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Lyusternik-Graves Theorem and Fixed Points II

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This work continues the studies in our previous paper [9]. It is written as a separate paper which extends the previous one in the direction of closing the gap between Lyusternik-Graves theorems and fixed point theorems. Here we introduce a new definition of global metric regularity on a set and associated definitions of Aubin continuity and linear openness that are equivalent to metric regularity on the same sets and with the same constant. When the sets are neighborhoods of a point in the graph of the mapping, these definitions reduce to the well studied properties at a point. We present Lyusternik-Graves type theorems in metric spaces for single-valued and set-valued perturbations, and show that they can be derived from, and some of them are even equivalent to, corresponding set-valued fixed point theorems.

Keywords: Set-valued analysis, metric regularity, Aubin property, linear openness, contraction mapping, Lyusternik-Graves theorem, Milyutin theorem

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1. Introduction

This paper continues the study of interrelations between Lyusternik-Graves theorems and fixed point theorems for set-valued mappings which we begun in [9]. Although it is basically on the same subject and [9] is a major reference, it is written as a separate paper and does not use any results from that previous paper. We consider it as part II of [9] (which is not titled Part I but actually is) to indicate that these two papers should be kept together, mainly because here we clarify and further develop ideas and results from the previous paper.

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There are also developments in this paper that do not originate from [9]. The first one, presented in this Introduction, is about global regularity properties and the relationships among them. It is commonly known that the metric regularity, as introduced and studied in most of the literature, is equivalent to the Aubin continuity of the inverse and the linear openness; however, when passing from one property to other the involved neighborhoods may change. When we consider global properties on fixed sets rather than neighborhoods, then we have to modify the definitions to keep the properties equivalent to each other. It turns out that there are two collections each consisting of three properties that are equivalent to each other with the same sets and same constant. To the authors' knowledge, not all of them have been noted in the previous works. In the further lines we introduce *metric regularity on a set* which is weaker than the standard global version of metric regularity and is equivalent to the standard Aubin property of the inverse and the linear openness with the same sets.

To put the stage, let us fix first the notation. Throughout the paper X, Y and P are metric spaces unless stated otherwise, with all metrics denoted by ρ. The set $I\!B_r(a) = \{x \,|\, \rho(x,a) \leq r\}$ is the closed ball of radius r centered at a; $\overset{\circ}{I\!B}_r(a)$ is the associated open ball. By convention, for any $x \in X$, $I\!B_\infty(x) = \overset{\circ}{I\!B}_\infty(x) = X$. We denote by $d(x,C)$ the distance from a point $x \in X$ to a subset $C \subset X$; that is $d(x,C) = \inf\{\rho(x,x') \,|\, x' \in C\}$ whenever $C \neq \emptyset$ and $d(x,\emptyset) = \infty$. The excess from a set C to a set D is $e(C,D) = \sup_{x\in C} d(x,D)$ under the convention $e(\emptyset,D) = 0$ for $D \neq \emptyset$ and $e(D,\emptyset) = +\infty$ for any D. The Pompeiu-Hausdorff distance between C and D is $\operatorname{haus}(C,D) = \max\{e(C,D), e(D,C)\}$. The smallest distance between two sets C and D is denoted by $\mathbf{d}(C,D)$, that is $\mathbf{d}(C,D) = \inf\{\, \rho(x',x) \,|\, x' \in C, x \in D\}$. If one of the sets C and D is empty, we set $\mathbf{d}(C,D) = +\infty$. We say that a set $C \subset X$ is locally closed (complete) at $\bar{x} \in C$ if there exists $a > 0$ such that the intersection $C \cap I\!B_a(\bar{x})$ is locally closed (complete).

A set-valued mapping F from X to Y, indicated by $F : X \rightrightarrows Y$, is identified with its graph $\operatorname{gph} F = \{(x,y) \in X \times Y \,|\, y \in F(x)\}$. It has effective domain $\operatorname{dom} F = \{\, x \in X \,|\, F(x) \neq \emptyset\}$ and effective range $\operatorname{rge} F = \{\, y \in Y \,|\, \exists\, x \text{ with } F(x) \ni y\}$. To avoid unnecessary complications in notation, in this paper we consider only mappings with nonempty domains. The inverse $F^{-1} : Y \rightrightarrows X$ of a mapping $F : X \rightrightarrows Y$ is obtained by reversing all pairs in the graph; then $\operatorname{dom} F^{-1} = \operatorname{rge} F$. By a composition of a mappings $B : Z \rightrightarrows Y$ with a mapping $A : X \times P \rightrightarrows Z$ we mean a mapping $C = B \circ A : X \times P \rightrightarrows Y$ with the following property: if $(x,p,y) \in \operatorname{gph} C$, then there exists $z \in Z$ such that $(x,p,z) \in \operatorname{gph} A$ and $(z,y) \in \operatorname{gph} B$; in other words, $C(x,p) = \{\, y \in B(z) \mid z \in A(x,p)\}$. We denote the set of fixed points of a mapping $F : X \rightrightarrows X$ by $\operatorname{Fix}(F)$, $\operatorname{Fix}(F) := \{x \in X \mid x \in F(x)\}$.

In this paper we introduce the following definition of global metric regularity on a set:

Definition 1.1 (metric regularity on a set). Let X, Y be metric spaces and U, V be nonempty subsets of X and Y respectively. A set-valued mapping F from X to Y is said to be metrically regular on U for V when there is a constant $\kappa > 0$

such that
$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x) \cap V) \text{ for all } (x, y) \in U \times V. \tag{1}$$

Observe that on the right side of (1) the value $F(x)$ is intersected with the set V; this makes Definition 1.1 more general than the well studied *metric regularity at a point.* A mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ when $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and *there exist neighborhoods* U of $\bar{x}$ and V of $\bar{y}$ and a constant $\kappa > 0$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } (x, y) \in U \times V. \tag{2}$$

The infimum of κ over all combinations of κ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ in (2) is called the *modulus of metric regularity* and is denoted by $\operatorname{reg}(F; \bar{x}\,|\,\bar{y})$; then the presence of metric regularity of F at $\bar{x}$ for $\bar{y}$ is identified with $\operatorname{reg}(F; \bar{x}\,|\,\bar{y}) < +\infty$. For some recent results concerning metric regularity see, e.g., [1], [2], [4], [6], [10], [16], [17], [19] and in particular the book [8].

Definition 1.1 for metric regularity has its origin in [8], Proposition 3C.1, where it is shown for the inverse F^{-1} of F and $U = X$, $V \subset \operatorname{dom} F^{-1}$ that (1) is equivalent to the Lipschitz continuity of F^{-1} on V with respect to the Pompeiu-Hausdorff distance, a result which is now a particular case of Proposition 1.5.

If F is metrically regular at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ with constant κ, then F is clearly metrically regular on U for V with the same constant κ. Conversely, when the sets U and V in Definition 1.1 are neighborhoods of points $\bar{x}$ and $\bar{y}$ with $\bar{y} \in F(\bar{x})$, then metric regularity on U for V becomes equivalent to metric regularity at $\bar{x}$ for $\bar{y}$. This equivalence can be extracted from statements in [8], and perhaps also from other works; here we supply it with a direct proof[1].

Proposition 1.2. *For positive scalars a, b and κ, and points $\bar{x} \in X$, $\bar{y} \in Y$ consider a mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$ and assume that F is metrically regular on $\mathbb{B}_a(\bar{x})$ for $\mathbb{B}_b(\bar{y})$ with constant κ. Then there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that* (2) *holds, that is, F is metrically regular at $\bar{x}$ for $\bar{y}$ with constant κ.*

Proof. Set $\beta = b/4$ and $\alpha = \min\{a, \kappa b/4\}$. Let $x \in \mathbb{B}_\alpha(\bar{x})$ and $y \in \mathbb{B}_\beta(\bar{y})$. If $F(x) = \emptyset$ the right side of (2) is $+\infty$ and we are done. If not, let $y' \in F(x)$. We consider two cases. First, if $\rho(y, y') > b/2$, then we have

$$\begin{aligned} d(x, F^{-1}(y)) &\leq \rho(x, \bar{x}) + d(\bar{x}, F^{-1}(y)) \leq \rho(x, \bar{x}) + \kappa d(y, F(\bar{x}) \cap \mathbb{B}_b(\bar{y})) \\ &\leq \rho(x, \bar{x}) + \kappa \rho(y, \bar{y}) \leq \kappa b/4 + \kappa b/4 = \kappa b/2 < \kappa \rho(y, y'). \end{aligned}$$

Further, if $\rho(y, y') \leq b/2$, then

$$\rho(y', \bar{y}) \leq \rho(y', y) + \rho(y, \bar{y}) \leq b/2 + b/4 = 3b/4,$$

hence
$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x) \cap \mathbb{B}_b(\bar{y})) \leq \kappa \rho(y, y').$$

[1]This proof is due to the referee of this paper; it is self-contained and shorter than the proof we gave in the first version of the paper.

