4.7.

Theorem 4.7. *Let X be a normed linear space, S a Banach space, $G \subset X$ an open set, and let $f : G \to S$ be an arbitrary mapping. Then the following conditions are equivalent.*

- (i) *f is generically Fréchet differentiable.*
- (ii) *There exists a rich family $\mathcal{F}$ of closed separable subspaces of X such that $f|_{Y \cap G}$ is generically Fréchet differentiable on $Y \cap G$ for each $Y \in \mathcal{F}$.*

Proof. First suppose that (i) holds. Then there exists a first category set $N \subset G$ such that f is Fréchet differentiable at each point of $G \setminus N$. Set $M := X \setminus N$ and let $\mathcal{F} := \mathcal{F}_M$ be a rich family from Lemma 4.6. Then, for each $Y \in \mathcal{F}$, the set $M \cap Y$ is residual in Y, and so $N \cap Y$ is a first category subset of Y. So (ii) holds, since $f|_{Y \cap G}$ is clearly Fréchet differentiable at each point of $(Y \cap G) \setminus N = (Y \cap G) \setminus (N \cap Y)$.

Now suppose that (ii) holds. Let t be the mapping from Lemma 4.1 and let $\mathcal{F}_t$ be the rich family from Lemma 4.5. Then $\mathcal{F}^* := \mathcal{F} \cap \mathcal{F}_t$ is a rich family by Lemma 4.3. Let now $H \neq \emptyset$ be an arbitrary open subset of G. Since $\mathcal{F}^*$ is a rich family, by condition (R2) we can clearly choose $Y \in \mathcal{F}^*$ with $Y \cap H \neq \emptyset$. Since $Y \in \mathcal{F}$, we have by (ii) that $f|_{Y \cap G}$ is generically Fréchet differentiable (and so also generically strictly differentiable by (5)) on $Y \cap G$. Thus we can choose $y_0 \in Y \cap H$ at which $f|_Y$ is strictly differentiable. Since $Y \in \mathcal{F}_t$, we obtain that f is strictly differentiable at y_0 by the definitions of t and $\mathcal{F}_t$. So f is strictly differentiable at all points of a dense subset of G. Using (4), we obtain (i). □

Lemma 4.8. *Let X be a Banach space, $G \subset X$ an open set, and let $f : G \to \mathbb{R}$ be a function. Let $0 \neq v \in X$ and suppose that f is essentially smooth on a generic line parallel to v. Then there exists a rich family $\mathcal{F}_v$ of closed separable subspaces Y of X such that, for each $Y \in \mathcal{F}_v$, the function $f|_{G \cap Y}$ is essentially smooth on a generic line in Y parallel to v.*

Proof. Let W be a topological complement of $V := \operatorname{span}\{v\}$; so $X = W \oplus V$. Using Lemma 2.1, we have that the set M of all $w \in W$ such that f is essentially smooth on the line $w + V$ is residual in W. Let $\mathcal{F}_M$ be the rich family of closed separable subspaces of W from Lemma 4.6. The family $\mathcal{F}^* := \{V\}$ is obviously a rich family in V, so Lemma 4.4 implies that

$$\mathcal{F}_v := \mathcal{F}_M \oplus \mathcal{F}^* = \{Z \oplus V : \ Z \in \mathcal{F}_M\}$$

is a rich family in X.

Let $Y \in \mathcal{F}_v$, i.e. $Y = Z \oplus V$ with $Z \in \mathcal{F}_M$. By the definition of $\mathcal{F}_M$, the set $Z \cap M$ is residual in Z. So, $f|_{G \cap Y}$ is essentially smooth on the generic line in Y parallel to v by Lemma 2.1. □

Lemma 4.9. *Let X be a Banach space and let $B \subset X$ be a Borel Haar null set. Then there exists a closed separable space $Z \subset X$ such that $B \cap Y$ is Haar null in Y for each closed separable space Y with $Z \subset Y \subset X$.*

Proof. Let μ be a Borel Radon probability measure on X such that $\mu(B + x) = 0$ for each $x \in X$. Since μ is Radon, the support of μ is a closed separable set. So, there exists a closed separable space $Z \subset X$ such that $\mu(X \setminus Z) = 0$.

Now consider an arbitrary closed separable space Y with $Z \subset Y \subset X$. The restriction ν of μ to the σ-algebra of all Borel subsets of Y is clearly a probability Radon measure on Y. If $y \in Y$, then clearly $(B \cap Y) + y = (B + y) \cap Y$, and so

$$\nu((B \cap Y) + y) = \nu((B + y) \cap Y) = \mu(B + y) = 0.$$

So $B \cap Y$ is Haar null in Y. □

Lemma 4.10. *Let X be a Banach space and let $D \subset S_X$ be a dense set. Then for each closed separable space $Y_0 \subset X$ there exists a closed separable space Y with $Y_0 \subset Y \subset X$ such that $D \cap Y$ is dense in S_Y.*

Proof. We will define inductively a sequence $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ of closed separable subspaces of X such that, for each $n \in \mathbb{N}$,

$$S_{Y_{n-1}} \subset \overline{D \cap S_{Y_n}}. \tag{30}$$

The space Y_0 is already defined. Let $n \in \mathbb{N}$ and suppose that Y_{n-1} is defined. Choose a countable set $B_{n-1} \subset D$ such that $S_{Y_{n-1}} \subset \overline{B_{n-1}}$ (to this end, choose a countable dense set A of $S_{Y_{n-1}}$, for each $v \in A$ a countable set $B_v \subset D$ with $v \in \overline{B_v}$, and set $B_{n-1} := \bigcup_{v \in A} B_v$) and define $Y_n := \overline{\operatorname{span} B_{n-1}}$. Then clearly $Y_n \supset Y_{n-1}$ and (30) holds. Setting $Y := \overline{\bigcup_{n=1}^{\infty} Y_n}$, we easily see that $Y_0 \subset Y$ and $D \cap Y$ is dense in S_Y. □

5. Differentiability results in non-separable Asplund spaces

Theorem 5.1. *Let $X = X_1 \oplus \cdots \oplus X_n$ be an Asplund space, $G \subset X$ an open set, and let $f \in \mathcal{S}_e^p(G)$. Then f is generically Fréchet differentiable on G.*

