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B-Convexity, Convexification of Minkowski Averages in a Banach Space, and SLLN for Random Sets

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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For an infinite-dimensional Banach space X, we demonstrate the equivalence of the following two properties. One, the space is B-convex, that is, it possesses a nontrivial type. Two, X possesses the convexification property, that is, the Hausdorff distance between the Minkowski average of k subsets of the unit ball, and the convex hull of the average, converges to 0 as k tends to infinity. A rate for the convergence is provided. The result is used to establish a general Strong Law of Large Numbers for random bounded subsets of the Banach space.

Keywords: Banach space, Minkowski averages, convexification, random sets.

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1. Introduction

Convexity in general, convexification procedures in particular, play a major role in mathematical analysis and applications. A prime example of a useful convexification phenomenon is the convergence to 0 as $k \to \infty$, of the distance between the Minkowski average $\frac{1}{k}(A_1 + \ldots + A_k)$ and its convex hull. Here A_k is a uniformly bounded sequence of subsets of the space. See the survey Fradelizi, Madiman, Marsiglietti and Zvavitc [7], and references therein for various approaches and the analyses of this convergence, and for various applications. This phenomenon was considered in connection with applications in mathematical economics by Shapley and Folkman, see [1], and by Starr [18]. Convergence rates were also worked out in detail, see [7]. Another key application of the phenomenon is within establishing limit statistical laws for random sets. We elaborate on the latter in the closing section.

The developments, described telegraphically above, are concerned primarily with sets in a finite-dimensional space, with some exceptions, though. Indeed, the convexification may not hold in an infinite-dimensional Banach space. Positive results for the convexification, e.g., the Hilbert space ℓ_2, and specific examples for the lack

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of convexification, e.g., the Minkowski average in ℓ_1, of replicas of $A = \{0, e_1, e_2, ...\}$, where e_i is the unit vector with 1 at the i-th position, are displayed in Artstein [2]. What was left open in [2], is a characterization of the Banach spaces in which the convexification is guaranteed. The present paper provides the answer. After introducing the setting, and relevant definitions, in the next section, we state in Section 3 the main result and prove it. An estimate for the convergence is provided as well. Key steps in the proof appeared already in Kadets, Kulykov and Shevchenko [11] in a different setting. The closing section is devoted to an application, namely, the role of the convexification in the Strong Law of Large Numbers for random sets.

2. The setting and basic definitions

In the text below the letter X is reserved for Banach spaces. For simplicity we consider only real Banach spaces (this is inessential) and use the standard Banach space theory notation, like B_X for the closed unit ball of X. Also, for $b \in X$ and A a subset of X denote by $\mathrm{dist}(b, A)$ the distance from b to A, namely $\inf\limits_{a \in A} \|b - a\|$.

Denote by $\mathrm{b}(X)$ the collection of all non-empty bounded subsets of X and by $\mathrm{b}(B_X)$ denote the non-empty subsets of B_X. The *one-sided Hausdorff distance* between sets $A, B \in \mathrm{b}(X)$ (also called the excess of B over A) is the number $\widetilde{\rho}_H(A, B) = \sup\limits_{b \in B} \mathrm{dist}(b, A)$. The *Hausdorff distance* between A and B is the number

$$\rho_H(A, B) = \max\{\widetilde{\rho}_H(A, B), \widetilde{\rho}_H(B, A)\}.$$

The Minkowski addition, and multiplication by a non-negative scalar λ, of sets $A, B \in \mathrm{b}(X)$, are defined in a natural way: $A + B := \{a + b \colon a \in A, b \in B\}$, and $\lambda A := \{\lambda a \colon a \in A\}$. By $\mathrm{conv}\, A$ we denote the convex hull of A. The Hausdorff distance does not distinguish between two sets that share the same closure. We, abusing rigor, shall not make this distinction as well. No confusion should arise.

Definition 2.1. A Banach space X has the *convexification property* if for every uniformly bounded collection $(A_k)_{k=1}^{\infty}$ of non-empty subsets of X

$$\rho_H\Big(\frac{1}{n}(A_1 + \ldots + A_n),\ \frac{1}{n}(\mathrm{conv}\, A_1 + \ldots + \mathrm{conv}\, A_n)\Big) \xrightarrow[n \to \infty]{} 0. \tag{1}$$

Definition 2.2. When the conditions for the convexification property (1) are required only when $A_1, \ldots, A_n, \ldots$ are replicas of a fixed set A, then we say that the Banach space X has *the weak convexification property.* □

Evidently, the convexification property of X implies its weak convexification property. The converse statement is not so obvious, but it holds, as we verify below. We also discover a surprising effect that these properties imply some uniformity of convergence with a power type of estimate. In order to state the corresponding result we need one definition more.

