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On Linear Isometries on Non-Archimedean Power Series Spaces*

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The non-archimedean power series spaces $A_p(a,t)$ are the most known and important examples of non-archimedean nuclear Fréchet spaces. We study when the spaces $A_p(a,t)$ and $A_q(b,s)$ are isometrically isomorphic. Next we determine all linear isometries on the space $A_p(a,t)$ and show that all these maps are surjective.

Keywords: Non-archimedean power series space, linear isometry, Schauder basis

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1. Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0,\infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [2], [4] and [6].

Let Γ be the family of all non-decreasing unbounded sequences of positive real numbers. Let $a = (a_n)$, $b = (b_n) \in \Gamma$. The power series spaces of finite type $A_1(a)$ and infinite type $A_\infty(b)$ were studied in [1] and [7]–[9]. In [7] it has been proved that $A_p(a)$ has the quasi-equivalence property i.e. any two Schauder bases in $A_p(a)$ are quasi-equivalent ([7], Corollary 6).

The problem when $A_p(a)$ has a subspace (or quotient) isomorphic to $A_q(b)$ was studied in [8]. In particular, the spaces $A_p(a)$ and $A_q(b)$ are isomorphic if and only if $p = q$ and the sequences a, b are equivalent i.e. $0 < \inf_n(a_n/b_n) \le \sup_n(a_n/b_n) < \infty$ ([8], Corollary 6).

For $p \in (0,\infty]$ we denote by Λ_p the family of all strictly increasing sequences $t = (t_k)$

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of real numbers such that $\lim_k t_k = \ln p$ (if $p = \infty$, then $\ln p := \infty$).

Let $p \in (0, \infty]$, $a = (a_n) \in \Gamma$ and $t = (t_k) \in \Lambda_p$. Then the following linear space $A_p(a, t) = \{(x_n) \subset \mathbb{K} : \lim_n |x_n| e^{t_k a_n} = 0 \text{ for all } k \in \mathbb{N}\}$ with the base $(\| \cdot \|_k)$ of the norms $\|(x_n)\|_k = \max_n |x_n| e^{t_k a_n}$, $k \in \mathbb{N}$, is a Fréchet space with a Schauder basis. Clearly, $A_1(a) = A_1(a, t)$ for $a = (a_n) \in \Gamma$, $t = (t_k) = (\ln \frac{k}{k+1})$, and $A_\infty(b) = A_\infty(b, s)$ for $b = (b_n) \in \Gamma$, $s = (s_k) = (\ln k)$. Let $q(p) = 1$ for $p \in (0, \infty)$ and $q(\infty) = \infty$. It is not hard to show that for every $p \in (0, \infty]$, $a = (a_n) \in \Gamma$ and $t = (t_k) \in \Lambda_p$ the space $A_p(a, t)$ is isomorphic to $A_{q(p)}(b)$ for some $b \in \Gamma$.

Thus we can consider the spaces $A_p(a, t)$ as power series spaces.

In this paper we study linear isometries on power series spaces.

First we show that the spaces $A_p(a, t)$ and $A_q(b, s)$, for $p, q \in (0, \infty]$, $t = (t_k) \in \Lambda_p$, $s = (s_k) \in \Lambda_q$ and $a = (a_n)$, $b = (b_n) \in \Gamma$, are isometrically isomorphic if and only if there exist $C, D \in \mathbb{R}$ such that $s_k = Ct_k + D$ and $a_k = Cb_k$ for all $k \in \mathbb{N}$, and for every $k \in \mathbb{N}$ there is $\psi_k \in \mathbb{K}$ with $|\psi_k| = e^{-(D/C)a_k}$ (Theorem 3.1).

Let $p \in (0, \infty]$, $t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$.

Let (N_s) be a partition of $\mathbb{N}$ into non-empty finite subsets such that (1) $a_i = a_j$ for all $i, j \in N_s$, $s \in \mathbb{N}$; (2) $a_i < a_j$ for all $i \in N_s$, $j \in N_{s+1}$, $s \in \mathbb{N}$.

We prove that a linear map $T : A_p(a, t) \to A_p(a, t)$ with $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i$, $j \in \mathbb{N}$, is an isometry if and only if (1) $|t_{i,j}| \le e^{(a_j - a_i)t_1}$ when $a_i < a_j$; (2) $|t_{i,j}| \le e^{(a_j - a_i)\ln p}$ when $a_i > a_j$ ($e^{-\infty} := 0$); (3) $\max_{(i,j) \in N_s \times N_s} |t_{i,j}| = 1$ and $|\det[t_{i,j}]_{(i,j) \in N_s \times N_s}| = 1$ for $s \in \mathbb{N}$; (Theorem 3.5 and Proposition 3.7).

In particular, if the sequence (a_n) is strictly increasing, then a linear map $T : A_p(a, t) \to A_p(a, t)$ with $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i$, $j \in \mathbb{N}$, is an isometry if and only if (1) $|t_{i,j}| \le e^{(a_j - a_i)t_1}$ when $i < j$; (2) $|t_{i,j}| \le e^{(a_j - a_i)\ln p}$ when $i > j$; (3) $|t_{i,i}| = 1$ for $i \in \mathbb{N}$.

Finally we show that every linear isometry on $A_p(a, t)$ is surjective (Corollary 3.10 and Theorem 3.12). Thus the family $\mathcal{I}_p(a, t)$ of all linear isometries on $A_p(a, t)$ forms a group by composition of maps.

2. Preliminaries

The linear span of a subset A of a linear space E is denoted by $\operatorname{lin} A$.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x + y) \le \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\{x \in E : p(x) = 0\} = \{0\}$.

If p is a seminorm on a linear space E and $x, y \in E$ with $p(x) \ne p(y)$, then $p(x + y) = \max\{p(x), p(y)\}$.

The set of all continuous seminorms on a lcs E is denoted by $\mathcal{P}(E)$. A non-decreasing sequence (p_k) of continuous seminorms on a metrizable lcs E is a *base* in $\mathcal{P}(E)$ if for any $p \in \mathcal{P}(E)$ there are $C > 0$ and $k \in \mathbb{N}$ such that $p \le Cp_k$. A complete metrizable lcs is called a *Fréchet space*.

Let E and F be locally convex spaces. A map $T : E \to F$ is called an *isomorphism* if it is linear, injective, surjective and the maps T, T^{-1} are continuous. If there exists an isomorphism $T : E \to F$, then we say that E is isomorphic to F. The family of all continuous linear maps from E to F we denote by $L(E, F)$.

Let E and F be Fréchet spaces with fixed bases $(\|\cdot\|_k)$ and $(|||\cdot|||_k)$ in $\mathcal{P}(E)$ and $\mathcal{P}(F)$, respectively. A map $T : E \to F$ is an *isometry* if $|||Tx - Ty|||_k = \|x - y\|_k$ for all $x, y \in E$ and $k \in \mathbb{N}$; clearly, a linear map $T : E \to F$ is an isometry if and only if $|||Tx|||_k = \|x\|_k$ for all $x \in E$ and $k \in \mathbb{N}$. A linear map $T : E \to F$ is a *contraction* if $|||Tx|||_k \le \|x\|_k$ for all $x \in E$ and $k \in \mathbb{N}$. A *rotation* on E is a surjective isometry $T : E \to E$ with $T(0) = 0$.