Thus, we obtain that for any $y' \in F(x)$ we have $d(x, F^{-1}(y)) \leq \kappa\rho(y, y')$. Taking infimum of the right side over $y' \in F(x)$ we obtain (2). $\square$

As an example showing the difference between metric regularity on a set and metric regularity at a point, consider the mapping $F : \mathbb{R} \to \mathbb{R}$ whose graph is $\{(x, y) \mid y = 0\}$. Then F is metrically regular on any set U for $V = \{0\}$ with any constant $\kappa > 0$ but it is not metrically regular at any $\bar{x}$ for any $\bar{y}$.

One may argue that the true definition of metric regularity of a mapping F on U for V should be as in (2) which is stronger than (1). It turns out that if we do that we may loose the equivalence of the metric regularity with the Aubin continuity of the inverse on the same sets; this is explained in detail in further lines.

Any function $f : X \to Y$, where X and Y are Banach spaces, which is continuously Fréchet differentiable around a point $\bar{x}$ and such that the derivative mapping $Df(\bar{x})$ is surjective, is metrically regular at $\bar{x}$ for $f(\bar{x})$. This follows from the Lyusternik-Graves theorem (Theorem 1.8) recalled below. Many more examples and applications can be found in [8], most of which is devoted to metric regularity at a point.

It is well known that metric regularity at a point is equivalent to both linear openness at a point and Aubin continuity at a point of the inverse. Parallel to metric regularity on a set given in Definition 1.1 we introduce Aubin continuity on a set in the following way:

Definition 1.3 (Aubin continuity on a set). Let U and V be nonempty subsets of X and Y respectively. A mapping $S : Y \rightrightarrows X$ is said to be Aubin continuous on V for U when there exists a constant $\kappa > 0$ such that

$$e(S(y) \cap U, S(y')) \leq \kappa\rho(y, y') \quad \text{for all } y, y' \in V. \tag{3}$$

If U and V are assumed to be neighborhoods of reference points $\bar{x}$ and $\bar{y}$, respectively, with $(\bar{y}, \bar{x}) \in \operatorname{gph} S$, the property in Definition 1.3 reduces to the original property introduced by Aubin [3] under the name "pseudo-Lipschitz continuity" of S around $(\bar{y}, \bar{x})$. According to the terminology in [8] which we use here, in this case S is said to be Aubin continuous at $\bar{y}$ for $\bar{x}$. Thus, Aubin continuity at $\bar{y}$ for $\bar{x}$ is equivalent to the existence of neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that S is Aubin continuous on V for U, according to Definition 1.3.

The infimum of κ over all combinations of κ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ in (3) is called the *Lipschitz modulus* of S and is denoted by $\operatorname{Lip}(S; \bar{y}|\bar{x})$. If S is single-valued, then $\operatorname{Lip}(S; \bar{y}|\bar{x})$ is the standard Lipschitz modulus $\operatorname{lip}(S; \bar{y})$ of the function S at $\bar{y}$. For a function of two variables $g : X \times P \to Y$, recall that g is Lipschitz continuous with respect to x uniformly in p around $(\bar{x}, \bar{p}) \in \operatorname{dom} g$ whenever there exist a constant ν and neighborhoods U of $\bar{x}$ and Q of $\bar{p}$ such that

$$\rho(g(x, p), g(x', p)) \leq \nu\rho(x, x') \quad \text{for all } x, x' \in U \text{ and } p \in Q.$$

The infimum of ν over all U and Q is called the partial uniform Lipschitz modulus of g with respect to x and is denoted by $\widehat{\operatorname{lip}}_x(g; (\bar{x}, \bar{p}))$. The property that g is Lipschitz continuous both with respect to x uniformly in p and with respect to p uniformly in x around $(\bar{x}, \bar{p})$ is of course equivalent to the property that g is Lipschitz continuous around $(\bar{x}, \bar{p})$.

If $U = X$ in Definition 1.3, then Aubin continuity becomes the usual Lipschitz continuity on V with respect to the Pompeiu-Hausdorff distance. Specifically, a mapping $S : Y \rightrightarrows X$ is said to be Lipschitz continuous on a set $V \subset \operatorname{dom} S$ when there exists a constant $\kappa \geq 0$ such that

$$\operatorname{haus}(S(y), S(y')) \leq \kappa\rho(y, y') \quad \text{for all } y, y' \in V.$$

Next comes a definition of openness with linear rate on a set.

Definition 1.4 (linear openness on a set). Let U and V be two nonempty subsets of X and Y respectively. A mapping $F : X \rightrightarrows Y$ is said to be open with linear rate (or linearly open) on U for V when there exists a constant $\kappa > 0$ such that

$$\overset{\circ}{\mathbb{B}}_r(y) \cap V \subset F(\overset{\circ}{\mathbb{B}}_{\kappa r}(x)) \quad \text{for all } r \in (0, \infty] \text{ and } (x, y) \in \operatorname{gph} F \cap (U \times V). \tag{4}$$

The standard definition of linear openness at a point $\bar{x}$ for $\bar{y}$ used e.g. in [8] assumes that $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and there exist a constant $\kappa > 0$ and neighborhoods U and V of $\bar{x}$ and $\bar{y}$, respectively, for which (4) is satisfied.

In the following proposition we show that metric regularity on a set of a mapping F in the sense of Definition 1.1 is equivalent to Aubin continuity on a set of the inverse F^{-1} in the sense of Definition 1.3 and also to linear openness on a set of F in the sense of Definition 1.4 *with the same sets U and V and constant κ.* This equivalence is well known for the properties in question at a point, but with perhaps different neighborhoods, see, e.g., Theorem 3E.6 in [8].

Proposition 1.5. *Let U and V be nonempty subsets of X and Y respectively, and consider a mapping $F : X \rightrightarrows Y$ such that*

$$\operatorname{gph} F \cap (U \times V) \neq \emptyset. \tag{$*$}$$

Then the following are equivalent:

(*i*) *F is metrically regular on U for V with constant κ;*
(*ii*) *F^{-1} is Aubin continuous on V for U with constant κ;*
(*iii*) *F is linearly open on U for V with constant κ.*

Proof. Let (*i*) hold. First, note that by $(*)$ there exists $\bar{x} \in U$ such that $F(\bar{x}) \cap V \neq \emptyset$. Then from (1) it follows that $d(\bar{x}, F^{-1}(y)) < \infty$ for any $y \in V$. Thus, $V \subset \operatorname{dom} F^{-1}$. Now, fix $y, y' \in V$. Since $V \subset \operatorname{dom} F^{-1}$, we have $F^{-1}(y') \neq \emptyset$. If $F^{-1}(y) \cap U = \emptyset$, then the left side of (3) is zero and then (3) is automatically satisfied. Let $x \in F^{-1}(y) \cap U$. Then, from (1),

$$d(x, F^{-1}(y')) \leq \kappa d(y', F(x) \cap V) \leq \kappa\rho(y', y)$$

since $y \in F(x)$. Taking the supremum of the left side with respect to $x \in F^{-1}(y) \cap U$ we obtain (3), that is, (*ii*).

Assume *(ii)*. By $(*)$ there exists $(x,y) \in \operatorname{gph} F \cap (U \times V)$. Let $r > 0$. If $\overset{\circ}{\mathbb{B}}_r(y) \cap V = \emptyset$ then (4) holds automatically. For any $w \in V$ from (3) we have that $e(F^{-1}(y) \cap U, F^{-1}(w)) \le \kappa\rho(y,w) < \infty$ and since $x \in F^{-1}(y) \cap U \neq \emptyset$, we get that $F^{-1}(w) \neq \emptyset$. But then $V \subset \operatorname{dom} F^{-1}$. Let $y' \in \overset{\circ}{\mathbb{B}}_r(y) \cap V$; then $y' \in \operatorname{rge} F = \operatorname{dom} F^{-1}$ and hence $F^{-1}(y') \neq \emptyset$. We have

$$d(x, F^{-1}(y')) \le e(F^{-1}(y) \cap U, F^{-1}(y')) \le \kappa\rho(y,y') < \kappa r.$$

Hence, there exists $x' \in F^{-1}(y')$ with $\rho(x,x') < \kappa r$, that is, $x' \in \overset{\circ}{\mathbb{B}}_{\kappa r}(x)$. Thus $y' \in F(x') \subset F(\overset{\circ}{\mathbb{B}}_{\kappa r}(x))$ and we obtain (4), that is, *(iii)* is satisfied.