Proof. Let $\mathcal{F}_i$ be the system of all closed separable subspaces of X_i, $i = 1, \ldots, n$. Then $\mathcal{F}_i$ is clearly a rich family in X_i, and so

$$\mathcal{F} := \{Y^1 \oplus \cdots \oplus Y^n : \ Y^i \in \mathcal{F}_i, \ 1 \le i \le n\}$$

is a rich family in X by Lemma 4.4. For each $Y \in \mathcal{F}$, the function $f|_{Y \cap G}$ clearly belongs to $\mathcal{S}_e^p(Y \cap G)$, and so it is generically Fréchet differentiable on $Y \cap G$ by (3) and Corollary 3.4(ii). Thus f is generically Fréchet differentiable on G by Theorem 4.7. □

Theorem 5.2. *Let X be an Asplund space, $\emptyset \neq G \subset X$ an open set, and let $f \in \mathcal{S}_e^{gd}(G)$ (i.e., f is locally Lipschitz on G and there exists a set $D \subset S_X$ dense in S_X such that, for each $0 \neq v \in D$, f is essentially smooth on a generic line parallel to v). Then f is generically Fréchet differentiable on G.*

Proof. Let t be the mapping from Lemma 4.1. Further, for each $0 \neq v \in D$, let $\mathcal{F}_v$ be the rich family of closed separable subspaces of X from Lemma 4.8. For each countable $C \subset D$ we define the rich family (cf. Lemma 4.3) $\mathcal{F}_C := \bigcap\{\mathcal{F}_v : v \in C\}$.

Now suppose that an arbitrary closed separable space $Y_0 \subset X$ is given. We will construct a closed separable space $Y \supset Y_0$ such that

(i) $f|_{G \cap Y} \in \mathcal{S}_e^{gd}(G \cap Y)$ (in Y), and
(ii) f is strictly differentiable at every point of $G \cap Y$, at which $f|_{G \cap Y}$ is strictly differentiable (in Y).

To this end, we will construct inductively closed separable spaces $Y_0 \subset Y_1 \subset \cdots$ and countable sets $D_n \subset S_{Y_n} \cap D$, $n = 0, 1, \dots$, so that, for each $n \in \mathbb{N}$:

$$t(Y_{n-1}) \subset Y_n, \tag{31}$$

$$S_{Y_{n-1}} \subset \overline{D_n}, \tag{32}$$

$$Y_n \in \mathcal{F}_{D_0 \cup \cdots \cup D_n}. \tag{33}$$

The space Y_0 is already given and we set $D_0 := \emptyset$. If $n \in \mathbb{N}$ and $Y_0, \dots, Y_{n-1}$ and $D_0, \dots, D_{n-1}$ are already defined, we first choose by Lemma 4.10 a separable closed space $Z_n \supset t(Y_{n-1})$ such that $D \cap Z_n$ is dense in S_{Z_n}, then a countable set D_n which is a dense subset of $D \cap Z_n$ and then $Y_n \in \mathcal{F}_{D_0 \cup \cdots \cup D_n}$ with $Z_n \subset Y_n$. Set $Y := \overline{\bigcup_{n=1}^{\infty} Y_n}$.

To prove (i), observe that $S_Y = \overline{\bigcup_{n=1}^{\infty} S_{Y_n}}$ and so (32) implies that $D^* := \bigcup_{n=1}^{\infty} D_n$ is dense in S_Y. If $v \in D^*$, then $v \in D_n$ for some $n \in \mathbb{N}$, and thus $Y_k \in \mathcal{F}_v$ for each $k \geq n$ by (33). Since $\mathcal{F}_v$ is a rich family, we obtain $Y \in \mathcal{F}_v$, which implies that f is essentially smooth on a generic line in Y parallel to v.

To prove (ii), observe that $\bigcup_{n=1}^{\infty} Y_n$ is dense in Y and $t(Y_n) \subset Y$ by (31). So (ii) holds by the choice of t (see Lemma 4.1).

Let now $H \neq \emptyset$ be an arbitrary open subset of G. Choose $h \in H$, set $Y_0 := \operatorname{span}\{h\}$ and find a closed separable space $Y \supset Y_0$ such that (i) and (ii) hold. By (i), (3) and Corollary 3.4(i), we obtain that $f|_{Y \cap G}$ is generically Fréchet differentiable (and so also generically strictly differentiable by (5)) on $Y \cap G$ (in Y). Since $Y \cap H \neq \emptyset$, we obtain by (ii) that f is strictly differentiable at a point of H. So f is strictly differentiable at all points of a dense subset of G. Using (4), we obtain that f is generically Fréchet differentiable on G. □

Remark 5.3. In the first version of the present article, which I presented at 38th Winter School in Abstract Analysis (January 2010), Theorem 5.1 was proved using methods of [35] only. However, in this time I did not know the method of rich families of [9] and [22] which was the reason why I was not able to prove Theorem 5.2. Using this method, I proved Theorem 5.2 in March 2010. Note that M. Cúth (independenty and in the same time) in his diploma thesis [11] (supervised by O. Kalenda) generalized Proposition 3.3 (using Proposition 3.3 for which he refers to my lecture) to the case of an arbitrary Asplund space X (and so obtained a result more general than Theorem 5.2). His proof does not use the method of rich families, but uses a separable reduction argument based on the (set-theoretic) method of elementary submodels. It seems that, in the differentiability theory, this method has similar strength as the method of rich families.

Using the notion of a rich family, the authors of [9] defined the class $\mathcal{R}_e(G)$. Now we recall this definition and define some related classes.

Definition 5.4. Let X be an arbitrary Banach space and let $\emptyset \neq G \subset X$ be an open set. We denote by $\mathcal{R}_e(G)$ ($\mathcal{R}_e^l(G)$, $\mathcal{R}_e^g(G)$, $\mathcal{R}_e^{gg}(G)$, $\mathcal{R}_e^{gd}(G)$) the class of all those locally Lipschitz functions f on G for which there is a rich family $\mathcal{F}$ of closed separable subspaces of X such that $f|_{G \cap Y}$ belongs to $\mathcal{S}_e(G \cap Y)$ ($\mathcal{S}_e^l(G \cap Y)$, $\mathcal{S}_e^g(G \cap Y)$, $\mathcal{S}_e^{gg}(G \cap Y)$, $\mathcal{S}_e^{gd}(G \cap Y)$) for each $Y \in \mathcal{F}$ with $G \cap Y \neq \emptyset$.