By [3], Corollary 1.7, we have the following

Proposition A. *Let $m \in \mathbb{N}$. Equip the linear space $\mathbb{K}^m$ with the maximum norm. Let $T : \mathbb{K}^m \to \mathbb{K}^m$ be a linear map with $Te_j = \sum_{i=1}^m t_{i,j} e_i$ for $1 \le j \le m$. Then T is an isometry if and only if $\max_{i,j} |t_{i,j}| = 1$ and $|\det[t_{i,j}]| = 1$.*

A sequence (x_n) in a lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^\infty \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$, and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n (n \in \mathbb{N})$ are continuous.

The coordinate sequence (e_n) is an unconditional Schauder basis in $A_p(a, t)$.

3. Results

First we show when the power series spaces $A_p(a, t)$ and $A_q(b, s)$ are isometrically isomorphic.

Theorem 3.1. *Let $p, q \in (0, \infty]$, $t = (t_k) \in \Lambda_p$, $s = (s_k) \in \Lambda_q$ and $a = (a_n)$, $b = (b_n) \in \Gamma$. Then the spaces $A_p(a, t)$ and $A_q(b, s)$ are isometrically isomorphic if and only if*

(1) there exist $C, D \in \mathbb{R}$ such that $s_k = Ct_k + D$ and $a_k = Cb_k$ for all $k \in \mathbb{N}$;
(2) for every $k \in \mathbb{N}$ there is $\psi_k \in \mathbb{K}$ with $|\psi_k| = e^{-(D/C)a_k}$.

In this case the linear map $P : A_p(a, t) \to A_q(b, s)$, $(x_n) \to (\psi_n x_n)$ is an isometric isomorphism.

Proof. Let $T : A_p(a, t) \to A_q(b, s)$ be an isometric isomorphism and let $Te_j = \sum_{i=1}^\infty t_{i,j} e_i$ for $j \in \mathbb{N}$. Then $\max_i |t_{i,j}| e^{s_k b_i} = e^{t_k a_j}$ for all $j, k \in \mathbb{N}$; so $\max_i |t_{i,j}| e^{s_k b_i - t_k a_j} = 1$ for $j, k \in \mathbb{N}$. Let $j, k \in \mathbb{N}$ with $k > 1$. Then for some $i \in \mathbb{N}$ we have $|t_{i,j}| = e^{t_k a_j - s_k b_i}$, $|t_{i,j}| \le e^{t_{k+1} a_j - s_{k+1} b_i}$ and $|t_{i,j}| \le e^{t_{k-1} a_j - s_{k-1} b_i}$.

Hence we get $(s_{k+1} - s_k) b_i \le (t_{k+1} - t_k) a_j$ and $(t_k - t_{k-1}) a_j \le (s_k - s_{k-1}) b_i$; so

$$\frac{s_{k+1} - s_k}{t_{k+1} - t_k} \le \frac{a_j}{b_i} \le \frac{s_k - s_{k-1}}{t_k - t_{k-1}}.$$

Thus the sequence $(\frac{s_{k+1} - s_k}{t_{k+1} - t_k})$ is non-increasing. Similarly we infer that the sequence $(\frac{t_{k+1} - t_k}{s_{k+1} - s_k})$ is non-increasing, since the map $T^{-1} : A_q(b, s) \to A_p(a, t)$ is an isometric isomorphism, too. It follows that the sequence $(\frac{s_{k+1}-s_k}{t_{k+1}-t_k})$ is constant, so there is $C>0$ such that $\frac{s_{k+1}-s_k}{t_{k+1}-t_k}=C$ for all $k\in\mathbb{N}$.

Moreover, for every $j\in\mathbb{N}$ there is $i\in\mathbb{N}$ with $a_j/b_i=C$ and for every $i\in\mathbb{N}$ there is $j\in\mathbb{N}$ with $b_i/a_j=1/C$. Thus $\{a_j : j\in\mathbb{N}\}=\{Cb_i : i\in\mathbb{N}\}$.

For $l>1$ we have $s_l-Ct_l=s_1-Ct_1$, since

$$s_l-s_1=\sum_{k=1}^{l-1}(s_{k+1}-s_k)=C\sum_{k=1}^{l-1}(t_{k+1}-t_k)=C(t_l-t_1).$$

Put $D=s_1-Ct_1$, then $s_k=Ct_k+D$ for $k\in\mathbb{N}$.

Let $(j_k)\subset\mathbb{N},(i_k)\subset\mathbb{N}$ be strictly increasing sequences such that $\{a_{j_k} : k\in\mathbb{N}\}=\{a_j : j\in\mathbb{N}\}$, $\{b_{i_k} : k\in\mathbb{N}\}=\{b_i : i\in\mathbb{N}\}$ and $a_{j_k}<a_{j_k+1}, b_{i_k}<b_{i_k+1}$ for $k\in\mathbb{N}$.

Hence we get $a_{j_k}=Cb_{i_k}$ for $k\in\mathbb{N}$, since $\{a_j : j\in\mathbb{N}\}=\{Cb_i : i\in\mathbb{N}\}$.

Put $j_0=i_0=0$ and $M_r=\{j\in\mathbb{N} : j_{r-1}<j\le j_r\}$, $W_r=\{i\in\mathbb{N} : i_{r-1}<i\le i_r\}$ for $r\in\mathbb{N}$; clearly $W_r=\{i\in\mathbb{N} : Cb_i=a_{j_r}\}$.

Let $r\in\mathbb{N}$ and $(\phi_j)_{j\in M_r}\subset\mathbb{K}$ with $\max_{j\in M_r}|\phi_j|>0$. Then we have

$$\begin{aligned}\max_{j\in M_r}|\phi_j|e^{t_k a_{j_r}} &= \max_{j\in M_r}|\phi_j|e^{t_k a_j} = \left\|\sum_{j\in M_r}\phi_j e_j\right\|_k \\ &= \left\|T\left(\sum_{j\in M_r}\phi_j e_j\right)\right\|_k = \left\|\sum_{j\in M_r}\phi_j\sum_{i=1}^{\infty}t_{i,j}e_i\right\|_k \\ &= \left\|\sum_{i=1}^{\infty}\left(\sum_{j\in M_r}t_{i,j}\phi_j\right)e_i\right\|_k = \max_i\left|\sum_{j\in M_r}t_{i,j}\phi_j\right|e^{s_k b_i}.\end{aligned}$$

Thus

$$\max_i\left|\sum_{j\in M_r}t_{i,j}\phi_j\right|e^{s_k b_i-t_k a_{j_r}}=\max_{j\in M_r}|\phi_j|.$$

Let $k>1$. For some $i\in\mathbb{N}$ we have

$$\left|\sum_{j\in M_r}t_{i,j}\phi_j\right|=\max_{j\in M_r}|\phi_j|e^{t_k a_{j_r}-s_k b_i},\qquad \left|\sum_{j\in M_r}t_{i,j}\phi_j\right|\le\max_{j\in M_r}|\phi_j|e^{t_{k+1}a_{j_r}-s_{k+1}b_i}$$

and

$$\left|\sum_{j\in M_r}t_{i,j}\phi_j\right|\le\max_{j\in M_r}|\phi_j|e^{t_{k-1}a_{j_r}-s_{k-1}b_i}.$$

Hence we get $(s_{k+1}-s_k)b_i\le(t_{k+1}-t_k)a_{j_r}$ and $(t_k-t_{k-1})a_{j_r}\le(s_k-s_{k-1})b_i$; so $Cb_i\le a_{j_r}$ and $a_{j_r}\le Cb_i$. Thus $a_{j_r}=Cb_i$, so $i\in W_r$.