Now, assume *(iii)*. Let $x \in U$, $y \in V$ and let $y' \in F(x) \cap V$; if there is no such y' the right side in (1) is ∞ and we are done. If $y = y'$ then (1) holds since both the left and the right sides are zero. Let $r := \rho(y,y') > 0$ and let $\varepsilon > 0$. Then of course $y \in \overset{\circ}{\mathbb{B}}_{r(1+\varepsilon)}(y') \cap V$. From *(iii)* there exists $x' \in \overset{\circ}{\mathbb{B}}_{\kappa r(1+\varepsilon)}(x) \cap F^{-1}(y)$. Then

$$d(x, F^{-1}(y)) \le \rho(x,x') \le \kappa r(1+\varepsilon) = \kappa(1+\varepsilon)\rho(y,y').$$

Taking infimum in the right side of this inequality with respect to $\varepsilon > 0$ and $y' \in F(x) \cap V$ we obtain (1) and hence *(i)*. The proof is complete. □

Remark 1.6. Note if condition $(*)$ is violated, then *(i)* and *(iii)* hold automatically while *(ii)* holds iff $V \subset \operatorname{rge} F$. Also note that for metric regularity at a point the condition $(*)$ is always satisfied.

Rockafellar proved[2] in [15] that when U and V are open sets the Aubin property of S on V for U in Definition 1.3 is equivalent to the property that there exist subsets U' of U and V' of V, which may be different from U and V, such that

$$e(S(y') \cap U', S(y)) \le \kappa\rho(y,y') \ \text{ for all } y' \in Y \text{ and } y \in V', \tag{5}$$

where κ is the same as in (3). A similar, but still different property is introduced in [13] and called "calmness".

In the next proposition we show that a property of linear openness associated with sets U and V which is stronger than (4) is equivalent to metric regularity in the form (2) as well as to a property stronger than the one in (3) but weaker than the one displayed in (5) for the inverse with the same sets U and V. Its proof is similar to that of Proposition 1.5; for completeness and because of some subtle differences we present it in full.

Proposition 1.7. *Let U and V be nonempty sets in X and Y respectively, let $\kappa > 0$ and consider a mapping $F : X \rightrightarrows Y$ such that condition $(*)$ is fulfilled. Then the following are equivalent:*

(i) $\quad d(x, F^{-1}(y)) \le \kappa d(y, F(x))$ *for all* $(x,y) \in U \times V$.

[2] The proof in [15] is for mapping acting in finite dimensions, but the extension to metric spaces is straightforward.

(*ii*) $e(F^{-1}(y') \cap U, F^{-1}(y)) \leq \kappa\rho(y, y')$ *for all* $y' \in \operatorname{rge} F$ *and* $y \in V$.

(*iii*) $\overset{\circ}{I\!B}_r(F(x)) \cap V \subset F(\overset{\circ}{I\!B}_{\kappa r}(x))$ *for all* $r \in (0, \infty]$ *and* $x \in U$.

Proof. Let (*i*) hold. By (∗) there exists $\bar{x} \in U$ such that $F(\bar{x}) \neq \emptyset$. Then from (*i*) $F^{-1}(y) \neq \emptyset$ for any $y \in V$, hence $V \subset \operatorname{dom} F^{-1}$. Now, let $y' \in \operatorname{rge} F$ and $y \in V$. Then $F^{-1}(y) \neq \emptyset$ and if $F^{-1}(y') \cap U = \emptyset$ then the left side of the inequality in (*ii*) is zero, hence (*ii*) holds automatically. If not, let $x \in U$ be such that $y' \in F(x)$. Applying (*i*) with so chosen x and y and taking supremum on the left with respect to $x \in F^{-1}(y') \cap U$ we obtain (*ii*).

Assume (*ii*). Let $x \in U$ and $r > 0$. If $\overset{\circ}{I\!B}_r(F(x)) \cap V = \emptyset$ then (*iii*) holds automatically. If not, let $y' \in V$ and $y \in F(x)$ be such that $\rho(y', y) < r$. Then, since $y \in \operatorname{rge} F$ we have from (*ii*) that

$$d(x, F^{-1}(y')) \leq e(F^{-1}(y) \cap U, F^{-1}(y')) \leq \kappa\rho(y, y') < \kappa r.$$

Then there exists $x' \in F^{-1}(y') \cap \overset{\circ}{I\!B}_{\kappa r}(x)$, that is, $y' \in F(x') \subset F(\overset{\circ}{I\!B}_{\kappa r}(x))$ and thus (*iii*) holds.

Assume (*iii*). Let $x \in U$, $y \in V$ and let $y' \in F(x)$; if there is no such y' the right side in (1) is ∞ and hence (*i*) holds. If $y = y'$ then (*i*) holds since both the left and the right sides are zero. Let $r := \rho(y, y') > 0$ and let $\varepsilon > 0$. Then $y \in \overset{\circ}{I\!B}_{r(1+\varepsilon)}(F(x)) \cap V$. It remains to repeat the last part of the proof of Proposition 1.5. □

Taking into account Proposition 1.2, we obtain that all six properties displayed in Propositions 1.5 and 1.7 become equivalent when they are considered at a point.

Metric regularity of a mapping F is preserved if one adds to F another mapping G with a sufficiently small Lipschitz constant. This general paradigm stems from the works of Lyusternik (1934) and Graves (1950). The following result is most known as the Lyusternik-Graves theorem:

Theorem 1.8 (Lyusternik-Graves). *Let* X *and* Y *be Banach spaces and let* $f : X \to Y$ *be strictly Fréchet differentiable at* $\bar{x}$. *Then the following are equivalent:*

(*i*) *the strict derivative mapping* $Df(\bar{x})$ *is surjective;*

(*ii*) f *is metrically regular at* $\bar{x}$ *for* $f(\bar{x})$.

The purely metric nature of this result was not fully understood, however, until the critical contribution of the paper [5], where the following result among many others, commonly attributed to Milyutin, was obtained:

Theorem 1.9. *Let* X *be a complete metric space and* Y *be a linear metric space with shift-invariant metric. Let* $f : X \to Y$ *be a continuous function which is metrically regular on* X *for* Y *with constant* κ *and let* $h : X \to Y$ *be a function which is Lipschitz continuous on* X *with Lipschitz constant* μ *such that* $\kappa\mu < 1$. *Then* $f + h$ *is metrically regular on* X *for* Y *with constant* $\kappa/(1 - \kappa\mu)$.

Among the numerous works inspired by the paper of Milyutin et al. [5] one should point out the works by Frankowska [11], Ursescu [18] and Ioffe [12]. More recently, Arutyunov [1] stated and proved a coincidence theorem which yields both Milyutin's theorem and the part regarding the existence of fixed point (but not the uniqueness) of the Banach contraction mapping principle. We should point out here that Ioffe and Arutyunov, as well as Lyusternik, Graves and Milyutin before them, employed in their proofs iterative procedures that resemble the Picard iteration, also used by Banach in the proof of his contraction mapping theorem. Arutyunov's result has been extended in different directions in [13] and [9] but still without determining the precise relation between this kind of theorems and associated contraction mapping theorems. In this paper we present extensions of the Lyusternik-Graves and Milyutin theorems in metric spaces without any linearity that easily follow from or are even equivalent to corresponding set-valued contraction mapping theorems.

In Section 2 we deal with the case when the reference mapping is perturbed by a function. We first state a contraction mapping theorem from [7] (Theorem 2.1), present an extension of it (Theorem 2.1A), and then show that Milyutin's theorem (Theorem 1.9) is an easy consequence of it. Then we prove a metric version of the Lyusternik-Graves theorem (Theorem 2.3) for mappings acting in metric spaces which is equivalent to the contraction mapping Theorem 2.1.