Obviously, $\mathcal{R}_e^l(G) \subset \mathcal{R}_e^g(G) \subset \mathcal{R}_e^{gg}(G) \subset \mathcal{R}_e^{gd}(G)$. Using Corollary 3.4($i$) and Theorem 4.7, we immediately obtain the following result.

Proposition 5.5. *Let X be an Asplund space, $\emptyset \neq G \subset X$ an open set, and let $f \in \mathcal{R}_e^{gd}(G)$ (resp. $f \in \mathcal{R}_e^{gg}(G)$, $f \in \mathcal{R}_e^g(G)$, $f \in \mathcal{R}_e^l(G)$). Then f is generically Fréchet differentiable on G.*

Remark 5.6. I do not know whether Theorem 5.2 can be deduced from Proposition 5.5, since I do not know whether $\mathcal{S}_e^{gd}(G) \subset \mathcal{R}_e^{gd}(G)$ (cf. Proposition 6.5, which shows that the inclusion $\mathcal{S}_e(G) \subset \mathcal{R}_e(G)$ holds).

6. Comparison of classes defined via essential smoothness and via linear essential smoothness

Recall that a function f defined on an open subset G of a Banach set X is called *essentially smooth* on G, if f is locally Lipschitz and, for each $v \in X$, f is strictly differentiable in the direction v at all points $x \in G$ except a Haar null set. The class of all essentially smooth functions on G is denoted (see [8]) by $\mathcal{S}_e(G)$. In the case $X = \mathbb{R}^n$, essentially smooth functions were considered in [29] under the name "primal functions" (see [2], where, in the case of a separable space X, the name "essentially strictly differentiable functions" was used).

As observed in [6, Proposition 4.1, Corollary 4.3] and [8, Theorem 3.3], essential smoothness is equivalent to some formally weaker properties. We will formulate only two consequences of these observations. The first one is immediate and the second quite easy.

Proposition BM. *Let f be a locally Lipschitz function defined on an open subset G of a Banach space X. Then the condition*

(i) *f is pseudoregular (resp. regular) on G except a Haar null set*

implies $f \in \mathcal{S}_e(G)$. If X is separable, then $f \in \mathcal{S}_e(G)$ if and only if (i) holds.

Corollary BM. *Let φ be a locally Lipschitz function defined on an open set $H \subset \mathbb{R}$. Then $\varphi \in \mathcal{S}_e(H)$ if and only if $D^+\varphi(x) = \limsup_{y \to x+} D^+\varphi(y)$ for a.e. $x \in H$ (where $D^+\varphi$ is the classical right upper Dini derivative of φ).*

Remark 6.1. Proposition BM and Corollary BM immediately imply (cf. Remark 1.2) that the following conditions are equivalent:

(i) f is essentially smooth on a line $L = a + \mathbb{R}v$,
(ii) $\varphi(t) := f(a + tv)$ is a.e. pseudoregular on its domain,
(iii) φ is a.e. regular on its domain,
(iv) $D^+\varphi(x) = \limsup_{y \to x+} D^+\varphi(y)$ for a.e. x from its domain.

So, also our classes defined via essential smoothness on lines have several equivalent (formally weaker) definitions.

The class $\mathcal{S}_e(G)$ was generalized in [3] (by the same definition, since the definition of strict differentiability in a direction has sense also for mappings, see [3, Definition 2.1]) to the class $\mathcal{S}_e(G, Y)$ of (locally Lipschitz) mappings $f : G \to Y$, where Y

is a Banach space. (Note that, in the case of a nonseparable space Y, there is an inconsistency in [3], since the theorems in [3] are proved under the definition described above which uses property (P5) from Introduction, but the definition given in [3, p. 15] uses property (P4), and so is probably strictly stronger.) Naturally (cf. Remark 1.2), we can define also corresponding classes

$$\mathcal{S}_e^l(G,Y),\ \mathcal{S}_e^g(G,Y),\ \mathcal{S}_e^{gg}(G,Y),\ \mathcal{S}_e^{gd}(G,Y),\ \mathcal{S}_e^p(G,Y). \tag{34}$$

However, in this article, we will use these classes of mapping only in Proposition 7.4 and Proposition 8.2, and so we omit the (obvious) definitions.

It is easy to see that

$$f = (f_1,\dots,f_m) \in \mathcal{S}_e(G,\mathbb{R}^m) \ \text{ if and only if } f_i \in \mathcal{S}_e(G),\ 1 \le i \le m. \tag{35}$$

The main aim of this section is to prove Proposition 6.4 which shows that the class $\mathcal{S}_e^g(G)$ (defined using one-dimensional Lebesgue measure and Baire category) is larger than the class $\mathcal{S}_e(G)$ (defined using Haar null sets). We start with an example in $\mathbb{R}^2$.

Lemma 6.2. *Let $\emptyset \neq A \subset H \subset \mathbb{R}^2$, and let A, H be open sets. Then there exists a function $f \in S_e^g(H)$ such that $f|_A \notin S_e(A)$.*

Proof. Let $B \subset \mathbb{R}^2$ be a ("line") Besicovitch set (see [23, Theorem 18.11]). Then $\lambda_2(B) = 0$ and for each $0 \neq v \in \mathbb{R}^2$ there exists a line $L_v \subset B$ parallel to v. Let $D \subset \mathbb{R}^2$ be a countable dense set and let $\tilde{B} := \bigcup_{d\in D}(B+d)$. Then $\lambda_2(\tilde{B}) = 0$ and so we can choose a compact set $F \subset A \setminus \tilde{B}$ with $\lambda_2(F) > 0$. Set $f(x) := \operatorname{dist}(x,F)$, $x \in H$. Then f is Lipschitz on H.

To prove that $f \in S_e^g(H)$, choose an arbitrary $0 \neq v \in \mathbb{R}^2$ and set $V := \operatorname{span}\{v\}$. Then $\tilde{F} := \{x + \mathbb{R}v :\ x \in F\}$ is a compact subset of $\mathbb{R}^2/V$. Further note that f is locally semiconcave (with linear modulus) on $H \setminus F$ (see [10, Proposition 2.2.2]) and therefore (see (19), (20), (10), and Remark 6.1) f is essentially smooth on each line which does not intersect F, i.e. on each line from $(\mathbb{R}^2/V) \setminus \tilde{F}$. Since the set $\{L_v + d :\ d \in D\}$ is dense in $\mathbb{R}^2/V$ and disjoint with $\tilde{F}$, we obtain that $\tilde{F}$ is nowhere dense, and so f is essentially smooth on a generic line parallel to v.