It follows that

$$\max_{i\in W_r}\left|\sum_{j\in M_r} t_{i,j}\phi_j\right| e^{s_k b_i - t_k a_{j_r}} = \max_{j\in M_r}|\phi_j|.$$

We have $s_k b_i - t_k a_{j_r} = (Ct_k + D)a_{j_r}/C - t_k a_{j_r} = (D/C)a_{j_r}$ for $i \in W_r$; so

$$\max_{i\in W_r}\left|\sum_{j\in M_r} t_{i,j}\phi_j\right| e^{(D/C)a_{j_r}} = \max_{j\in M_r}|\phi_j|.$$

Thus $e^{-(D/C)a_{j_r}} = |\gamma_r|$ for some $\gamma_r \in \mathbb{K}$. Put $\psi_j = \gamma_r$ for every $j \in M_r$. Then $|\psi_j| = e^{-(D/C)a_j}$ for $j \in M_r$. Since $\max_{i\in W_r}|\sum_{j\in M_r} t_{i,j}\phi_j||\psi_j^{-1}| = \max_{j\in M_r}|\phi_j|$, the linear map

$$U : \mathbb{K}^{M_r} \to \mathbb{K}^{W_r},\quad (\phi_j)_{j\in M_r} \to \left(\sum_{j\in M_r} t_{i,j}\psi_j^{-1}\phi_j\right)_{i\in W_r}$$

is an isometry, so $|M_r| \le |W_r|$. We have shown that $j_r - j_{r-1} \le i_r - i_{r-1}$ for every $r \in \mathbb{N}$. Similarly we get $i_r - i_{r-1} \le j_r - j_{r-1}$ for every $r \in \mathbb{N}$, since T^{-1} is an isometric isomorphism. Thus $j_r - j_{r-1} = i_r - i_{r-1}$ for every $r \in \mathbb{N}$; so $j_r = i_r$ for $r \in \mathbb{N}$. It follows that $a_j = Cb_j$ for $j \in \mathbb{N}$.

Now we assume that *(1)* and *(2)* hold. Then the linear map

$$P : A_p(a,t) \to A_q(b,s),\quad (x_j) \to (\psi_j x_j)$$

is an isometric isomorphism. Indeed, P is surjective since for any $y = (y_j) \in A_q(b,s)$ we have $x = (\psi_j^{-1}y_j) \in A_p(a,t)$ and $Px = y$. For $x \in A_p(a,t)$ and $k \in \mathbb{N}$ we have

$$\|Px\|_k = \max_j |\psi_j||x_j|e^{s_k b_j} = \max_j |x_j|e^{-(D/C)a_j + s_k b_j} = \max_j |x_j|e^{t_k a_j} = \|x\|_k.$$

□

By obvious modifications of the proof of Theorem 3.1 we get the following two propositions.

Proposition 3.2. *Let $p \in (0,\infty]$, $t \in \Lambda_p$ and $a = (a_n)$, $b = (b_n) \in \Gamma$. Then $A_p(b,t)$ contains a linear isometric copy of $A_p(a,t)$ if and only if a is a subsequence of b.*

If $(n_j) \subset \mathbb{N}$ is a strictly increasing sequence with $a_j = b_{n_j}$, $j \in \mathbb{N}$, then the map $T : A_p(a,t) \to A_p(b,t)$, $(x_j) \to (y_j)$, where $y_j = x_k$ if $j = n_k$ for some $k \in \mathbb{N}$, and $y_j = 0$ for all other $j \in \mathbb{N}$, is a linear isometry.

Proposition 3.3. *Let $p, q \in (0,\infty]$, $t \in \Lambda_p$, $s \in \Lambda_q$ and $a, b \in \Gamma$. If there exist linear isometries $T : A_p(a,t) \to A_q(b,s)$ and $S : A_q(b,s) \to A_p(a,t)$, then $A_p(a,t)$ and $A_q(b,s)$ are isometrically isomorphic.*

Remark 3.4. Let $p, q \in (0,\infty]$, $t \in \Lambda_p$, $s \in \Lambda_q$ and $a, b \in \Gamma$. If $P : A_p(a,t) \to A_q(b,s)$ is an isometric isomorphism, then every isometric isomorphism $T : A_p(a,t) \to A_q(b,s)$ is of the form $P \circ S$ where S is an isometric automorphism of $A_p(a,t)$.

Now we determine all linear isometries on the space $A_p(a,t)$. Recall that (N_s) is a partition of $\mathbb{N}$ into non-empty finite subsets such that (1) $a_i = a_j$ for all $i,j \in N_s$, $s \in \mathbb{N}$; (2) $a_i < a_j$ for all $i \in N_s$, $j \in N_{s+1}$, $s \in \mathbb{N}$.

Theorem 3.5. *Let $p \in (0,\infty]$, $t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$. Let $T : A_p(a,t) \to A_p(a,t)$ be a continuous linear map and let $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i$ for $j \in \mathbb{N}$.*
Then T is an isometry if and only if

(1) $|t_{i,j}| \le e^{(a_j - a_i)t_1}$ *when* $a_i < a_j$, *and* $|t_{i,j}| \le e^{(a_j-a_i)\ln p}$ *when* $a_i > a_j$;
(2) $\max_{(i,j)\in N_s \times N_s} |t_{i,j}| = 1$ *and* $|\det[t_{i,j}]_{(i,j)\in N_s\times N_s}| = 1$ *for all* $s \in \mathbb{N}$.

Proof. ($\Rightarrow$) For $k,j \in \mathbb{N}$ we have $\|Te_j\|_k = \max_i |t_{i,j}| e^{t_k a_i}$ and $\|e_j\|_k = e^{t_k a_j}$. Thus $\max_i |t_{i,j}| e^{t_k(a_i - a_j)} = 1$ for all $j,k \in \mathbb{N}$. Hence $|t_{i,j}| \le e^{t_k(a_j - a_i)}$ for all $i,j,k \in \mathbb{N}$; so $|t_{i,j}| \le \inf_k e^{t_k(a_j-a_i)}$ for all $i,j \in \mathbb{N}$. It follows (*1*); moreover $|t_{i,j}| \le 1$ when $a_i = a_j$.

Let $s \in \mathbb{N}$, $j_s = \min N_s$ and $(\beta_j)_{j\in N_s} \subset \mathbb{K}$ with $\max_{j\in N_s} |\beta_j| > 0$. Then we have

$$\left\| T\left(\sum_{j\in N_s} \beta_j e_j\right)\right\|_k = \left\| \sum_{j\in N_s} \beta_j \sum_{i=1}^{\infty} t_{i,j} e_i \right\|_k = \left\| \sum_{i=1}^{\infty} \left(\sum_{j\in N_s} \beta_j t_{i,j}\right) e_i \right\|_k = \max_i \left| \sum_{j\in N_s} \beta_j t_{i,j}\right| e^{t_k a_i}$$

and $\| \sum_{j\in N_s} \beta_j e_j \|_k = \max_{j\in N_s} |\beta_j| e^{t_k a_j} = (\max_{j\in N_s} |\beta_j|) e^{t_k a_{j_s}}$ for all $k \in \mathbb{N}$. Thus

$$\max_i \left| \sum_{j\in N_s} \beta_j t_{i,j} \right| e^{t_k(a_i - a_{j_s})} = \max_{j\in N_s} |\beta_j|, \quad k \in \mathbb{N};$$

hence $\max_{i\in N_s} |\sum_{j\in N_s} \beta_j t_{i,j}| \le \max_{j\in N_s} |\beta_j|$.