Definition 2.3. A Banach space X has the *uniform convexification property* if there is a function $\varphi : \mathbb{N} \to (0, +\infty)$ such that $\lim_{n\to\infty} \varphi(n) = 0$ and

$$\rho_H\Big(\frac{1}{n}(A_1 + \ldots + A_n),\ \frac{1}{n}(\mathrm{conv}\, A_1 + \ldots + \mathrm{conv}\, A_n)\Big) \le \varphi(n), \tag{2}$$

for every collection $(A_k)_{k=1}^{n}$ of non-empty subsets of B_X. The infimum of $\varphi(n)$ in the right hand side of (2) is referred to as the *convexification rate* of X.

We now recall the notions of type and infratype for a Banach space and the notion of finite representability. These notions are standard in Banach space theory. Here we follow the developments in the monograph by Kadets and Kadets [10]; other adequate references are Ledoux and Talagrand [12, Chapter 9], the survey by Piesier [16], and the book Wojtaszczyk [19, Sections II.E, III.A].

A Banach space X is said to be of type p with constant $C > 0$ if for every finite collection $(x_k)_{k=1}^n \in X^n$ the following inequality holds:

$$\frac{1}{2^n} \sum_{(\alpha_i)_{i=1}^n \in \{-1,1\}^n} \Big\| \sum_{i=1}^n \alpha_i x_i \Big\| \;\leq\; C\Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}. \tag{3}$$

In the sequel, we say that X is of type p if there exists a $C > 0$ such that X is of type p with the constant C. Trivially, every Banach space is of type 1, so spaces with a type $p > 1$ are called *spaces with a non-trivial type*.

We say that the space Z is *finitely representable* in X if for any $\varepsilon > 0$ and any finite-dimensional subspace Z_1 of Z there exists a finite-dimensional subspace X_1 of X with $\dim X_1 = \dim Z_1$ such that both $\|T\| < 1 + \varepsilon$ and $\|T^{-1}\| < 1 + \varepsilon$ for a certain isomorphism T between X_1 and Z_1 (see [10, page 59]).

Definition 2.4. A Banach space X is said to be B-convex if it is of non-trivial type, equivalently, if the space ℓ_1 is not finitely representable in X.

The equivalence of the two properties mentioned in the definition of B-convexity, is far from being obvious. This important result was established by Gilles Pisier [15]. The letter B in the previous definition stands for Anatole Beck; we display the connection in the closing section.

Another tool that we shall use is *infratype*, which is a little bit weaker property than the type. A Banach space X is said to be of infratype s with constant $C > 0$ if

$$\min_{\theta_i = \pm 1} \Big\| \sum_{i=1}^N \theta_i x_i \Big\| \;\leq\; C\Big(\sum_{i=1}^N \|x_i\|^s \Big)^{1/s} \tag{4}$$

for every finite collection $(x_k)_{k=1}^N \in X^N$. The space X is said to be of infratype s if there exists a $C > 0$ such that X is of infratype s with the constant C. Since the minimum among the terms that define the type, is not greater than the average in (3), a type p implies infratype p with the same constant. Due to the same paper [15] by Pisier, X is B-convex if and only if it has an infratype $p > 1$.

3. The main result

We start by stating the main equivalence result.

Theorem 3.1. *For a Banach space X the following properties are equivalent:*

(1) *X is B-convex.*

(2) *X has the uniform convexification property.*

(3) *X has the convexification property.*

(4) *X has the weak convexification property.*

The proof will follow some useful observations. An important technical tool for us is the following known result (Kadets [9, Lemma before Theorem 3], or Kadets and Kadets [10, pp. 133–134, Lemma 3]).