To prove that $f|_A \notin S_e(A)$, it is (by (1) and (15)) sufficient to show that f is strictly differentiable at no $x \in F$. So, suppose on the contrary that f is strictly differentiable at some $x \in F$. Then clearly $f'(x) = 0$, which contradicts the well-known fact that $\|f'(y)\| = 1$ for almost all $y \in H \setminus F$. $\square$

Lemma 6.3. *Let X be a Banach space, $\emptyset \neq G \subset X$ an open set, and let f be a locally Lipschitz function on G. Let $0 \neq v \in X$, and let W be a topological complement of* $\operatorname{span}\{v\}$. *Let D be the set of all $w \in W$ for which there exists a Lebesgue null set $N_w \subset \mathbb{R}$, such that f is strictly differentiable in the direction v at all points of $G \cap \{w + tv : t \in \mathbb{R} \setminus N_w\}$. Suppose that D is dense in W. Then D is residual in W. Consequently, f is essentially smooth on a generic line parallel to v.*

Proof. Suppose, on the contrary, that $S := W \setminus D$ is of the second category in W. For each $w \in W$ and $n \in \mathbb{N}$, denote by $A(w,n)$ the set of all $t \in \mathbb{R}$, for which $w + tv \in G$ and $\omega(f, w+tv, v) \geq 1/n$ (see (13) for the definition of ω). Using (13), we easily obtain that all $A(w,n)$ are Borel. So, for each $w \in S$, there exists $n \in \mathbb{N}$ such that $\lambda A(w,n) > 0$. Set $\mathcal{I} := \{[r,s] : r < s \text{ are rational}\}$. Further, for $k \in \mathbb{N}$ and $I \in \mathcal{I}$, let

$$S(k,I) := \{w \in S : \ w + Iv \subset G \text{ and } \lambda(A(w,k) \cap I) \geq 1/k\}.$$

It is easy to see that $S = \bigcup\{S(k,I) : k \in \mathbb{N}, I \in \mathcal{I}\}$, and so we can choose $k \in \mathbb{N}$ and $I \in \mathcal{I}$ such that $S(k,I)$ is of the second category in W. Thus there exists a nonempty $U \subset W$ open in W such that $S(k,I)$ is dense in U. Choosing an arbitrary $w_0 \in S(k,I) \cap U$, we can clearly find an open subset V of W such that $w_0 \in V \subset U$ and

$$w + Iv \subset G \ \text{ for each } w \in V. \tag{36}$$

Obviously, we can choose $d \in D \cap V$ and a sequence (s_i) in $S(k,I)$ which converge to d. Set

$$A := \limsup_{i \to \infty} A(s_i, k) = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} A(s_i, k).$$

Obviously, $\lambda(A \cap I) \geq 1/k$. Using (36) and (13), we obtain $A \cap I \subset A(d,k)$. So $A \cap I \subset N_d$, which is a contradiction.

Obviously, if $w \in D$, then f is essentially smooth on the line $w + \mathbb{R}v$ (cf. Remark 1.2). So, using Lemma 2.1, we obtain that f is essentially smooth on a generic line parallel to v. □

Proposition 6.4. *Let X be a Banach space with $\dim X > 1$ and let $\emptyset \neq G \subset X$ be an open set. Then $S_e(G) \subset S_e^g(G)$ and the inclusion is strict.*

Proof. Let $f \in S_e(G)$, $0 \neq v \in X$, and let W be a topological complement of $\operatorname{span}\{v\}$. Applying Lemma 2.4 to the Borel Haar null set $B := \{x \in G : \ f^0(x,v) \neq -f^0(x,-v)\}$, we obtain that the assumptions of Lemma 6.3 are satisfied. So f is essentially smooth on a generic line parallel to v. Thus $f \in S_e^g(G)$ and the inclusion is proved.

To prove that it is strict, choose a linear space $V \subset X$ with $\dim V = 2$, choose a topological complement W to V and denote by π the projection of X on V along W. Set $H := \pi(G)$ and choose nonempty open sets $A \subset V$, $B \subset W$ with $A \times B \subset G$. Since $H \supset A$ is open and the classes $S_e(A)$ and $S_e^g(H)$ clearly do not depend on the choice of an equivalent norm on V, by Lemma 6.2 there exists a function $\varphi \in S_e^g(H)$ such that $\varphi|_A \notin S_e(A)$. Set $f(x) := \varphi(\pi(x)), \ x \in G$.

To prove $f \in S_e^g(G)$, consider an arbitrary $0 \neq v \in S_X$. If $v \in W$, then f is constant (and so essentially smooth) on each line parallel to v. If $v \notin W$, then $u := \pi(v) \neq 0$. Let S be a topological complement of $\operatorname{span}\{u\}$ in V and $T := S + W$. Then T is a topological complement of $\operatorname{span}\{v\}$ in X. Since $\varphi \in S_e^g(H)$, there exists (cf. Lemma 2.1) a set $P \subset S$ residual in S such that φ is essentially smooth on each line $p + \mathbb{R}u$, $p \in P$. It is easy to see that the set $P + W$ is residual in the space T. Now consider an arbitrary line $L = a + \mathbb{R}v$, where $a = p + w$ with $p \in P$ and $w \in W$. Then $g(t) := f(a+tv) = \varphi(p+tu)$ for each t from the domain of g, and so f is essentially smooth on L. By Lemma 2.1, we obtain $f \in S_e^g(G)$.

Since $\varphi|_A \notin S_e(A)$, there exists $c \in V$, such that $N := \{x \in A : \varphi^0(x,-c) \neq -\varphi^0(x,c)\}$ has positive Lebesgue (Haar) measure in V. Since clearly $f^0(z,-c) \neq -f^0(z,c)$ for each $z \in N+B$ and $N+B$ is not Haar null in X by [8, Theorem 2.3], we obtain $f \notin S_e(G)$. □

As an immediate corollary we obtain the inclusion $\mathcal{R}_e(G) \subset \mathcal{R}_e^g(G)$. We finish this section with the following result.