In Section 3 we follow a similar path but now for perturbations represented by set-valued mappings. This case turns out to be considerably more involved than the single-valued case. To deal with set-valued perturbations we state and prove a "double contraction mapping" theorem (Theorem 3.2). We obtain in Theorem 3.1 an extension of the main result in [9] with a proof based on Theorem 3.2; then we state a special case of it (Theorem 3.3) which is equivalent to the double contraction Theorem 3.2. In our final result, Theorem 3.5, we present a counterpart of Theorem 3.3 which employs metric regularity at a point

2. Single-valued perturbations

Our starting point is the following set-valued contraction mapping theorem established in [7], see also [8], Theorem 5E.2:

Theorem 2.1 (set-valued contraction). *Let X be a complete metric space, and consider a set-valued mapping $\Phi : X \rightrightarrows X$ and a point $\bar{x} \in X$. Suppose that there exist scalars $c > 0$ and $\lambda \in (0,1)$ such that the set* $\operatorname{gph}\Phi \cap (I\!B_c(\bar{x}) \times I\!B_c(\bar{x}))$ *is closed and*

(*i*) $d(\bar{x}, \Phi(\bar{x})) < c(1-\lambda)$;

(*ii*) $e(\Phi(u) \cap I\!B_c(\bar{x}), \Phi(v)) \le \lambda\, \rho(u,v)$ *for all* $u, v \in I\!B_c(\bar{x})$.

Then Φ has a fixed point in $I\!B_c(\bar{x})$.

Theorem 2.1 is a generalization of the well known Nadler theorem which says that when $\Phi : X \rightrightarrows X$ is closed-valued and Lipschitz continuous with constant $\lambda \in (0,1)$, then Φ has a fixed point; for a proof that Theorem 2.1 implies Nadler's theorem, see [8], Theorem 5E.8, p. 291. If Φ is single-valued then the fixed point is unique in $I\!B_c(\bar{x})$, and both Nadler's theorem and its generalization in Theorem 2.1 become the usual (Banach) contraction mapping principle.

Theorem 2.1 is a particular case of the following contraction mapping theorem, which we haven't seen in the literature and therefore we supply it with a proof.

Theorem 2.1A (extended set-valued contraction). *Consider a complete metric space X, a set-valued mapping $\Phi : X \rightrightarrows X$ and a point $\bar{x} \in X$. Suppose that there exist $c > 0$ and $\lambda \in (0,1)$ such that the set $\operatorname{gph}\Phi \cap (\mathbb{B}_c(\bar{x}) \times \mathbb{B}_c(\bar{x}))$ is closed and*

(i) $d(\bar{x}, \Phi(\bar{x})) < c(1-\lambda)$;

(ii) $e(\Phi(u) \cap \mathbb{B}_{\rho(u,v)}(u), \Phi(v)) \leq \lambda\, \rho(u,v)$ *for all* $u, v \in X$ *such that* $\rho(\bar{x}, u) + \rho(u,v) < c$.

Then Φ has a fixed point in $\mathbb{B}_c(\bar{x})$.

Proof. By (i) and (ii) there exists $\lambda < \bar{\lambda} < 1$ such that

$(i)'$ $d(\bar{x}, \Phi(\bar{x})) < c(1-\bar{\lambda})$;

$(ii)'$ $e(\Phi(u) \cap \mathbb{B}_{\rho(u,v)}(u), \Phi(v)) < \bar{\lambda}\, \rho(u,v)$ for all $u \neq v \in X$ such that $\rho(\bar{x}, u) + \rho(u,v) < c$.

Let $x_0 \in \Phi(\bar{x})$ be such that $\rho(x_0, \bar{x}) < c(1-\bar{\lambda})$. If $x_0 = \bar{x}$ there is nothing to prove. If not, by $(ii)'$, there exists $x_1 \in \Phi(x_0)$ such that $\rho(x_0, x_1) < \bar{\lambda}\rho(\bar{x}, x_0)$. Then

$$\rho(\bar{x}, x_1) \leq \rho(\bar{x}, x_0) + \rho(x_0, x_1) < c(1-\bar{\lambda})(1+\bar{\lambda}) < c.$$

If $x_0 = x_1$, then $x_0 \in \Phi(x_0)$ and we are done. If not, by $(ii)'$, for some $x_2 \in \Phi(x_1)$ we have $\rho(x_1, x_2) < \bar{\lambda}\rho(x_0, x_1) < \bar{\lambda}^2 c(1-\bar{\lambda})$. Therefore

$$\rho(\bar{x}, x_1) + \rho(x_1, x_2) < c(1-\bar{\lambda})(1+\bar{\lambda}+\bar{\lambda}^2) < c.$$

If $x_2 = x_1$, then, again, there is no more to prove.

We proceed using an induction argument. Assume that for some $k \geq 2$ we have constructed $x_1, \ldots, x_k$ such that for every integer $1 \leq j \leq k-1$ we have $x_{j+1} \in \Phi(x_j)$, $x_j \neq x_{j+1}$, $\rho(x_j, x_{j+1}) < \bar{\lambda}\rho(x_{j-1}, x_j)$ and $\rho(\bar{x}, x_j) + \rho(x_j, x_{j+1}) < c(1-\bar{\lambda})(1+\bar{\lambda}+ ... +\bar{\lambda}^{j+1})$. From $(ii)'$ we deduce the existence of $x_{k+1} \in \Phi(x_k)$ such that

$$\rho(x_k, x_{k+1}) < \bar{\lambda}\rho(x_{k-1}, x_k) < \bar{\lambda}^k \rho(x_0, x_1) < \bar{\lambda}^{k+1}\rho(x_0, \bar{x}) < \bar{\lambda}^{k+1} c(1-\bar{\lambda}).$$

Therefore

$$\begin{aligned}\rho(\bar{x}, x_k) + \rho(x_k, x_{k+1}) &\leq \rho(\bar{x}, x_{k-1}) + \rho(x_{k-1}, x_k) + \rho(x_k, x_{k+1}) \\ &< c(1-\bar{\lambda})(1+\bar{\lambda}+ ... +\bar{\lambda}^{k+1}) < c.\end{aligned}$$

If $x_{k+1} = x_k$, then $x_k \in \Phi(x_k)$ and the conclusion of the theorem follows.

The induction argument implies that either we obtain a fixed point of Φ as required, or we can construct a sequence $\{x_j\}_{j=0}^{\infty}$ as above such that $x_j \neq x_{j+1}$ for every j. The sequence $\{x_j\}_{j=0}^{\infty}$ is Cauchy and hence converges to some $x \in \mathbb{B}_c(\bar{x})$. Since $(x_j, x_{j+1}) \in \operatorname{gph}\Phi \cap (\mathbb{B}_c(\bar{x}) \times \mathbb{B}_c(\bar{x}))$, passing to the limit we get $(x,x) \in \operatorname{gph}\Phi \cap (\mathbb{B}_c(\bar{x}) \times \mathbb{B}_c(\bar{x}))$. The proof is complete. □

The following Theorem 2.2 is an easy consequence of Theorem 5E.3 in [8] which was proved there by utilizing the contraction mapping Theorem 2.1.

Theorem 2.2 (metric Lyusternik-Graves). *Let X be a complete metric space, Y and P be metric spaces and let κ, μ and ν be positive constants such that $\kappa\mu < 1$. Consider a mapping $F : X \rightrightarrows Y$ and a function $g : X \times P \to Y$, and let $\bar{x} \in X$, $\bar{p} \in P$ and $\bar{y} \in Y$ be such that $\bar{y} \in F(\bar{x})$ and $\bar{y} = g(\bar{x}, \bar{p})$. Assume that $\operatorname{gph} F$ is locally closed at $(\bar{x}, \bar{y})$, that F is metrically regular at $\bar{x}$ for $\bar{y}$ with $\operatorname{reg}(F; \bar{x}|\bar{y}) < \kappa$ and g is Lipschitz continuous around $(\bar{x}, \bar{p})$ with $\widehat{\operatorname{lip}}_x(g; (\bar{x}, \bar{p})) < \mu$ and $\widehat{\operatorname{lip}}_p(g; (\bar{x}, \bar{p})) < \nu$. Then the mapping $p \mapsto \operatorname{Fix}(F^{-1}(g(\cdot, p)))$ is Aubin continuous at $\bar{p}$ for $\bar{x}$ with constant $\kappa\nu/(1 - \kappa\mu)$.*

The implication $(i) \Rightarrow (ii)$ in the Lyusternik-Graves theorem (Theorem 1.8) follows from Theorem 2.2 for $P = Y$, $\bar{p} = f(\bar{x})$, $F(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x})$ and $g(x, p) = p - f(x) + f(\bar{x}) + Df(\bar{x})(x - \bar{x})$ while the opposite implication comes from taking $F(x) = f(x)$ and $g(x, p) = p + f(x) - f(\bar{x}) - Df(\bar{x})(x - \bar{x})$.

We will present next a short proof of Theorem 1.9 which follows in an elementary way from the contraction mapping Theorem 2.1.