Let $k > 1$. For some $i_k \in \mathbb{N}$ we have

$$\left| \sum_{j\in N_s} \beta_j t_{i_k,j} \right| e^{t_k(a_{i_k} - a_{j_s})} = \max_{j\in N_s} |\beta_j|.$$

If $a_{i_k} < a_{j_s}$, then

$$\max_{j\in N_s} |\beta_j| \ge \left| \sum_{j\in N_s} \beta_j t_{i_k,j} \right| e^{t_{k-1}(a_{i_k} - a_{j_s})} > \left| \sum_{j\in N_s} \beta_j t_{i_k,j} \right| e^{t_k(a_{i_k} - a_{j_s})} = \max_{j\in N_s} |\beta_j|;$$

if $a_{i_k} > a_{j_s}$, then

$$\max_{j\in N_s} |\beta_j| \ge \left| \sum_{j\in N_s} \beta_j t_{i_k,j} \right| e^{t_{k+1}(a_{i_k} - a_{j_s})} > \left| \sum_{j\in N_s} \beta_j t_{i_k,j} \right| e^{t_k(a_{i_k} - a_{j_s})} = \max_{j\in N_s} |\beta_j|.$$

It follows that $a_{i_k} = a_{j_s}$, so $i_k \in N_s$ and $|\sum_{j\in N_s} \beta_j t_{i_k,j}| = \max_{j\in N_s} |\beta_j|$.

Thus the following linear map is an isometry

$$S : \mathbb{K}^{N_s} \to \mathbb{K}^{N_s}, \quad (\beta_j)_{j\in N_s} \to \left(\sum_{j\in N_s} \beta_j t_{i,j}\right)_{i\in N_s}.$$

By Proposition A we get $\max_{(i,j)\in N_s\times N_s} |t_{i,j}| = 1$ and $|\det[t_{i,j}]_{(i,j)\in N_s\times N_s}| = 1$.

($\Leftarrow$) Let $x = (\beta_j) \in A_p(a,t)$ and $k \in \mathbb{N}$. Clearly, $\|Tx\|_k = \lim_m \|T(\sum_{j=1}^m \beta_j e_j)\|_k$ and $\|x\|_k = \lim_m \|\sum_{j=1}^m \beta_j e_j\|_k$. Thus to prove that $\|Tx\|_k = \|x\|_k$ it is enough to show that $\|T(\sum_{j=1}^m \beta_j e_j)\|_k = \|\sum_{j=1}^m \beta_j e_j\|_k$ for all $m \in \mathbb{N}$. Let $m \in \mathbb{N}$. We have

$$T\left(\sum_{j=1}^m \beta_j e_j\right) = \sum_{j=1}^m \beta_j \sum_{i=1}^\infty t_{i,j} e_i = \sum_{i=1}^\infty \left(\sum_{j=1}^m \beta_j t_{i,j}\right) e_i,$$

so $L := \|T(\sum_{j=1}^m \beta_j e_j)\|_k = \max_i |\sum_{j=1}^m \beta_j t_{i,j}| e^{t_k a_i}$; clearly $P := \|\sum_{j=1}^m \beta_j e_j\|_k = \max_{1\le j\le m} |\beta_j| e^{t_k a_j}$. We shall prove that $L = P$.

By (*1*) and (*2*) we have $|t_{i,j}| \le e^{t_k(a_j - a_i)}$ for all $i, j \in \mathbb{N}$. Hence for $i \in \mathbb{N}$ we get

$$\left|\sum_{j=1}^m \beta_j t_{i,j}\right| e^{t_k a_i} \le \max_{1\le j\le m} |\beta_j| e^{t_k a_j} = P;$$

so $L \le P$. If $P = 0$, then $L = P$. Assume that $P > 0$.

Put $j_0 = \max\{1 \le j \le m : |\beta_j| e^{t_k a_j} = P\}$ and $\beta_j = 0$ for $j > m$. Let $q, s \in \mathbb{N}$ with $m \in N_q$, $j_0 \in N_s$. Put $W_s = \bigcup\{N_l : 1 \le l < s\}$ and $M_s = \bigcup\{N_l : s < l \le q\}$. Then $|\beta_j| e^{t_k a_j} \le |\beta_{j_0}| e^{t_k a_{j_0}}$ for $j \in W_s$, $|\beta_j| e^{t_k a_j} < |\beta_{j_0}| e^{t_k a_{j_0}}$ for $j \in M_s$ and $\max_{j\in N_s} |\beta_j| = |\beta_{j_0}| > 0$. By (*2*) and Proposition A, the linear map

$$S : \mathbb{K}^{N_s} \to \mathbb{K}^{N_s}, \quad (x_j)_{j\in N_s} \to \left(\sum_{j\in N_s} t_{i,j} x_j\right)_{i\in N_s}$$

is an isometry, so $\max_{i\in N_s} |\sum_{j\in N_s} t_{i,j}\beta_j| = \max_{j\in N_s} |\beta_j| = |\beta_{j_0}|$. Thus for some $i_0 \in N_s$, we have $|\sum_{j\in N_s} t_{i_0,j}\beta_j| = |\beta_{j_0}|$; clearly $a_{i_0} = a_{j_0}$. If $j \in W_s$, then

$$|\beta_j||t_{i_0,j}| \le |\beta_{j_0}| e^{t_k(a_{j_0} - a_j)} e^{(a_j - a_{j_0})\ln p} = |\beta_{j_0}| e^{(a_j - a_{j_0})(\ln p - t_k)} < |\beta_{j_0}|,$$

so $|\sum_{j\in W_s} \beta_j t_{i_0,j}| < |\beta_{j_0}|$. If $j \in M_s$, then

$$|\beta_j||t_{i_0,j}| < |\beta_{j_0}| e^{t_k(a_{j_0} - a_j)} e^{t_1(a_j - a_{j_0})} = |\beta_{j_0}| e^{(a_j - a_{j_0})(t_1 - t_k)} \le |\beta_{j_0}|,$$

so $|\sum_{j\in M_s} \beta_j t_{i_0,j}| < |\beta_{j_0}|$.
Thus

$$\left|\sum_{j=1}^m \beta_j t_{i_0,j}\right| = \left|\sum_{j\in W_s} \beta_j t_{i_0,j} + \sum_{j\in N_s} \beta_j t_{i_0,j} + \sum_{j\in M_s} \beta_j t_{i_0,j}\right| = |\beta_{j_0}|,$$

so $|\sum_{j=1}^m \beta_j t_{i_0,j}| e^{t_k a_{i_0}} = |\beta_{j_0}| e^{t_k a_{j_0}} = P$. Hence $P \le L$. Thus $L = P$. □

By the proof of Theorem 3.5 we get the following.