Lemma 3.2. *Let $p > 1$, $C > 0$, and let X be a Banach space with infratype p and constant C. Let $(A_i)_{i=1}^n$ be an arbitrary collection of bounded subsets of X and $(b_i)_{i=1}^n$ be a collection of points such that $b_i \in \operatorname{conv} A_i$, for $i = 1, \dots, n$. Then one can choose points $a_i \in A_i$, so that*

$$\Big\| \sum_{i=1}^n a_i - \sum_{i=1}^n b_i \Big\| \le C_1 \Big(\sum_{i=1}^n \operatorname{diam}(A_i)^p \Big)^{1/p}, \tag{5}$$

where $C_1 = \frac{2C}{2^{1-\frac{1}{p}}-1}$.

Now we are ready for our first result. The theorem below means that infratype p is "almost equivalent" to an estimate of the convexification rate φ_X of the form $\varphi_X(n) \le Dn^{\frac{1-p}{p}}$.

Theorem 3.3. *For a Banach space X and a number $p > 1$, consider the following three assertions:*

(1) *X has infratype p.*

(2) *There is a constant $D > 0$ such that $\varphi_X(n) \le Dn^{\frac{1-p}{p}}$ for all $n \in \mathbb{N}$.*

(3) *X has infratype s for every $s < p$.*

Then, (1)⇒(2)⇒(3).

Proof. (1)⇒(2) We will prove (2) with $D = 2C_1$ where C_1 is the constant from Lemma 3.2. Fix $n \in \mathbb{N}$ and a collection $(A_i)_{i=1}^n$ of non-empty subsets of B_X. Then consider an arbitrary $b \in \frac{1}{n}(\operatorname{conv} A_1 + \ldots + \operatorname{conv} A_n)$. It can be written in the form $b = \frac{1}{n}\sum_{i=1}^n b_i$ with $b_i \in \operatorname{conv} A_i$. Applying Lemma 3.2 to the collection of sets $(A_i)_{i=1}^n$ and to the corresponding collection of points $(b_i)_{i=1}^n$, we obtain $(a_i)_{i=1}^n$ such that $a_i \in A_i$ and (5) holds true. Then, $a := \frac{1}{n}\sum_{i=1}^n a_i$ lies in $\frac{1}{n}(A_1 + \ldots + A_n)$ and

$$\begin{aligned} \|a-b\| &= \Big\| \frac{1}{n}\sum_{i=1}^n a_i - \frac{1}{n}\sum_{i=1}^n b_i \Big\| \le\ C_1 \frac{1}{n} \Big(\sum_{i=1}^n (\operatorname{diam} A_i)^p \Big)^{1/p} \le\ C_1 \frac{1}{n} (n \cdot 2^p)^{1/p} \\ &= 2C_1 \cdot n^{\frac{1-p}{p}}. \end{aligned}$$

So, for every $b \in \frac{1}{n}(\operatorname{conv} A_1 + \ldots + \operatorname{conv} A_n)$ there is an $a \in \frac{1}{n}(A_1 + \ldots + A_n)$ such that $\|a - b\| \le 2C_1 \cdot n^{\frac{1-p}{p}}$. Consequently,

$$\begin{aligned} &\widetilde{\rho}_H \left(\frac{1}{n}(A_1 + \ldots + A_n),\ \frac{1}{n}(\operatorname{conv} A_1 + \ldots + \operatorname{conv} A_n) \right) \\ &= \sup_{b \in \frac{1}{n}(\operatorname{conv} A_1 + \ldots + \operatorname{conv} A_n)} \operatorname{dist}\Big(b, \frac{1}{n}(A_1 + \ldots + A_n)\Big) \ \le\ 2C_1 \cdot n^{\frac{1-p}{p}}. \end{aligned}$$

On the other hand, $\frac{1}{n}(A_1 + \ldots + A_n) \ \subset\ \frac{1}{n}(\operatorname{conv} A_1 + \ldots + \operatorname{conv} A_n)$, so

$$\widetilde{\rho}_H \Big(\frac{1}{n}(\operatorname{conv} A_1 + \ldots + \operatorname{conv} A_n),\ \frac{1}{n}(A_1 + \ldots + A_n) \Big) = 0.$$

This means that

$$\rho_H\Big(\frac{1}{n}(A_1+\ldots+A_n),\ \frac{1}{n}(\operatorname{conv} A_1+\ldots+\operatorname{conv} A_n)\Big) \le Dn^{\frac{1-p}{p}}$$

as claimed.