Proposition 6.5. *Let X be a Banach space and let $\emptyset \neq G \subset X$ be an open set. Then $\mathcal{S}_e(G) \subset R_e(G)$.*

Proof. Let $f \in S_e(G)$. To prove $f \in R_e(G)$, we define the family $\mathcal{F}$ of all closed separable spaces $Y \subset X$ with the following property:

($*$) There exists a dense set $D_Y \subset Y$ such that, for each $v \in D_Y$ and each closed separable space Z with $Y \subset Z \subset X$, the set
$$A_{v,Z} := \{z \in G \cap Z : \; f^0(z,-v) \neq -f^0(z,v)\}$$
is Haar null in Z.

First observe that, applying ($*$) to $Z := Y$, we obtain (see [8, Theorem 3.7]) that $f|_Y \in S_e(G \cap Y)$ for each $Y \in \mathcal{F}$ with $G \cap Y \neq \emptyset$.

So, to prove $f \in R_e(G)$, it is sufficient to show that $\mathcal{F}$ is a rich family.

To prove (R1), suppose that $Y_n \in \mathcal{F}$, $n \in \mathbb{N}$, and $Y_1 \subset Y_2 \subset \cdots$. For each $n \in \mathbb{N}$, choose a dense subset D_{Y_n} of Y_n by ($*$). Obviously, $D := \bigcup_{n=1}^\infty D_{Y_n}$ is dense in $Y := \overline{\bigcup_{n=1}^\infty Y_n}$. Since $A_{v,Z}$ is Haar null in Z for each $v \in D$ and each closed separable space Z with $Y \subset Z \subset X$ by the choice of D_{Y_n}, we have $Y \in \mathcal{F}$.

To prove (R2), let $Y_0 \subset X$ be a closed separable space. We will define inductively a sequence $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ of closed separable subspaces of X. The space Y_0 is already defined.

If Y_{n-1}, $n \in \mathbb{N}$, is defined, we choose a countable dense subset D_{n-1} of Y_{n-1}. Since $f \in S_e(G)$, we know that
$$A_v := \{x \in G : \; f^0(x,-v) \neq -f^0(x,v)\}$$
is Haar null in X for each $v \in X$. So, by Lemma 4.9, there exists a closed separable space Z_v such that
$$A_v \cap Z \text{ is Haar null in } Z \text{ for each closed separable space } Z \supset Z_v. \tag{37}$$
Now define
$$Y_n := \overline{\operatorname{span}(Y_{n-1} \cup \bigcup_{v \in D_{n-1}} Z_v)}.$$
Denote $Y := \overline{\bigcup_{n=1}^\infty Y_n}$. Then clearly $D_Y := \bigcup_{n=0}^\infty D_n$ is dense in Y and, for each $v \in D_Y$ and each closed separable space Z with $Y \subset Z \subset X$, we have that $A_v \cap Z$ is Haar null in Z by (37), since $Z_v \subset Y \subset Z$. So $Y \in \mathcal{F}$. □

7. Properties of classes defined via linear essential smoothness

The Borwein-Moors class $\mathcal{S}_e(G)$ is a linear lattice closed under multiplication and division (when it is defined); see [6, Corollary 4.4.] and [8, Theorem 3.9]. Since our classes are defined by essential smoothness on lines, and the intersection of two residual sets is a residual set, using the stability of $\mathcal{S}_e(H)$ for suitable open sets $H \subset \mathbb{R}$ and functions of the form $\varphi(t) = f(a + tv)$, $t \in H$, (cf. Remark 1.2), we easily obtain the following proposition.

Proposition 7.1. *Let X be a Banach space (resp. a Banach space written as $X = X_1 \oplus \cdots \oplus X_n$) and $G \subset X$ an open set. Then the classes $\mathcal{S}_e^l(G)$, $\mathcal{S}_e^g(G)$, $\mathcal{S}_e^{gg}(G)$ (resp. $\mathcal{S}_e^p(G)$) are linear lattices closed under multiplication and division (when it is defined).*

The stability of $\mathcal{S}_e(G)$ was inferred from the fact that $g \circ f \in \mathcal{S}_e(G)$ whenever $f \in \mathcal{S}_e(G, \mathbb{R}^n)$ and g is regular [2], or, more generally, g is arc-wise essentially smooth [6], [8], on a neigbourhood of $f(G)$.

The arc-wise essentially smooth functions in $\mathbb{R}^n$ were defined (under the name "saine fonctions") and used by Valadier [30], [31]. They were studied and applied in [7] (using another, but equivalent definition; see Remark 7.3 below). We present here the more general definition of [3].

Definition 7.2. Let X, Y be Banach spaces, $\emptyset \neq G \subset X$ an open set, and $f : G \to Y$ a mapping. We will say that f is arc-wise essentially smooth and write $f \in \mathcal{A}_e(G, Y)$ if f is locally Lipschitz and, for each $x \in \mathcal{S}_e((0,1), X)$ with $x((0,1)) \subset G$, the set $\{t \in (0,1) : \ x'(t)$ exists and f is not strictly differentiable in the direction $x'(t)\}$ is Lebesgue null. We set $\mathcal{A}_e(G) := \mathcal{A}_e(G, \mathbb{R})$.

Remark 7.3. The above definition coincides, in the case $X = \mathbb{R}^n$, $Y = \mathbb{R}$, with the definition of [7]. Valadier's definition (in the same case) is the same, but works with absolutely continuous x. It is not difficult to show, that, in Definition 7.2, we can equivalently work with C^1 smooth x or with continuous x which are almost everywhere differentiable. This easily follows from the fact that if x is continuous and almost everywhere differentiable then for each $\varepsilon > 0$ there exists a C^1 smooth $\tilde{x} : (0,1) \to X$ such that $\lambda(\{t \in (0,1) : \ x(t) \neq \tilde{x}(t)\}) < \varepsilon$. This fact immediately follows from [16, Theorem 3.1.16] in the case $X = \mathbb{R}^n$, and in the general case it can be proved using the vector-valued Whitney C^1 extension theorem (see [16]), Lusin theorem, and Egoroff theorem, as in the proof of [16, Theorem 3.1.16].

Note that

$$\mathcal{A}_e(G) \text{ is a linear space containing all regular (resp. pseudoregular) functions on } G. \tag{38}$$

(See [30] and [7] for the case $X = \mathbb{R}^n$; the case of a general space X is similar.)