Corollary 3.6. *Let $p \in (0,\infty]$, $t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$. Let $T \in L(A_p(a,t))$ and $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i$ for $j \in \mathbb{N}$. Then T is a contraction if and only if $|t_{i,j}| \le e^{(a_j - a_i)t_1}$ when $a_i \le a_j$ and $|t_{i,j}| \le e^{(a_j-a_i)\ln p}$ when $a_i > a_j$.*

Proposition 3.7. *Let $p \in (0,\infty]$, $t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$. Let $(t_{i,j}) \subset \mathbb{K}$ with*

(1) $|t_{i,j}| \le e^{(a_j-a_i)t_1}$ when $a_i < a_j$, and $|t_{i,j}| \le e^{(a_j-a_i)\ln p}$ when $a_i > a_j$;
(2) $\max_{(i,j)\in N_s \times N_s} |t_{i,j}| = 1$ and $|\det[t_{i,j}]_{(i,j)\in N_s\times N_s}| = 1$ for all $s \in \mathbb{N}$.

Then there exists a linear isometry T on $A_p(a,t)$ such that $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i, j \in \mathbb{N}$.

Proof. Let $j \in \mathbb{N}$ and $k \in \mathbb{N}$. For $i \in \mathbb{N}$ with $a_i > a_j$ we have $|t_{i,j}| e^{t_k a_i} \le e^{(a_j-a_i)\ln p + t_k a_i} = e^{a_j \ln p + a_i(t_k - \ln p)}$ if $p \in (0,\infty)$, and $|t_{i,j}|e^{t_k a_i} = 0$, if $p = \infty$. Thus $\lim_i \|t_{i,j} e_i\|_k = 0$ for $k \in \mathbb{N}$; so $\lim_i t_{i,j} e_i = 0$. Therefore the series $\sum_{i=1}^{\infty} t_{i,j} e_i$ is convergent in $A_p(a,t)$ to some element Te_j. Let $x = (x_j) \in A_p(a,t)$.

We shall prove that $\lim_j x_j Te_j = 0$ in $A_p(a,t)$. By (1) and (2) we have $|t_{i,j}| \le e^{t_k(a_j - a_i)}$ for all $i,j,k \in \mathbb{N}$. Let $k \in \mathbb{N}$ and $j \in \mathbb{N}$. Then $|x_j| \le e^{-t_{k+1}a_j} \|x\|_{k+1}$; moreover $\|Te_j\|_k = \max_i |t_{i,j}| e^{t_k a_i} \le e^{t_k a_j}$. Hence $\|x_j Te_j\|_k \le e^{(t_k - t_{k+1})a_j} \|x\|_{k+1}$ for $j,k \in \mathbb{N}$; so $\lim_j x_j Te_j = 0$.

Thus the series $\sum_{j=1}^{\infty} x_j Te_j$ is convergent in $A_p(a,t)$ to some Tx for every $x \in A_p(a,t)$. Clearly $Tx = \lim_n T_n x$, where $T_n : A_p(a,t) \to A_p(a,t)$, $T_n x = \sum_{j=1}^{n} x_j Te_j$. The linear operators $T_n, n \in \mathbb{N}$, are continuous, so using the Banach-Steinhaus theorem we infer that the operator $T : A_p(a,t) \to A_p(a,t)$, $x \to Tx$ is linear and continuous. By Theorem 3.5, T is an isometry. □

By Proposition 3.7 and the proof of Theorem 3.5 we get the following.

Corollary 3.8. *Let $p \in (0,\infty]$, $t \in \Lambda_p$ and $a \in \Gamma$. Then a linear map $T : A_p(a,t) \to A_p(a,t)$ is an isometry if and only if $\|Te_j\|_k = \|e_j\|_k$ for all $j,k \in \mathbb{N}$.*

Finally we shall show that every linear isometry on the space $A_p(a,t)$ is a surjection. For $p = \infty$ it follows from Theorem 3.5 and our next proposition. For $p \in (0,\infty)$ the proof is much more complicated.

Let $p \in (0,\infty]$, $t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$. Put $W_k = \bigcup_{i=1}^{k} N_i$, $M_k = \bigcup_{i=k}^{\infty} N_i$ for $k \in \mathbb{N}$ and $N_{k,m} = N_k \times N_m$ for all $k,m \in \mathbb{N}$. For every $m \in \mathbb{N}$ there is $v(m) \in \mathbb{N}$ with $m \in N_{v(m)}$.

Proposition 3.9. *Let $p \in (0,\infty], t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$. Let $D \in L(A_p(a,t))$ with $De_j = \sum_{i=1}^{\infty} d_{i,j} e_i$ for $j \in \mathbb{N}$. Assume that*

(1) $|d_{i,j}| \le e^{t_1(a_j - a_i)}$ when $a_i < a_j$, and $d_{i,j} = 0$ when $a_i > a_j$;
(2) $\max_{(i,j)\in N_{s,s}} |d_{i,j}| = 1$ and $|\det[d_{i,j}]_{(i,j)\in N_{s,s}}| = 1$ for all $s \in \mathbb{N}$.

Then D is surjective.

Proof. We have $\mathrm{lin}\{De_j : j \in W_k\} \subset \mathrm{lin}\{e_i : i \in W_k\}$ for $k \in \mathbb{N}$, since $De_j = \sum_{i \in W_k} d_{i,j} e_i$ for $j \in N_k$, $k \in \mathbb{N}$. By Theorem 3.5 the operator D is a linear isometry, so $D(A_p(a,t))$ is a closed subspace of $A_p(a,t)$ and the sequence $(De_j)_{j\in W_k}$ is linearly independent for every $k\in\mathbb{N}$. Thus $\operatorname{lin}\{De_j : j\in W_k\} = \operatorname{lin}\{e_i : i\in W_k\}$, $k\in\mathbb{N}$; so $D(A_p(a,t)) \supset \operatorname{lin}\{e_i : i\in\mathbb{N}\}$. It follows that D is surjective. □

Corollary 3.10. *Let $t=(t_k)\in\Lambda_\infty$ and $a=(a_n)\in\Gamma$. Every linear isometry on $A_\infty(a,t)$ is surjective.*

Proposition 3.11. *Let $p\in(0,\infty]$, $t=(t_k)\in\Lambda_p$ and $a=(a_n)\in\Gamma$. Let $S\in L(A_p(a,t))$ with $Se_j=\sum_{i=1}^\infty s_{i,j}e_i$ for $j\in\mathbb{N}$. Assume that*

(1) $s_{i,j}=0$ *when* $a_i<a_j$, *and* $|s_{i,j}|\le e^{(a_j-a_i)\ln p}$ *when* $a_i>a_j$;
(2) $\max_{(i,j)\in N_{k,k}}|s_{i,j}|=1$ *and* $|\det[s_{i,j}]_{(i,j)\in N_{k,k}}|=1$ *for* $k\in\mathbb{N}$.

Then S is surjective.