Proof of Theorem 1.9 from Theorem 2.1. Let $y \in Y$ and consider the mapping

$$X \ni x \mapsto \Phi_y(x) = f^{-1}(-h(x) + y) \in X.$$

By Proposition 1.5, f^{-1} is Lipschitz continuous on Y with constant κ and, by assumption, h is Lipschitz continuous on X with constant μ. Then the composite mapping Φ_y is Lipschitz continuous on X with constant $\kappa\mu < 1$. Nadler's theorem yields the existence of a fixed point $x \in \Phi_y(x)$, that is, $x \in f^{-1}(-h(x) + y)$. But then $\operatorname{gph}(f + h) \neq \emptyset$ and hence condition $(*)$ is satisfied.

Let $\kappa^+ > \kappa$ and $\mu^+ > \mu$ be such that $\kappa^+\mu^+ < 1$. We will prove that $(f + h)^{-1}$ is Lipschitz continuous on Y. Let $y, y' \in Y$, $y' \neq y$ and let $x' \in (f + h)^{-1}(y')$. We will show now that the mapping Φ_y defined above has a fixed point $x \in \Phi_y(x)$ in the closed ball $I\!B_\gamma(x')$ centered at x' with radius

$$\gamma := \frac{\kappa^+ \rho(y, y')}{1 - \kappa^+\mu^+}.$$

To do that we apply Theorem 2.1 with $\bar{x} = x'$ and $c = \gamma$. Clearly, since both f and h have closed graphs, the set $\operatorname{gph} \Phi_y \cap (I\!B_\gamma(x') \times I\!B_\gamma(x'))$ is closed. Furthermore, we have

$$\begin{aligned} d(x', \Phi_y(x')) &= d(x', f^{-1}(-h(x') + y)) \leq \kappa\rho(-h(x') + y, f(x')) \\ &= \kappa\rho(y, (f + h)(x')) = \kappa\rho(y, y') < \kappa^+\rho(y, y') = \gamma(1 - \kappa^+\mu^+). \end{aligned}$$

Moreover, we have that for all $u, v \in I\!B_\gamma(x')$,

$$\begin{aligned} e(\Phi_y(u) \cap I\!B_\gamma(x'), \Phi_y(v)) &\leq \operatorname{haus}(f^{-1}(-h(u) + y), f^{-1}(-h(v) + y)) \\ &\leq \kappa^+ \rho(h(u), h(v)) \leq \kappa^+\mu^+ \rho(u, v). \end{aligned}$$

Hence, by Theorem 2.1 we obtain the existence of a fixed point $x \in \Phi_y(x)$ within distance at most γ from x'. Since $x \in (h+f)^{-1}(y)$, we have

$$d(x', (f+h)^{-1}(y)) \leq \gamma = \frac{\kappa^+}{1-\kappa^+\mu^+}\rho(y', y). \tag{6}$$

Since $y, y' \in Y$ and $x' \in (f+h)^{-1}(y')$ were arbitrarily chosen, (6) tells us that $(h+f)^{-1}$ is Lipschitz continuous with constant $\kappa^+/(1-\kappa^+\mu^+)$ on Y. Hence, by Proposition 1.5, the mapping $f+h$ is metrically regular on X for Y with constant $\kappa^+/(1-\kappa^+\mu^+)$. Since κ^+ and μ^+ can be arbitrarily close to κ and μ, respectively, the proof is complete. □

Now we present a *global* Lyusternik-Graves theorem which is in line with Theorem 2.2 and which is equivalent to Theorem 2.1.

Theorem 2.3. *Let X be a complete metric space, Y and P be metric spaces and let κ, μ be positive constants such that $\kappa\mu < 1$. Let $\alpha > 0$ and consider a mapping $F : X \rightrightarrows Y$ and a function $g : X \times P \to Y$, and let $(\bar{x}, \bar{p}) \in X \times P$ and $(\bar{\bar{x}}, \bar{y}) \in X \times Y$ be such that $U := I\!B_\alpha(\bar{x}) \cap I\!B_\alpha(\bar{\bar{x}}) \neq \emptyset$. Assume that the set $\operatorname{gph} F \cap (U \times I\!B_\alpha(\bar{y}))$ is closed, the set $\operatorname{gph} F \cap (I\!B_\alpha(\bar{\bar{x}}) \times I\!B_\alpha(\bar{y}))$ is nonempty, and F is metrically regular on $I\!B_\alpha(\bar{\bar{x}})$ for $I\!B_\alpha(\bar{y})$ with constant κ. Also, assume that there exists a neighborhood Q of $\bar{p}$ such that g is continuous in $U \times Q$, Lipschitz continuous with respect to x in $I\!B_\alpha(\bar{x})$ uniformly in $p \in Q$ with constant μ, and satisfies*

$$\rho(\bar{y}, g(x,p)) \leq \alpha \quad \textit{for every } x \in U \textit{ and } p \in Q. \tag{7}$$

Let a and ε be any positive constants such that

$$a + \varepsilon \leq \alpha. \tag{8}$$

Then for any $x \in I\!B_a(\bar{x}) \cap I\!B_a(\bar{\bar{x}})$ and $p \in Q$ that satisfy

$$\frac{\kappa}{1-\kappa\mu} d(g(x,p), F(x) \cap I\!B_\alpha(\bar{y})) < \varepsilon \tag{9}$$

one has

$$d(x, \operatorname{Fix}(F^{-1}(g(\cdot,p)))) \leq \frac{\kappa}{1-\kappa\mu} d(g(x,p), F(x) \cap I\!B_\alpha(\bar{y})). \tag{10}$$

In particular, there exists a fixed point of the mapping $F^{-1}(g(\cdot,p))$ which is at distance from x less than ε.

Proof. From the assumed metric regularity of F and Proposition 1.5 with condition $(*)$ satisfied, we obtain

$$e(F^{-1}(y') \cap I\!B_\alpha(\bar{\bar{x}}), F^{-1}(y)) \leq \kappa\rho(y', y) \quad \text{for all } y', y \in I\!B_\alpha(\bar{y}). \tag{11}$$

We also have that

$$\rho(g(x',p), g(x,p)) \leq \mu\rho(x', x) \quad \text{for all } x', x \in I\!B_\alpha(\bar{x}),\ p \in Q. \tag{12}$$

Pick $a > 0$ and $\varepsilon > 0$ that satisfies (8) and then choose $x \in I\!B_a(\bar{x}) \cap I\!B_a(\bar{\bar{x}})$ and $p \in Q$ such that (9) holds. If there is no such x and p the theorem is formally true. If $d(g(x,p), F(x) \cap I\!B_\alpha(\bar{y})) = 0$, then $x \in \mathrm{Fix}(F^{-1}(g(\cdot,p)))$, the left side of (10) is zero and there is nothing to prove. Let $d(g(x,p), F(x) \cap I\!B_\alpha(\bar{y})) > 0$ and let $\kappa^+ > \kappa$ be such that

$$\frac{\kappa^+}{1-\kappa\mu} d(g(x,p), F(x) \cap I\!B_\alpha(\bar{y})) \leq \varepsilon,$$

and denote

$$\gamma := \frac{\kappa^+}{1-\kappa\mu} d(g(x,p), F(x) \cap I\!B_\alpha(\bar{y})). \tag{13}$$

Then $\gamma \leq \varepsilon$ and for any $u \in I\!B_\gamma(x)$ we have

$$\rho(u,\bar{x}) \leq \rho(u,x) + \rho(x,\bar{x}) \leq \gamma + a \leq \varepsilon + a \leq \alpha.$$

In the same way, $\rho(u,\bar{\bar{x}}) \leq \alpha$, and hence

$$I\!B_\gamma(x) \subset U. \tag{14}$$

We apply Theorem 2.1 to the mapping $x \mapsto \Phi_p(x) := F^{-1}(g(x,p))$ with $\bar{x} = x$, $c = \gamma$ and $\lambda = \kappa\mu$. Clearly, the set $\mathrm{gph}\,\Phi_p \cap (I\!B_\gamma(x) \times I\!B_\gamma(x))$ is closed. Furthermore, utilizing metric regularity of F, (7) and (13), we obtain

$$\begin{aligned} d(x, \Phi_p(x)) = d(x, F^{-1}(g(x,p))) &\leq \kappa d(g(x,p), F(x) \cap I\!B_\alpha(\bar{y})) \\ &< \kappa^+ d(g(x,p), F(x) \cap I\!B_\alpha(\bar{y})) = \gamma(1-\kappa\mu). \end{aligned}$$