Theorem 3.3 of [3] asserts that if X, Y, Z are Banach spaces, $G \subset X$, $H \subset Y$ are open, $f \in \mathcal{S}_e(G, Y)$, $f(G) \subset H$ and $g \in \mathcal{A}_e(H, Z)$, then $g \circ f \in \mathcal{S}_e(G, Z)$.

This result clearly implies (cf. Remark 1.2) the following proposition.

Proposition 7.4. *Let X, Y, Z be Banach spaces, $G \subset X$, $H \subset Y$ be open, $f : G \to H$, and $g \in \mathcal{A}_e(H, Z)$.*

(*i*) *If $f \in \mathcal{S}_e^l(G, Y)$ (resp. $f \in \mathcal{S}_e^g(G, Y)$, $f \in \mathcal{S}_e^{gg}(G, Y)$, $f \in \mathcal{S}_e^{gd}(G, Y)$), then $g \circ f \in \mathcal{S}_e^l(G, Z)$ (resp. $g \circ f \in \mathcal{S}_e^g(G, Z)$, $g \circ f \in \mathcal{S}_e^{gg}(G, Z)$, $g \circ f \in \mathcal{S}_e^{gd}(G, Z)$).*

(*ii*) *If X is written as $X = X_1 \oplus \cdots \oplus X_n$ and $f \in \mathcal{S}_e^p(G, Y)$, then $g \circ f \in \mathcal{S}_e^p(G, Z)$.*

Proposition 7.5. *Let X be an Asplund space, $G \subset X$ a connected open set, and let f_1, f_2 be functions on G such that either*

(*i*) *$f_1, f_2 \in \mathcal{S}_e^{gg}(G)$ or*

(*ii*) *$f_1, f_2 \in \mathcal{S}_e^p(G)$.*

Suppose that $\partial f_1 = \partial f_2$ generically on G. Then $g := f_1 - f_2$ is a constant function.

Proof. First observe that, by Theorem 5.2, Theorem 5.1 and (5), both f_1 and f_2 are generically strictly differentiable on G, and therefore (see (14)) there exists a first category set $M \subset G$ such that $\partial f_1(x) = \partial f_2(x) = \{f_1'(x)\} = \{f_2'(x)\}$ for each $x \in G \setminus M$. Consequently,

$$g'(x, u) = 0, \text{ whenever } x \in G \setminus M \text{ and } u \in X. \tag{39}$$

Choose $c \in G$ and set $A := \{a \in G : g(c) = g(a)\}$. To prove $A = G$, it is sufficient to show that A is both closed an open in G. The continuity of g implies that A is closed in G. To prove that A is open, choose an arbitrary $a \in A$. Now we will distinguish the cases (i) and (ii).

Ad (i): Choose $r > 0$ such that $B(a, r) \subset G$ and consider an arbitrary $b \in B(a, r)$. Write $b = a + su$, where $s \geq 0$ and $u \in S_X$. Since $g \in \mathcal{S}_e^{gg}(G)$ by Proposition 7.1, we can find vectors $u_n \in S_X$, such that $u_n \to u$, $b_n := a + su_n \in B(a, r)$, $n \in \mathbb{N}$, and g is essentially smooth on a generic line parallel to u_n. Using (39) and Lemma 3.1 for g and $\tilde{g} := -g$ (with $\alpha = 0$), we obtain $g(b_n) - g(a) \geq 0$ and $-g(b_n) + g(a) \geq 0$. Since $b_n \to b$, we obtain $g(b) = g(a)$ and $B(a, r) \subset A$.

Ad (ii): Choose $K > 0$ such that $K\|v\| \geq \sum_{i=1}^n \|v_i\|$, whenever $v = \sum_{i=1}^n v_i$ and $v_i \in X_i$ for each $1 \leq i \leq n$. Choose $r > 0$ such that $B(a, 2Kr) \subset G$ and consider an arbitrary $b \in B(a, r)$. Write $b - a = \sum_{i=1}^n s_i u_i$, where $s_i \geq 0$ and $u_i \in S_{X_i}$, $i = 1, \dots, n$. Set $c_0 := a$, $c_k := a + \sum_{i=1}^k s_i u_i$ for $1 \leq k \leq n$. Clearly $c_k = c_{k-1} + s_k u_k$ and $c_k \in B(a, 2Kr)$ for $1 \leq k \leq n$. Since $g \in \mathcal{S}_e^p(G)$ by Proposition 7.1, we can use (39) and Lemma 3.1 for g and $\tilde{g} := -g$ on the segment $\overline{c_{k-1}, c_k}$ and obtain $g(c_{k-1}) = g(c_k)$ for $1 \leq k \leq n$. Therefore $g(b) = g(c_n) = g(c_0) = g(a)$ and $B(a, r) \subset A$. □

8. Some consequences of main results

First we will present some consequences (which are formulated using standard notions) of Theorem 5.1. For the sake of simplicity, we will formulate these consequences for functions defined on the whole space.

The following consequence covers the cases of biconvex and saddle functions (proved in [1]), and also the (probably new) case of "partially DC functions".

Corollary 8.1. *Let $X = X_1 \times \cdots \times X_n$ be an Asplund space and let f be a locally Lipschitz function on X. Let each partial function of the form*

$$f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n), \quad 1 \le j \le n,$$

be a difference of two approximately convex functions. Then f is generically Fréchet differentiable on X.

A more general (see (20)) result (which is a special case of Corollary 8.3 on composite functions) we obtain, if we suppose that each partial function is a difference of two regular (or, more generally, two pseudoregular) functions.

The following general result on composite functions immediately follows from Proposition 7.4, Theorem 5.1 and Theorem 5.2.

Proposition 8.2. *Let X be an Asplund space (resp. an Asplund space written as $X = X_1 \oplus \cdots \oplus X_n$). Let Y be a Banach space and let $G \subset X$, $H \subset Y$ be open sets. Let $f \in \mathcal{S}_e^{gd}(G, Y)$ (resp. $f \in \mathcal{S}_e^{p}(G, Y)$), $f(G) \subset H$, and $g \in \mathcal{A}_e(H)$. Then $g \circ f$ is generically Fréchet differentiable on X.*

A more concrete corollary is the following.