Proof. For $x=(x_j)\in A_p(a,t)$ we have

$$Sx=\sum_{j=1}^\infty x_jSe_j=\sum_{j=1}^\infty x_j\sum_{i=1}^\infty s_{i,j}e_i=\sum_{i=1}^\infty\left(\sum_{j=1}^\infty s_{i,j}x_j\right)e_i=\sum_{i=1}^\infty\left(\sum_{j\in W_{v(i)}} s_{i,j}x_j\right)e_i.$$

Let $y=(y_i)\in A_p(a,t)$. By (2) and Proposition A, there exists $(x_i)_{i\in N_1}\subset\mathbb{K}$ with $\max_{i\in N_1}|x_i|=\max_{i\in N_1}|y_i|$ such that $\sum_{j\in N_1}s_{i,j}x_j=y_i$ for $i\in N_1$.
Assume that for some $l\in\mathbb{N}$ with $l>1$ we have chosen $(x_j)_{j\in N_s}\subset\mathbb{K}$ for $1\le s<l$.
By (2) and Proposition A, there exists $(x_j)_{j\in N_l}\subset\mathbb{K}$ with

$$\max_{i\in N_l}|x_i|=\max_{i\in N_l}\left|y_i-\sum_{j\in W_{l-1}}s_{i,j}x_j\right|$$

such that $\sum_{j\in N_l}s_{i,j}x_j=y_i-\sum_{j\in W_{l-1}}s_{i,j}x_j$ for $i\in N_l$. Thus by induction we get $x=(x_j)\in\mathbb{K}^{\mathbb{N}}$ such that $\sum_{j\in W_l}s_{i,j}x_j=y_i$ for all $i\in N_l, l\in\mathbb{N}$ and

$$\max_{i\in N_1}|x_i|=\max_{i\in N_1}|y_i|,\quad\text{and}\quad\max_{i\in N_l}|x_i|=\max_{i\in N_l}\left|y_i-\sum_{j\in W_{l-1}}s_{i,j}x_j\right|\quad\text{for } l>1.$$

Let $k\in\mathbb{N}$. Clearly, $\max_{i\in W_1}|x_i|e^{t_ka_i}=\max_{i\in W_1}|y_i|e^{t_ka_i}$. For $l>1$, $i\in N_l$, $j\in W_{l-1}$ we have

$$|s_{i,j}||x_j|e^{t_ka_i}\le e^{(a_j-a_i)\ln p+t_ka_i}|x_j|=e^{(a_j-a_i)(\ln p-t_k)}|x_j|e^{t_ka_j}\le|x_j|e^{t_ka_j}.$$

Thus by induction we get $\max_{i\in W_l}|x_i|e^{t_ka_i}\le\max_{i\in W_l}|y_i|e^{t_ka_i}$ for all $l\in\mathbb{N}$.
It follows that $x\in A_p(a,t)$. We have

$$Sx=\sum_{l=1}^\infty\sum_{i\in N_l}\left(\sum_{j\in W_l}s_{i,j}x_j\right)e_i=\sum_{l=1}^\infty\sum_{i\in N_l}y_ie_i=\sum_{i=1}^\infty y_ie_i=y.$$

Thus S is a surjection. □

Theorem 3.12. *Let $p \in (0,\infty), t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$. Every linear isometry T on $A_p(a,t)$ is surjective.*

Proof. Let $k, m \in \mathbb{N}$. Denote by $\mathcal{M}_{k,m}$ the family of all matrixes $B = [\beta_{i,j}]_{(i,j)\in N_{k,m}}$ with $(\beta_{i,j}) \subset \mathbb{K}$ such that

a) $|\beta_{i,j}| \le e^{(a_j-a_i)\ln p}$ for $(i,j) \in N_{k,m}$, if $k > m$;
b) $|\beta_{i,j}| \le e^{t_1(a_j-a_i)}$ for $(i,j) \in N_{k,m}$, if $k < m$;
c) $|\beta_{i,j}| \le 1$ for $(i,j) \in N_{k,m}$ and $|\det[\beta_{i,j}]_{(i,j)\in N_{k,m}}| = 1$, if $k = m$.

By Proposition A, for every $k \in \mathbb{N}$ and $B \in \mathcal{M}_{k,k}$ we have $B^{-1} \in \mathcal{M}_{k,k}$. Let $Te_j = \sum_{i=1}^{\infty} t_{i,j}e_i$ for $j \in \mathbb{N}$. Put $T_{k,m} = [t_{i,j}]_{(i,j)\in N_{k,m}}$ and $I_{k,m} = [\delta_{i,j}]_{(i,j)\in N_{k,m}}$ for all $k, m \in \mathbb{N}$. We define matrixes $D_{k,m}, S_{k,m} \in \mathcal{M}_{k,m}$ for $k \in \mathbb{N}$ and $m = 1, 2, 3, \ldots$.

Put $D_{k,1} = I_{k,1}$ and $S_{k,1} = T_{k,1}$ for $k \in \mathbb{N}$; clearly $D_{k,1}, S_{k,1} \in \mathcal{M}_{k,1}$ for $k \in \mathbb{N}$. Assume that for some $m \in \mathbb{N}$ with $m > 1$ we have $D_{k,j}, S_{k,j} \in \mathcal{M}_{k,j}$ for $k \in \mathbb{N}$ and $1 \le j < m$. Let $D_{1,m} = S_{1,1}^{-1}T_{1,m}$. It is easy to see that $D_{1,m} \in \mathcal{M}_{1,m}$, since $S_{1,1}^{-1} \in \mathcal{M}_{1,1}$ and $T_{1,m} \in \mathcal{M}_{1,m}$.

Let $C_{k,m} = \sum_{v=1}^{k-1} S_{k,v}D_{v,m}$ and $D_{k,m} = S_{k,k}^{-1}[T_{k,m} - C_{k,m}]$ for $k = 2, 3, ..., m-1$. Let $1 < k < m$. Let $[s_{i,n}]_{(i,n)\in N_{k,v}} = S_{k,v}$ and $[d_{n,j}]_{(n,j)\in N_{v,m}} = D_{v,m}$ for $1 \le v < k$. Put $[c_{i,j}]_{(i,j)\in N_{k,m}} = C_{k,m}$. Then

$$|c_{i,j}| = \left|\sum_{v=1}^{k-1}\sum_{n\in N_v} s_{i,n}d_{n,j}\right| \le \max_{n\in W_{k-1}} |s_{i,n}d_{n,j}|$$

for $(i,j) \in N_{k,m}$. For $i \in N_k$, $j \in N_m$ and $n \in W_{k-1}$ we have

$$|s_{i,n}d_{n,j}| \le e^{(a_n-a_i)\ln p + t_1(a_j-a_n)} = e^{(a_n-a_i)(\ln p - t_1) + t_1(a_j-a_i)} \le e^{t_1(a_j-a_i)};$$

hence $C_{k,m} \in \mathcal{M}_{k,m}$. Since $S_{k,k}^{-1} \in \mathcal{M}_{k,k}$ and $T_{k,m} \in \mathcal{M}_{k,m}$, we infer that $D_{k,m} \in \mathcal{M}_{k,m}$ for $k = 2, ..., m-1$.

Let $D_{k,m} = I_{k,m}$ for $k \ge m$; clearly $D_{k,m} \in \mathcal{M}_{k,m}$. Let $S_{k,m} = I_{k,m}$ for $1 \le k < m$; then $S_{k,m} \in \mathcal{M}_{k,m}$. Let $C_{k,m} = \sum_{v=1}^{m-1} S_{k,v}D_{v,m}$ and $S_{k,m} = T_{k,m} - C_{k,m}$ for $k \ge m$.