Also, for any $u, v \in I\!B_\gamma(x)$, from (7), (11), (12) and (14) we have

$$\begin{aligned} e(\Phi_p(u) \cap I\!B_\gamma(x), \Phi_p(v)) &\leq e(F^{-1}(g(u,p)) \cap I\!B_\alpha(\bar{\bar{x}}), F^{-1}(g(v,p))) \\ &\leq \kappa\rho(g(u,p), g(v,p)) \leq \kappa\mu\rho(u,v). \end{aligned}$$

Thus, Theorem 2.1 applies and we obtain the existence of $\tilde{x} \in \Phi_p(\tilde{x}) \cap I\!B_\gamma(x)$, that is $\tilde{x} \in F^{-1}(g(\tilde{x},p))$ and is at distance at most γ from x. Utilizing (13) and noting that κ^+ can be arbitrarily close to κ, we complete the proof. □

Proof of Theorem 2.1 from Theorem 2.3. We apply Theorem 2.3 with $X = Y = P$, $F = \Phi^{-1}$, $g(x,p) = x$, $\kappa = \lambda$, $\mu = 1$ and $\alpha = c$. Then we choose $\bar{\bar{x}}$, $\bar{x}$ and $\bar{y}$ in Theorem 2.3 equal $\bar{x}$ in Theorem 2.1, $a = c(1-\kappa)$ and $\varepsilon = c\kappa$. By assumption, $\Phi = F^{-1}$ is Aubin continuous on $I\!B_c(\bar{x})$ for $I\!B_c(\bar{x})$; hence, $I\!B_c(\bar{x}) \subset \mathrm{dom}\,\Phi = \mathrm{rge}\,F$ and then by Proposition 1.5, F is metrically regular on $I\!B_c(\bar{x})$ for $I\!B_c(\bar{x})$ with constant $\kappa = \lambda$. Note that conditions (7) and (8) hold trivially. From the condition $d(\bar{x}, F^{-1}(\bar{x})) < c(1-\kappa)$ it follows that there exists $\tilde{x} \in F^{-1}(\bar{x})$ such that $\rho(\tilde{x},\bar{x}) < c(1-\kappa) = a$. Then $\bar{x} \in F(\tilde{x})$ and

$$\frac{\kappa}{1-\kappa} d(\tilde{x}, F(\tilde{x}) \cap I\!B_\alpha(\bar{x})) \leq \frac{\kappa}{1-\kappa}\rho(\bar{x},\tilde{x}) < \kappa c = \varepsilon,$$

thus condition (9) holds for $x = \tilde{x}$. Then, by the last claim in the statement of Theorem 2.3, $\Phi = F^{-1}$ has a fixed point in $I\!B_\varepsilon(\tilde{x})$. But $I\!B_\varepsilon(\tilde{x}) \subset I\!B_c(\bar{x})$ and hence the proof is complete. □

3. Set-valued perturbations

Theorem 2.2 does not hold in general when the perturbation g is a set-valued mapping, and the reason is that adding two set-valued mappings may result in mismatching their reference points, see [8], Example 5E.6, p. 291. In our previous paper [9] we showed a way to go around this difficulty. The following result is a generalization of Theorem 6 in [9].

Theorem 3.1. *Let X, Y be metric spaces and κ and μ be positive constants such that $\kappa\mu < 1$. Let $\alpha \in (0, \infty]$ and $\beta \in (0, \infty]$. Consider mappings $F : X \rightrightarrows Y$, $\Psi : X \rightrightarrows Y$ and points $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, $(\bar{\bar{x}}, \bar{\bar{y}}) \in X \times Y$. Assume that either one of the sets $\operatorname{gph} F \cap (I\!B_\alpha(\bar{x}) \times I\!B_\beta(\bar{y}))$ and $\operatorname{gph} \Psi \cap (I\!B_\alpha(\bar{\bar{x}}) \times I\!B_\beta(\bar{\bar{y}}))$ is complete while the other is closed, or both sets $\operatorname{gph}(F^{-1} \circ \Psi) \cap ((I\!B_\alpha(\bar{x}) \cap I\!B_\alpha(\bar{\bar{x}})) \times (I\!B_\alpha(\bar{x}) \cap I\!B_\alpha(\bar{\bar{x}})))$ and $\operatorname{gph}(\Psi^{-1} \circ F) \cap (I\!B_\beta(\bar{y}) \cap I\!B_\beta(\bar{\bar{y}})) \times (I\!B_\beta(\bar{y}) \cap I\!B_\beta(\bar{\bar{y}}))$ are complete. Also assume that F is metrically regular on $I\!B_\alpha(\bar{x})$ for $I\!B_\beta(\bar{y})$ with constant κ and Ψ has the Aubin property on $I\!B_\alpha(\bar{\bar{x}})$ for $I\!B_\beta(\bar{\bar{y}})$ with constant μ. Let a and b be any positive scalars that satisfy*

$$\frac{a + \kappa b}{1 - \kappa\mu} + a < \alpha \quad \textit{and} \quad \frac{a + \kappa b}{\mu(1 - \kappa\mu)} + b < \beta \tag{15}$$

and denote

$$U := I\!B_a(\bar{x}) \cap I\!B_a(\bar{\bar{x}}) \quad \textit{and} \quad V := I\!B_b(\bar{y}) \cap I\!B_b(\bar{\bar{y}}).$$

Then for any $x \in U$,

$$d(x, \operatorname{Fix}(F^{-1} \circ \Psi)) \leq \frac{\kappa}{1 - \kappa\mu} \mathbf{d}(\Psi(x) \cap V, F(x) \cap I\!B_b(\bar{y})), \tag{16}$$

and for any $y \in \Psi(x) \cap V$

$$d(y, \operatorname{Fix}(\Psi \circ F^{-1})) \leq \frac{\kappa}{\mu(1 - \kappa\mu)} \mathbf{d}(\Psi(x) \cap V, F(x) \cap I\!B_b(\bar{y})). \tag{17}$$

If the constants a and b are chosen so that in addition F is metrically regular at $\bar{x}$ for $\bar{y}$ according to (2) and F^{-1} is Aubin continuous at $\bar{y}$ for $\bar{x}$ according to (3), both with neighborhoods $I\!B_a(\bar{x})$ and $I\!B_b(\bar{y})$, then the intersection with $I\!B_b(\bar{y})$ in the right sides of (16) and (17) can be dropped.
In the case when $\alpha = \beta = +\infty$, the constants a and b can be taken equal to $+\infty$ and then $U = X$ and $V = Y$.

We only stated Theorem 6 in [9] indicating that it can be proved by closely following the argument in another proof there using an iterative procedure in line with the proofs of Lyusternik, Graves and Milyutin. Theorem 3.1 stated above is more general than Theorem 6 in [9]. We will supply it with a proof utilizing a contraction mapping theorem.

We will present next a double contraction mapping theorem. Since we were not able to identify such a result in the literature, we give a full proof of it.

Theorem 3.2 (double contraction). *Let X and Y be two metric spaces. Consider a set-valued mapping $\Phi : X \rightrightarrows Y$ and a set-valued mapping $\Upsilon : Y \rightrightarrows X$. Let $\bar{x} \in X$ and $\bar{y} \in Y$ and let c, κ and μ be positive scalars such that $\kappa\mu < 1$. Assume that one of the sets $\operatorname{gph}\Phi \cap (\mathbb{B}_c(\bar{x}) \times \mathbb{B}_{c/\mu}(\bar{y}))$ and $\operatorname{gph}\Upsilon \cap (\mathbb{B}_{c/\mu}(\bar{y}) \times \mathbb{B}_c(\bar{x}))$ is closed while the other is complete, or both sets $\operatorname{gph}(\Phi \circ \Upsilon) \cap (\mathbb{B}_c(\bar{x}) \times \mathbb{B}_c(\bar{x}))$ and $\operatorname{gph}(\Upsilon \circ \Phi) \cap (\mathbb{B}_{c/\mu}(\bar{y}) \times \mathbb{B}_{c/\mu}(\bar{y}))$ are complete. Also, suppose that the following conditions hold:*

(a) $d(\bar{y}, \Phi(\bar{x})) < c(1-\kappa\mu)/\mu$;
(b) $d(\bar{x}, \Upsilon(\bar{y})) < c(1-\kappa\mu)$;
(c) $e(\Phi(u) \cap \mathbb{B}_{c/\mu}(\bar{y}), \Phi(v)) \le \kappa\, \rho(u,v)$ *for all* $u, v \in \mathbb{B}_c(\bar{x})$ *such that* $\rho(u,v) \le c(1-\kappa\mu)/\mu$;
(d) $e(\Upsilon(u) \cap \mathbb{B}_c(\bar{x}), \Upsilon(v)) \le \mu\, \rho(u,v)$ *for all* $u, v \in \mathbb{B}_{c/\mu}(\bar{y})$ *such that* $\rho(u,v) \le c(1-\kappa\mu)$.