Corollary 8.3. *Let $X = X_1 \times \cdots \times X_n$ be an Asplund space and let $f : X \to \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$ be a locally Lipschitz function on X. Let each function of the form*

$$f_i(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n), \quad 1 \le i \le m, \ 1 \le j \le n,$$

be a difference of two regular (or, more generally, of two pseudoregular) functions. Let $g : \mathbb{R}^m \to \mathbb{R}$ be a difference of two regular functions (or, more generally, of two pseudoregular) functions. Then $g \circ f$ is generically Fréchet differentiable on X.

Proof. Identifying $v \in X_i$ with $(0, \ldots, 0, v, 0, \ldots, 0) \in X$, we have $X = X_1 \oplus \cdots \oplus X_n$. By (38) and Proposition 8.2, it is sufficient to show that $f \in \mathcal{S}_e^p(X, \mathbb{R}^m)$. To this end, consider $a \in X$, $1 \le j \le n$, $v \in X_j \subset X$, and the function $\varphi(t) := f(a + tv) = (f_1(a + tv), \ldots, f_m(a + tv))$, $t \in \mathbb{R}$. By our assumptions and (10), we obtain that each function $f_i(a+tv)$, $1 \le i \le m$, is a difference of two pseudoregular function, and so it belongs to the linear space $\mathcal{S}_e(\mathbb{R})$ by Proposition BM. Thus $\varphi \in \mathcal{S}_e(\mathbb{R}, \mathbb{R}^m)$ by (35), which proves $f \in \mathcal{S}_e^p(X, \mathbb{R}^m)$. □

Another corollary works with the notion (see [32]) of DC (= delta-convex) mappings between Banach spaces.

Corollary 8.4. *Let $X = X_1 \times \cdots \times X_n$ be an Asplund space, Y a Banach space, and let $f : X \to Y$ be a locally Lipschitz mapping. Let each partial mapping*

$$f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n), \quad 1 \le j \le n,$$

be a locally DC mapping from X_j to Y. Let $g : Y \to \mathbb{R}$ be a difference of two regular functions (or, more generally, of two pseudoregular) functions. Then $g \circ f$ is generically Fréchet differentiable on X.

Proof. Identifying $v \in X_i$ with $(0,\dots,0,v,0,\dots,0 \in X$, we have $X = X_1 \oplus \cdots \oplus X_n$. By (38) and Proposition 8.2, it is sufficient to show that $f \in \mathcal{S}_e^p(X,Y)$. To this end, consider $a \in X$, $1 \le j \le n$, $v \in X_i \subset X$, and the mapping $\varphi(t) := f(a+tv)$, $t \in \mathbb{R}$. By our assumptions and [32, Lemma 1.5(b)], we obtain that $\varphi : \mathbb{R} \to Y$ is a locally DC mapping. So φ is a DC mapping by [32, Theorem 1.20] and thus we can choose a control convex function $c : \mathbb{R} \to \mathbb{R}$ of φ. Since c is Fréchet differentiable except a countable set, we obtain that φ is strictly differentiable except a countable set by [32, Proposition 3.9(i)]. So $\varphi \in \mathcal{S}_e(\mathbb{R},Y)$, and thus $f \in \mathcal{S}_e^p(X,Y)$. □

It is possibly a new observation, that generic differentiability of a locally Lipschitz function f on an Asplund space can be implied by an assumption concerning only behaviour of restrictions of f to lines. Of course, the verification of such assumption can be sometimes easier that the verification of directional strict differentiability (or regularity) with respect to the whole space. As an illustration, we present an alternative proof of the following well-known result.

Theorem UG. *Let X be an Asplund space with UG (uniformly Gâteaux smooth) norm and let $\emptyset \neq F \subset X$ be a closed set. Then the distance function $d_F(x) :=$* $\mathrm{dist}(x,F)$ *is generically Fréchet differentiable on $X \setminus F$.*

This result follows from the fact that $-d_F$ is Lipschitz and regular on $G := X \setminus F$ (see [4]), since regularity is a "(GFD) property" (cf. Introduction); see [12] and [17].

Using Corollary 8.1 (for $n = 1$), we need not regularity of $-d_F$; to prove Theorem UG it is sufficient to show that d_F is locally semiconcave (cf. (19)) on lines outside F (which is essentially easier). More precisely:

Lemma 8.5. *Let X, F and d_F be as in Theorem UG. Let $a \in G := X \setminus F$ and $v \in S_X$. Then the function $\varphi(t) := d_F(a+tv)$ is locally semiconcave (with a general modulus, cf.* [10]*) on $H := \{t \in \mathbb{R} : a+tv \in G\}$.*

Proof. Denote $q(x) := \|x\|$ and recall ([33, Proposition 7(iii)]) that

$$\begin{gathered}\text{the function } x \mapsto q'(x,v) \text{ is uniformly continuous}\\ \text{on } \{x \in X : \|x\| > r\} \text{ for each } r > 0.\end{gathered} \tag{40}$$

Consider an arbitrary $t_0 \in H$, set $x_0 := a + t_0 v$ and $d_0 := \mathrm{dist}(x_0, F) > 0$. Let (cf. (40)) ω be a modulus of continuity of the function $x \mapsto q'(x,v)$ on $\{x \in X : \|x\| > d_0/2\}$. For each $y \in F$, set $d_y(x) := q(x-y)$, $x \in X$. Then clearly $(d_y)'(x,v) = q'(x-y,v)$ for $x \in G$, which easily implies that, for each $y \in F$, the function $(d_y)'(\cdot,v)$ is uniformly continuous on $B(x_0, d_0/2)$ with modulus of continuity ω. For each $y \in F$, set $\varphi_y(t) := d_y(a+tv)$, $t \in \mathbb{R}$. Since clearly $(\varphi_y)'(t) = (d_y)'(a+tv,v)$ for $t \in H$, we easily see that, for each $y \in F$, the function $(\varphi_y)'$ is uniformly continuous with modulus of continuity ω on $(t_0 - d_0/2, t_0 + d_0/2)$. Since $\varphi = \inf_{y \in F} \varphi_y$, one of the basic facts on semiconcave functions ([10, Corollary 2.1.6]) implies that φ is semiconcave (with modulus ω) on $(t_0 - d_0/2, t_0 + d_0/2)$. □

Acknowledgements. I thank the authors of [22] for the opportunity to learn the method of rich families in the differentiability theory from a preliminary form of their monograph.