Let $k \ge m$. Let $[s_{i,n}]_{(i,n)\in N_{k,v}} = S_{k,v}$ and $[d_{n,j}]_{(n,j)\in N_{v,m}} = D_{v,m}$ for $1 \le v < m$. Put $[c_{i,j}]_{(i,j)\in N_{k,m}} = C_{k,m}$. Then

$$|c_{i,j}| = \left|\sum_{v=1}^{m-1}\sum_{n\in N_v} s_{i,n}d_{n,j}\right| \le \max_{n\in W_{m-1}} |s_{i,n}d_{n,j}|$$

for $(i,j) \in N_{k,m}$. For $i \in N_k$, $j \in N_m$ and $n \in W_{m-1}$ we have

$$|s_{i,n}d_{n,j}| \le e^{(a_n-a_i)\ln p + t_1(a_j-a_n)} = e^{(a_n-a_j)(\ln p - t_1) + \ln p(a_j-a_i)} < e^{(a_j-a_i)\ln p};$$

hence $C_{k,m} \in \mathcal{M}_{k,m}$. Since $|t_{i,j}| \le e^{(a_j-a_i)\ln p}$ for $(i,j) \in N_{k,m}$, we get $S_{k,m} \in \mathcal{M}_{k,m}$ for $k > m$ and $|c_{i,j}| < 1$ for all $(i,j) \in N_{m,m}$.

Thus for some $(\varphi_\sigma)_{\sigma\in S(N_m)} \subset \{\alpha \in \mathbb{K} : |\alpha| < 1\}$ we have

$$\begin{aligned}
|\det S_{m,m}| &= \left| \sum_{\sigma\in S(N_m)} \operatorname{sgn}\sigma \prod_{i\in N_m} (t_{i,\sigma(i)} - c_{i,\sigma(i)}) \right| \\
&= \left| \sum_{\sigma\in S(N_m)} \operatorname{sgn}\sigma \left[\left(\prod_{i\in N_m} t_{i,\sigma(i)} \right) - \varphi_\sigma \right] \right| \\
&= \left| \det(T_{m,m}) - \sum_{\sigma\in S(N_m)} \operatorname{sgn}\sigma\, \varphi_\sigma \right| = |\det(T_{m,m})| = 1.
\end{aligned}$$

It follows that $S_{m,m} \in \mathcal{M}_{m,m}$.

By definition of $D_{k,m}$ and $S_{k,m}$ we get

a) $T_{k,1} = S_{k,1} = \sum_{v=1}^{k} S_{k,v} D_{v,1}$ for $k \in \mathbb{N}$;

b) $S_{1,1}D_{1,m} = T_{1,m}$ for $m \geq 2$ and $S_{k,k}D_{k,m} = T_{k,m} - \sum_{v=1}^{k-1} S_{k,v}D_{v,m}$ for $2 \leq k < m$,

so $T_{k,m} = \sum_{v=1}^{k} S_{k,v}D_{v,m}$ for $1 \leq k < m$;

c) $S_{k,m}D_{m,m} = S_{k,m} = T_{k,m} - \sum_{v=1}^{m-1} S_{k,v}D_{v,m}$ for $k \geq m > 1$,

so $T_{k,m} = \sum_{v=1}^{m} S_{k,v}D_{v,m} = \sum_{v=1}^{k} S_{k,v}D_{v,m}$ for $k \geq m > 1$.
Thus $(*)$ $T_{k,m} = \sum_{v=1}^{k} S_{k,v}D_{v,m} = \sum_{v=1}^{\infty} S_{k,v}D_{v,m}$ for all $k, m \in \mathbb{N}$.

Let $[s_{i,j}]_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ and $[d_{i,j}]_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ be matrixes such that $[s_{i,j}]_{(i,j)\in N_{k,m}} = S_{k,m}$ and $[d_{i,j}]_{(i,j)\in N_{k,m}} = D_{k,m}$ for all $k, m \in \mathbb{N}$.

By Theorem 3.5 and Proposition 3.7, there exist linear isometries S and D on $A_p(a,t)$ such that $Se_j = \sum_{i=1}^{\infty} s_{i,j}e_i$ and $De_j = \sum_{i=1}^{\infty} d_{i,j}e_i$ for all $j \in \mathbb{N}$; by Propositions 3.9 and 3.11, these isometries are surjective. Using $(*)$ we get

$$t_{i,j} = \sum_{v=1}^{k} \sum_{n\in N_v} s_{i,n}d_{n,j} = \sum_{v=1}^{\infty} \sum_{n\in N_v} s_{i,n}d_{n,j} = \sum_{n=1}^{\infty} s_{i,n}d_{n,j}$$

for $(i,j) \in N_{k,m}$ and $k, m \in \mathbb{N}$. Hence for $j \in \mathbb{N}$ we get

$$\begin{aligned}
SDe_j &= S\left(\sum_{n=1}^{\infty} d_{n,j}e_n \right) = \sum_{n=1}^{\infty} d_{n,j} \left(\sum_{i=1}^{\infty} s_{i,n}e_i \right) \\
&= \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} s_{i,n}d_{n,j} \right) e_i = \sum_{i=1}^{\infty} t_{i,j}e_i = Te_j;
\end{aligned}$$

so $T = SD$. Thus T is surjective. □

Let $p \in (0,\infty]$, $t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$.

For every $m \in \mathbb{N}$ there is $v(m)$ with $m \in N_{v(m)}$.

Denote by $\mathcal{D}_p(a,t)$, $\mathcal{K}_p(a,t)$ and $\mathcal{S}_p(a,t)$ the families of all linear isometries on $A_p(a,t)$ such that $Te_j = \sum_{i\in W_{v(j)}} t_{i,j}e_i$, $Te_j = \sum_{i\in N_{v(j)}} t_{i,j}e_i$ and $Te_j = \sum_{i\in M_{v(j)}} t_{i,j}e_i$ for $j \in \mathbb{N}$, respectively.

We have the following two propositions.

Proposition 3.13. *$\mathcal{D}_p(a,t)$, $\mathcal{K}_p(a,t)$ and $\mathcal{S}_p(a,t)$ are subgroups of the group $\mathcal{I}_p(a,t)$ of all linear isometries on $A_p(a,t)$. Moreover $\mathcal{D}_\infty(a,t) = \mathcal{I}_\infty(a,t)$ and $\mathcal{S}_\infty(a,t) = \mathcal{K}_\infty(a,t)$. For every $T \in \mathcal{I}_p(a,t)$ there exist $D \in \mathcal{D}_p(a,t)$ and $S \in \mathcal{S}_p(a,t)$ such that $T = S \circ D$.*

Proof. The last part of the proposition follows by the proof of Theorem 3.12. Clearly, $\mathcal{I}_p(a,t)$ is a subgroup of the group of all automorphisms of $A_p(a,t)$; moreover $\mathcal{D}_\infty(a,t) = \mathcal{I}_\infty(a,t)$ and $\mathcal{S}_\infty(a,t) = \mathcal{K}_\infty(a,t)$.

Let $S, T \in \mathcal{S}_p(a,t)$. Let $j \in \mathbb{N}$. We have

$$STe_j = S\left(\sum_{i=1}^{\infty} t_{i,j}e_i\right) = \sum_{i=1}^{\infty} t_{i,j}\left(\sum_{k=1}^{\infty} s_{k,i}e_k\right) = \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} s_{k,i}t_{i,j}\right)e_k.$$

If $a_k < a_j$, then for every $i \in \mathbb{N}$ we have $a_k < a_i$ or $a_i < a_j$; so $s_{k,i} = 0$ or $t_{i,j} = 0$ for $i \in \mathbb{N}$. Thus $\sum_{i=1}^{\infty} s_{k,i}t_{i,j} = 0$ for $k \in \mathbb{N}$ with $a_k < a_j$; so $ST \in \mathcal{S}_p(a,t)$.