Then there exist $\hat{x} \in \mathbb{B}_c(\bar{x})$ and $\hat{y} \in \mathbb{B}_{c/\mu}(\bar{y})$ such that $\hat{y} \in \Phi(\hat{x})$ and $\hat{x} \in \Upsilon(\hat{y})$.
If the mappings $\mathbb{B}_c(\bar{x}) \ni x \mapsto \Phi(x) \cap \mathbb{B}_{c/\mu}(\bar{y})$ and $\mathbb{B}_{c/\mu}(\bar{y}) \ni y \mapsto \Upsilon(y) \cap \mathbb{B}_c(\bar{x})$ are single-valued, then the points $\hat{x}$ and $\hat{y}$ are unique in $\mathbb{B}_c(\bar{x})$ and $\mathbb{B}_{c/\mu}(\bar{y})$, respectively.

Proof. By assumptions (a) and (b) there exists $y^1 \in \Phi(\bar{x})$ such that $\rho(y^1, \bar{y}) < c(1-\kappa\mu)/\mu$ and there exists $x^1 \in \Upsilon(\bar{y})$ such that $\rho(x^1, \bar{x}) < c(1-\kappa\mu)$. Proceeding by induction, let $x^0 = \bar{x}$, $y^0 = \bar{y}$ and suppose that for $k = 0, 1, \ldots, j-1$ there exist

$$y^{k+1} \in \Phi(x^k) \cap \mathbb{B}_{c/\mu}(\bar{y}) \quad \text{and} \quad x^{k+1} \in \Upsilon(y^{k+1}) \cap \mathbb{B}_c(\bar{x})$$

with

$$\rho(x^{k+1}, x^k) < c(1-\kappa\mu)(\kappa\mu)^k \quad \text{and} \quad \rho(y^{k+1}, y^k) < \frac{c}{\mu}(1-\kappa\mu)(\kappa\mu)^k.$$

By assumption (c),

$$\begin{aligned} d(y^j, \Phi(x^j)) &\le e(\Phi(x^{j-1}) \cap \mathbb{B}_{c/\mu}(\bar{y}), \Phi(x^j)) \\ &\le \kappa\, \rho(x^j, x^{j-1}) < \kappa c(1-\kappa\mu)(\kappa\mu)^{j-1} \le \frac{c}{\mu}(1-\kappa\mu)(\kappa\mu)^j. \end{aligned}$$

This implies there is an $y^{j+1} \in \Phi(x^j)$ such that

$$\rho(y^{j+1}, y^j) < \frac{c}{\mu}(1-\kappa\mu)(\kappa\mu)^j.$$

Also, by assumption (d),

$$d(x^j, \Upsilon(y^{j+1})) \le e(\Upsilon(y^j) \cap \mathbb{B}_c(\bar{x}), \Upsilon(y^{j+1})) \le \mu\, \rho(y^j, y^{j+1}) < \mu\frac{c}{\mu}(1-\kappa\mu)(\kappa\mu)^j,$$

which yields the existence of an $x^{j+1} \in \Upsilon(y^{j+1})$ such that

$$\rho(x^{j+1}, x^j) < c(1-\kappa\mu)(\kappa\mu)^j.$$

By the triangle inequality,

$$\rho(y^{j+1},\bar{y}) \le \sum_{i=0}^{j} \rho(y^{i+1},y^i) < \frac{c}{\mu}(1-\kappa\mu)\sum_{i=0}^{j}(\kappa\mu)^i \le \frac{c}{\mu}.$$

and

$$\rho(x^{j+1},\bar{x}) \le \sum_{i=0}^{j} \rho(x^{i+1},x^i) < c(1-\kappa\mu)\sum_{i=0}^{j}(\kappa\mu)^i \le c.$$

Hence $y^{j+1} \in \Phi(x^j) \cap I\!\!B_{c/\mu}(\bar{y})$ and $x^{j+1} \in \Upsilon(y^j) \cap I\!\!B_c(\bar{x})$. The induction step is complete.

For any $k > m > 1$ we then have

$$\rho(x^k,x^m) \le \sum_{i=m}^{k-1} \rho(x^{i+1},x^i) < c(1-\kappa\mu)\sum_{i=m}^{k-1}(\kappa\mu)^i < c(\kappa\mu)^m.$$

and

$$\rho(y^k,y^m) \le \sum_{i=m}^{k-1} \rho(y^{i+1},y^i) < \frac{c}{\mu}(1-\kappa\mu)\sum_{i=m}^{k-1}(\kappa\mu)^i < \frac{c}{\mu}(\kappa\mu)^m.$$

Thus $\{(x^k,y^k)\}$ is a Cauchy sequence. Note that $(x^{k-1},y^k) \in \operatorname{gph}\Phi \cap (I\!\!B_c(\bar{x}) \times I\!\!B_{c/\mu}(\bar{y}))$, a closed set, and similarly $(y^k,x^k) \in \operatorname{gph}\Upsilon \cap (I\!\!B_{c/\mu}(\bar{y}) \times I\!\!B_c(\bar{x}))$, a closed set. Let one of these sets, say $\operatorname{gph}\Phi \cap (I\!\!B_c(\bar{x}) \times I\!\!B_{c/\mu}(\bar{y}))$, be complete. Then (x^{k-1},y^k) is convergent to a point $(\hat{x},\hat{y})$ with the desired properties. Otherwise, since both (x^k,x^{k+1}) and (y^k,y^{k+1}) belong to both sets $\operatorname{gph}(\Phi\circ\Upsilon)\cap(I\!\!B_c(\bar{x})\times I\!\!B_c(\bar{x}))$ and $\operatorname{gph}(\Upsilon\circ\Phi)\cap(I\!\!B_{c/\mu}(\bar{y})\times I\!\!B_{c/\mu}(\bar{y}))$, respectively, that are complete, we conclude that the sequence (x^k,y^k) is convergent to $(\hat{x},\hat{y})$ that again satisfies the conditions in the statement of the theorem.

If the mappings $I\!\!B_c(\bar{x}) \ni x \mapsto \Phi(x) \cap I\!\!B_{c/\mu}(\bar{y})$ and $I\!\!B_{c/\mu} \ni y \mapsto \Upsilon(y) \cap I\!\!B_c(\bar{x})$ are single-valued, then the uniqueness of the points $\hat{x}$ and $\hat{y}$ in $I\!\!B_c(\bar{x})$ and $I\!\!B_{c/\mu}(\bar{y})$, respectively, follows directly from the standard contraction mapping principle. □

We should point out that Theorem 3.2 is somewhat similar to the "fixed double-point" Theorem 2 in [13] which however uses assumptions involving points in the graphs of the mappings considered that are at certain distances from each other. In addition, in [13] a property of "covering" at a point in the graph of a mapping is utilized which is different from the conditions (*c*), (*d*). We do not exclude the possibility that these two theorems are connected much closer than it appears to us; however, finding the exact relationship between them may require lengthy derivations that go beyond the scope of the present paper.

Note also that Theorem 3.2 is a straightforward generalization of Theorem 2.1, which can be obtained from it by taking $X = Y$, $\bar{y} = \bar{x}$, and $\Upsilon(y) = y$ for all $y \in Y$. An obvious generalization of Theorem 3.2 could be in line with Theorem 2.1A, but we shall not go here any further.