References

[1] J. M. Borwein: Partially monotone operators and the generic differentiability of convex-concave and biconvex mappings, Isr. J. Math. 54 (1986) 42–50.

[2] J. M. Borwein: Minimal cuscos and subgradients of Lipschitz functions, in: Fixed Point Theory and Applications (Marseille, 1989), J.-B. Baillion, M. Théra (eds.), Pitman Research Notes in Math. 252, Longman, Harlow (1991) 57–82.

[3] M. Bounkhel: On arc-wise essentially smooth mappings between Banach spaces, Optimization 51 (2002) 11–29.

[4] J. M. Borwein, S. P. Fitzpatrick, J. R. Giles: The differentiability of real functions on normed linear space using generalized subgradients, J. Math. Anal. Appl. 128 (1987) 512–534.

[5] J. M. Borwein, S. Fitzpatrick, P. Kenderov: Minimal convex uscos and monotone operators on small sets, Can. J. Math. 43 (1991) 461–476.

[6] J. M. Borwein, W. B. Moors: Essentially smooth Lipschitz functions, J. Funct. Anal. 149 (1997) 305–351.

[7] J. M. Borwein, W. B. Moors: A chain rule for essentially smooth Lipschitz functions, SIAM J. Optim. 8 (1998) 300–308.

[8] J. Borwein, W. B. Moors: Null sets and essentially smooth Lipschitz functions, SIAM J. Optim. 8 (1998) 309–323.

[9] J. Borwein, W. B. Moors: Separable determination of integrability and minimality of the Clarke subdifferential mapping, Proc. Amer. Math. Soc. 128 (1999) 215–221.

[10] P. Cannarsa, C. Sinestrari: Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control, Progress in Nonlinear Differential Equations and their Applications 58, Birkhäuser, Boston (2004).

[11] M. Cúth: Separable Reduction in Functional Analysis, Diploma Thesis, Charles University, Prague (2010) (in Czech).

[12] G. de Barra, S. Fitzpatrick, J. R. Giles: On generic differentiability of locally Lipschitz functions on Banach spaces, in: Functional Analysis and Optimization, Workshop / Miniconference (Canberra, 1988), Proc. Cent. Math. Anal. Aust. Natl. Univ. 20, Australian National University, Canberra (1988) 39–49.

[13] J. P. R. Christensen, P. S. Kenderov: Dense strong continuity of mappings and the Radon-Nikodym property, Math. Scand. 54 (1984) 70–78.

[14] J. Duda, L. Zajíček: Semiconvex functions: representations as suprema of smooth functions and extensions, J. Convex Analysis 16 (2009) 239–260.

[15] I. Ekeland, G. Lebourg: Generic Fréchet-differentiability and perturbed optimization problems in Banach spaces, Trans. Amer. Math. Soc. 224 (1976) 193–216.

[16] H. Federer: Geometric Measure Theory, Springer, Berlin (1969).

[17] P. G. Georgiev: Locally Lipschitz and regular functions are Fréchet differentiable almost everywhere in Asplund spaces, C. R. Acad. Bulg. Sci. 42 (1989) 13–15.

[18] M. Gieraltowska-Kedzierska, F. S. Van Vleck: Fréchet differentiability of regular locally Lipschitzian functions, J. Math. Anal. Appl. 159 (1991) 147–157.

[19] J. R. Giles, S. Sciffer: Locally Lipschitz functions are generically pseudo-regular on separable Banach spaces, Bull. Aust. Math. Soc. 47 (1993) 205–212.

[20] W. B. Johnson, J. Lindenstrauss, D. Preiss, G. Schechtman: Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces, Proc. Lond. Math. Soc., III. Ser. 84 (2002) 711–746.

[21] G. Lebourg: Generic differentiability of Lipschitzian functions, Trans. Amer. Math. Soc. 256 (1979) 125–144.

[22] J. Lindenstrauss, D. Preiss, J. Tišer: Fréchet differentiability of Lipschitz maps and porous sets in Banach spaces, to appear.

[23] P. Mattila: Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge (1995).

[24] B. S. Mordukhovich: Variational Analysis and Generalized Differentiation. I: Basic Theory, Grundlehren der Mathematischen Wissenschaften 330, Springer, Berlin (2006).

[25] H. V. Ngai, D. T. Luc, M. Théra: Approximate convex functions, J. Nonlinear Convex Anal. 1 (2000) 155–176.

[26] H. V. Ngai, J.-P. Penot: In Asplund spaces, approximately convex functions and regular functions are generically differentiable, Taiwanese J. Math. 12 (2008) 1477-1492.

[27] J. C. Oxtoby: Measure and Category. A Survey of the Analogies between Topological and Measure Spaces, 2nd Ed., Graduate Texts in Mathematics 2, Springer, New York (1980).

[28] D. Preiss: Gâteaux differentiable Lipschitz functions need not be Fréchet differentiable on a residual subset, Suppl. Rend. Circ. Mat. Palermo, II. Ser. (1982) 217–222.

[29] L. Qi: The maximal normal operator space and integration of subdifferentials of nonconvex functions, Nonlinear. Anal. 13 (1989) 1003–1012.

[30] M. Valadier: Lignes de descente de fonctions lipschitziennes non pathologiques, Sémin. d'Analyse Convexe Montpellier 18(9) (1988) 10 pp.

[31] M. Valadier: Entraîtement unilatéral, lignes de descente, fonctions Lipschitziennes non pathologiques, C. R. Acad. Sci. Paris 308, Series I (1989) 241–244.

[32] L. Veselý, L. Zajíček: Delta-convex mappings between Banach spaces and applications, Diss. Math. 289 (1989) 48 pp.

[33] L. Zajíček: Differentiability of the distance function and points of multi-valuedness of the metric projection in Banach space, Czech. Math. J. 33 (1983) 292–308.

[34] L. Zajíček: A generalization of an Ekeland-Lebourg theorem and the differentiability of distance functions, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 3 (1984) 403–410.

[35] L. Zajíček: Fréchet differentiability, strict differentiability and subdifferentiability, Czech. Math. J. 41(116) (1991) 471–489.

[36] L. Zajíček: Differentiability of approximately convex, semiconcave and strongly paraconvex functions, J. Convex Analysis 15 (2008) 1–15.