Let $k \in \mathbb{N}$. For some $x_k = (x_{j,k}) \in A_p(a,t)$ we have $Sx_k = e_k$. By the proof of Proposition 3.11 we have $\max\{|x_{j,k}|e^{t_1 a_j} : a_j < a_k\} = 0$, so $x_{j,k} = 0$ for $j \in \mathbb{N}$ with $a_j < a_k$. Hence $S^{-1}(e_k) = \sum_{j\in M_{v(k)}} x_{j,k}e_j$, so $S^{-1} \in \mathcal{S}_p(a,t)$. We have shown that $\mathcal{S}_p(a,t)$ is a subgroup of $\mathcal{I}_p(a,t)$.

Let $D, T \in \mathcal{D}_p(a,t)$. Let $j \in \mathbb{N}$. We have $DTe_j = \sum_{k=1}^{\infty}(\sum_{i=1}^{\infty} d_{k,i}t_{i,j})e_k$. If $a_k > a_j$, then for every $i \in \mathbb{N}$ we have $a_k > a_i$ or $a_i > a_j$; so $d_{k,i} = 0$ or $t_{i,j} = 0$ for $i \in \mathbb{N}$. Thus $\sum_{i=1}^{\infty} d_{k,i}t_{i,j} = 0$ for every $k \in \mathbb{N}$ with $a_k > a_j$, so $DT \in \mathcal{D}_p(a,t)$.

Let $k \in \mathbb{N}$. Put $F_k = \operatorname{lin}\{e_i : a_i \le a_k\}$. We know that $D(F_k) = F_k$. Thus there exists $x_k = (x_{j,k}) \in F_k$ such that $Dx_k = e_k$. Then $x_{j,k} = 0$ for $j \in \mathbb{N}$ with $a_j > a_k$ and $D^{-1}(e_k) = x_k = \sum_{j\in W_{v(k)}} x_{j,k}e_j$, so $D^{-1} \in \mathcal{D}_p(a,t)$. Thus $\mathcal{D}_p(a,t)$ is a subgroup of $\mathcal{I}_p(a,t)$. Clearly, $\mathcal{K}_p(a,t) = \mathcal{S}_p(a,t) \cap \mathcal{D}_p(a,t)$, so $\mathcal{K}_p(a,t)$ is subgroup of $\mathcal{I}_p(a,t)$. □

Proposition 3.14. *$\mathcal{I}_p(a,t) \subset \mathcal{I}_p(a,s)$ if and only if $t_1 \le s_1$. In particular, $\mathcal{I}_p(a,t) = \mathcal{I}_p(a,s)$ if and only if $t_1 = s_1$.*

Proof. If $t_1 \le s_1$, then using Theorem 3.5 we get $\mathcal{I}_p(a,t) \subset \mathcal{I}_p(a,s)$. Assume that $t_1 > s_1$. Then $\lim_j e^{(t_1-s_1)(a_j-a_1)} = \infty$, so there exists $j_0 > 1$ and $\beta_0 \in \mathbb{K}$ such that $e^{s_1(a_{j_0}-a_1)} < |\beta_0| \le e^{t_1(a_{j_0}-a_1)}$. Let $T \in L(A_p(a,t))$ with $Te_j = e_j + \beta_0\delta_{j_0,j}e_1$ for $j \in \mathbb{N}$. By Theorem 3.5, we have $T \in \mathcal{I}_p(a,t)$ and $T \notin \mathcal{I}_p(a,s)$. □

In relation with Corollary 3.10 and Theorem 3.12 we give the following two examples and state one open problem.

Let $p \in (0,\infty]$, $t = (t_k) \in \Lambda_p$ and $a = (a_n) \in \Gamma$.

For every isometry F on $\mathbb{K}$ the map $T_F : A_p(a,t) \to A_p(a,t)$, $(x_n) \to (Fx_n)$ is an isometry on $A_p(a,t)$.

Example 3.15. Assume that the field $\mathbb{K}$ is not spherically complete or the residue class field of $\mathbb{K}$ is infinite. Then there exists an isometry on $A_p(a,t)$ which is not a surjection.

Indeed, by [5], Theorem 2, there is an isometry F on $\mathbb{K}$ which is not surjective. Then the map T_F is an isometry on $A_p(a,t)$ which is not a surjection. □

Problem. *Assume that $\mathbb{K}$ is spherically complete with finite residue class. Does every isometry on $A_p(a,t)$ is surjective?*

Example 3.16. On $A_p(a,t)$ there exists a non-linear rotation.

Indeed, put $S_{\mathbb{K}} = \{\beta \in \mathbb{K} : |\beta| = 1\}$ and let $f : [0,\infty) \to S_{\mathbb{K}}$ be a function which is not constant on the set $\{|\alpha| : \alpha \in \mathbb{K} \text{ with } |\alpha| > 0\}$. Then the map $F : \mathbb{K} \to \mathbb{K}$, $F(x) = f(|x|)x$ is a non-linear surjective isometry with $F(0) = 0$.

In fact, let $x, y \in \mathbb{K}$. If $|x| = |y|$, then

$$|F(x) - F(y)| = |f(|x|)x - f(|y|)y| = |f(|x|)||x - y| = |x - y|.$$

If $|x| \neq |y|$, then $|F(x)| = |x| \neq |y| = |F(y)|$, so

$$|F(x) - F(y)| = \max\{|F(x)|, |F(y)|\} = \max\{|x|, |y|\} = |x - y|.$$

If $\alpha \in S_{\mathbb{K}}$, then $F(\alpha x) = \alpha F(x)$, so $F(x/f(|x|)) = (1/f(|x|))f(|x|)x = x$ for every $x \in \mathbb{K}$. Let $\alpha \in (\mathbb{K} \setminus \{0\})$ with $f(|\alpha|) \neq f(1)$, then $F(\alpha 1) \neq \alpha F(1)$.

Then T_F is an nonlinear surjective isometry on $A_p(a,t)$ with $T_F(0) = 0$. □

References

[1] N. De Grande-De Kimpe: Non-archimedean Fréchet spaces generalizing spaces of analytic functions, Indag. Math. 44 (1982) 423–439.

[2] C. Perez-Garcia, W. H. Schikhof: Locally Convex Spaces over Non-Archimedean Valued Fields, Cambridge Studies in Advanced Mathematics 119, Cambridge University Press (2010).

[3] A. C. M. van Rooij: Notes on p-adic Banach spaces, Report 7633, Department of Mathematics, University of Nijmegen (1976).

[4] A. C. M. van Rooij: Non-Archimedean Functional Analysis, Marcel Dekker, New York (1978).

[5] W. H. Schikhof: Isometrical embeddings of ultrametric spaces into non-archimedean valued fields, Indag. Math. 87 (1984) 51–53.

[6] W. H. Schikhof: Locally convex spaces over non-spherically complete valued fields, Bull. Belg. Math. Soc. 38 (1986) 187–224.

[7] W. Śliwa: On the quasi equivalence of orthogonal bases in non-archimedean metrizable locally convex spaces, Bull. Belg. Math. Soc. 9 (2002) 465–472.

[8] W. Śliwa: On relations between non-archimedean power series spaces, Indag. Math., New Ser. 17 (2006) 627–639.

[9] W. Śliwa, A. Ziemkowska: On complemented subspaces of non-archimedean power series spaces, Can. J. Math. 63 (2011) 1188–1200.