Proof of Theorem 3.1 from Theorem 3.2. Choose a and b to satisfy (15) and let $\varepsilon > 0$ be such that

$$\frac{(1+\varepsilon)(a+\kappa b)}{1-\kappa\mu} + a < \alpha \quad \text{and} \quad \frac{(1+\varepsilon)(a+\kappa b)}{\mu(1-\kappa\mu)} + b < \beta. \tag{18}$$

If U is empty then the theorem is formally true. Let $x \in U$. If $\Psi(x) \cap V$ is empty, then (16) and (17) are trivially satisfied. Let $y \in \Psi(x) \cap V$. Then

$$d(\bar{x}, F^{-1}(y)) \leq \kappa d(y, F(\bar{x}) \cap I\!B_b(\bar{y})) < \kappa\rho(y, \bar{y}) < +\infty,$$

and hence $F^{-1}(y) \neq \emptyset$. Thus, there exists $z \in F^{-1}(y)$ such that $\rho(x, z) \leq d(x, F^{-1}(y)) + \varepsilon$. If $z = x$ then $x \in \mathrm{Fix}(F^{-1} \circ \Psi)$ and (16) holds because its left side is zero. Let $z \neq x$. Then we have

$$d(x, F^{-1}(y)) < (1+\varepsilon)\rho(z, x). \tag{19}$$

Also note that

$$\begin{aligned} \rho(z, x) &\leq \rho(x, \bar{x}) + d(\bar{x}, F^{-1}(y)) + \varepsilon \\ &\leq a + \kappa d(y, F(\bar{x}) \cap I\!B_b(\bar{y})) \leq a + \kappa\rho(y, \bar{y}) \leq a + \kappa b. \end{aligned}$$

Let

$$c := \frac{1+\varepsilon}{1-\kappa\mu}\rho(z, x). \tag{20}$$

Then

$$c \leq \frac{1+\varepsilon}{1-\kappa\mu}(a+\kappa b)$$

and, from (18), any $u \in I\!B_c(x)$ satisfies

$$\rho(u, \bar{x}) \leq \rho(u, x) + \rho(x, \bar{x}) \leq c + a \leq \frac{1+\varepsilon}{1-\kappa\mu}(a+\kappa b) + a < \alpha.$$

Further, from (18) again, any $v \in I\!B_{c/\mu}(y)$ satisfies

$$\rho(v, \bar{y}) \leq \rho(v, y) + \rho(y, \bar{y}) \leq c/\mu + b \leq \frac{(1+\varepsilon)(a+\kappa b)}{\mu(1-\kappa\mu)} + b < \beta.$$

Thus, we have

$$I\!B_c(x) \subset I\!B_\alpha(\bar{x}) \quad \text{and} \quad I\!B_{c/\mu}(y) \subset I\!B_\beta(\bar{y}). \tag{21}$$

We now apply Theorem 3.2 with $\Upsilon = F^{-1}$, $\Phi = \Psi$, $\bar{x} = x$ and $\bar{y} = y$ and c defined in (20). Since $y \in \Psi(x)$, condition (a) is automatically satisfied. From (19),

$$d(x, F^{-1}(y)) < (1+\varepsilon)\rho(x, z) = c(1-\kappa\mu)$$

and hence condition (b) holds. Taking into account (21) and Proposition 1.5, the assumptions for F and Ψ imply that both conditions (c) and (d) are satisfied. Hence, there exists $\hat{x}$ and $\hat{y}$ such that $\hat{x} \in F^{-1}(\hat{y})$ and $\hat{y} \in \Psi(\hat{x})$ and moreover

$$\rho(x, \hat{x}) \leq c \quad \text{and} \quad \rho(y, \hat{y}) \leq c/\mu. \tag{22}$$

But then $\hat{x} \in \mathrm{Fix}(F^{-1} \circ \Psi)$ and

$$d(x, \mathrm{Fix}(F^{-1} \circ \Psi)) \leq \rho(x, \hat{x}) \leq \frac{1+\varepsilon}{1-\kappa\mu}\rho(z, x)$$

$$\leq \frac{1+\varepsilon}{1-\kappa\mu}(d(x, F^{-1}(y)) + \varepsilon) \leq \frac{1+\varepsilon}{1-\kappa\mu}(\kappa d(y, F(x) \cap I\!B_b(\bar{y})) + \varepsilon).$$

Since y is arbitrarily chosen in $\Psi(x) \cap V$, passing to the limit with $\varepsilon \to 0$ leads to (16). In the same way we obtain (17) from the second estimate in (22). In the case $\alpha = \infty$ we can take $a = b = \infty$ and then $I\!B_a(\bar{x}) = I\!B_a(\bar{\bar{x}}) = U = X$ and $I\!B_b(\bar{y}) = I\!B_b(\bar{\bar{y}}) = V = Y$. If we assume that a and b are such that F is metrically regular at $\bar{x}$ for $\bar{y}$ in the sense of (2) with neighborhoods $I\!B_a(\bar{x})$ and $I\!B_b(\bar{y})$, then we can remove the intersection with $I\!B_b(\bar{y})$ in the right side of (16) and (17). □

From the proof of Theorem 3.1 we can deduce the following theorem which turns out to be equivalent to the double contraction Theorem 3.2:

Theorem 3.3. *Assume that the conditions of Theorem* 3.1 *hold. Then for any* $x \in U \cap \mathrm{dom}\, F$, *any* $y \in \Psi(x) \cap V$ *and any* $\delta > 0$ *such that*

$$\frac{1}{1-\kappa\mu} d(x, F^{-1}(y)) < \delta,$$

there exists $\hat{x}$ *and* $\hat{y}$ *such that* $\hat{y} \in F(\hat{x})$ *and* $\hat{x} \in \Psi(\hat{y})$, *and moreover*

$$\rho(x, \hat{x}) \leq \delta \quad \textit{and} \quad \rho(y, \hat{y}) \leq \frac{\delta}{\mu}. \tag{23}$$

Proof. Choose $\delta > 0$ and repeat the proof of Theorem 3.1 above with $\varepsilon > 0$ small enough so that

$$\frac{1+\varepsilon}{1-\kappa\mu}(d(x, F^{-1}(y)) + \varepsilon) < \delta.$$

Then $\delta \geq c$ and (23) follows from (22). □

Remark 3.4. Note that if $\bar{\bar{y}} \in \Psi(\bar{\bar{x}})$, then the mappings F and Ψ play symmetric roles in Theorem 3.1 and can switch sides, obtaining that the distance from any $y \in V$ to the set $\mathrm{Fix}(\Psi \circ F^{-1})$ is bounded by the constant $\mu/(1-\kappa\mu)$ times the minimal distance between the truncated $F^{-1}(y)$ and $\Psi(y)$.

Our next and final theorem is a version of Theorem 3.1 which easily follows from its proof and in which metric regularity on a set is replaced by metric regularity at a point, and the same for the Aubin continuity.

Theorem 3.5. *Let* X, Y *be metric spaces and* κ, μ *be positive constants such that* $\kappa\mu < 1$. *Consider* $F : X \rightrightarrows Y$, $\Psi : X \rightrightarrows Y$ *and* $(\bar{x}, \bar{y}) \in \mathrm{gph}\, F$, $(\bar{\bar{x}}, \bar{\bar{y}}) \in \mathrm{gph}\, \Psi$ *at which either one of the sets* $\mathrm{gph}\, F$ *and* $\mathrm{gph}\, \Psi$ *is locally complete while the other is locally closed, or both sets* $\mathrm{gph}(F^{-1} \circ \Psi)$ *and* $\mathrm{gph}(\Psi^{-1} \circ F)$ *are locally complete at* $(\bar{\bar{x}}, \bar{x})$ *and* $(\bar{x}, \bar{\bar{x}})$, *respectively. Also assume that* F *is metrically regular at* $\bar{x}$ *for* $\bar{y}$ *with constant* κ *and* Ψ *has the Aubin property at* $\bar{\bar{x}}$ *for* $\bar{\bar{y}}$ *with constant* μ. *Then there* *are neighborhoods U' of $\bar{x}$, U'' of $\bar{\bar{x}}$, V' of $\bar{y}$ and V'' of $\bar{\bar{y}}$ such that for $U := U' \cap U''$ and $V := V' \cap V''$ and for any $x \in U$*

$$d(x, \mathrm{Fix}(F^{-1} \circ \Psi)) \leq \frac{\kappa}{1 - \kappa\mu} \mathbf{d}(\Psi(x) \cap V, F(x)) \tag{24}$$

and for any $y \in \Psi(x) \cap V$

$$d(y, \mathrm{Fix}(\Psi \circ F^{-1})) \leq \frac{\kappa}{\mu(1 - \kappa\mu)} \mathbf{d}(\Psi(x) \cap V, F(x)). \tag{25}$$

Proof. Clearly, there exist positive scalars α and β such that all the assumptions in Theorem 3.1 regarding local closedness/completeness as well as for metric regularity of F and Aubin continuity of Ψ are satisfied on balls with radii α and β, respectively. Choosing a and b to satisfy (15) we repeat the argument in the proof of Theorem 3.1 using the definition of metric regularity at a point (2) instead of (1), in which case the neighborhood $\mathbb{B}_b(\bar{y})$ in the right sides of (16) and (17) can be dropped. Then, taking $U' = \mathbb{B}_a(\bar{x})$, $U'' = \mathbb{B}_a(\bar{\bar{x}})$, $V' = \mathbb{B}_b(\bar{y})$ and $V'' = \mathbb{B}_b(\bar{\bar{y}})$, the statement follows from Theorem 3.1. □

Theorem 3.5 generalizes Theorem 5 in [9] where it is assumed that $\bar{x} = \bar{\bar{x}}$ and $\bar{y} = \bar{\bar{y}}$.

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