(2)⇒(3) At first, notice that our assumption (2) implies that for every finite collection of vectors $(y_i)_{i=1}^n \in X^n$ there is a collection of signs $(\alpha_i)_{i=1}^n \in \{-1,1\}^n$ such that

$$\Big\|\sum_{i=1}^n \alpha_i y_i\Big\| \le D \max_{k\in\{1,\ldots,n\}} \|y_k\| \cdot n^{1/p}. \tag{6}$$

To see this we denote $M = \max_{k\in\{1,\ldots,n\}} \|y_k\|$ and consider the sets

$$A_i = \Big\{-\frac{y_i}{M}, \frac{y_i}{M}\Big\} \subset B_X,\ i=1,\ldots,n.$$

Applying Definition 2.3 to $0 \in \frac{1}{n}(\operatorname{conv} A_1+\ldots+\operatorname{conv} A_n)$ one gets an element of the form

$$a = \frac{1}{n}\sum_{i=1}^n \alpha_i \frac{y_i}{M} \ \in\ \frac{1}{n}(A_1+\ldots+A_n),\ \alpha_i = \pm 1,$$

such that $\|a\| = \|a-0\| \le \varphi_X(n) \le Dn^{\frac{1-p}{p}}$. So, we obtain the claimed inequality

$$\Big\|\sum_{i=1}^n \alpha_i y_i\Big\| \ =\ n\cdot M\cdot\|a\| \le n\cdot M\cdot Dn^{\frac{1-p}{p}} \ =\ D \max_{k\in\{1,\ldots,n\}} \|y_k\|\cdot n^{1/p}.$$

We can now pass to the main part of the proof. Fix a finite collection $(x_k)_{k=1}^N \in X^N$ and $1 < s < p$. Our goal is to demonstrate (4) with constant $C = 2D\sum_{k=1}^\infty 2^{-k(1-s/p)}$. At this stage we borrow an idea by Gilles Pisier from [15].

Denote $M_0 = \Big(\sum_{i=1}^N \|x_i\|^s\Big)^{1/s}$, $M_k = 2^{-k}M_0$, and by

$$J_k = \{i \in \{1,\ldots,N\} : M_k < \|x_i\| \le M_{k-1}\},$$

this for $k = 1,2,\ldots$ (note that only finitely many of J_k are not empty). Then for each k we have

$$\Big(\sum_{i=1}^N \|x_i\|^s\Big)^{1/s} \ \ge\ \Big(\sum_{i\in J_k} \|x_i\|^s\Big)^{1/s} \ \ge\ |J_k|^{1/s}\Big(\sum_{i=1}^N \|x_i\|^s\Big)^{1/s} 2^{-k},$$

hence

$$|J_k| \le 2^{ks}. \tag{7}$$

For each collection of $(x_i)_{i\in J_k}$ we choose the corresponding collection of signs $(\alpha_i)_{i\in J_k}$ such that (6) holds true:

$$\Big\|\sum_{i\in J_k} \alpha_i x_i\Big\| \ \le\ D\max_{k\in J_k} \|x_k\|\cdot|J_k|^{1/p} \ \le\ DM_{k-1}|J_k|^{1/p}.$$

Then

$$\min_{\theta_i=\pm1}\Big\|\sum_{i=1}^{N}\theta_i x_i\Big\| \;\le\; \Big\|\sum_{i=1}^{N}\alpha_i x_i\Big\| \;=\; \Big\|\sum_{k=1}^{\infty}\sum_{i\in J_k}\alpha_i x_i\Big\| \;\le\; \sum_{k=1}^{\infty} D M_{k-1}|J_k|^{1/p}.$$

Substituting the definition of M_{k-1} and the inequality (7) we get the result:

$$\min_{\theta_i=\pm1}\Big\|\sum_{i=1}^{N}\theta_i x_i\Big\| \;\le\; 2D\sum_{k=1}^{\infty}2^{-k(1-s/p)}\Big(\sum_{i=1}^{N}\|x_i\|^s\Big)^{1/s} \;=\; C\Big(\sum_{i=1}^{N}\|x_i\|^s\Big)^{1/s}. \qquad \square$$

We are ready to provide the proof of Theorem 3.1.

Proof. The implication (1)⇒(2) of this theorem follows immediately from the implication (1)⇒(2) of Theorem 3.3. The implications (2)⇒(3)⇒(4) are evident. It remains to show that (4)⇒(1). We will verify the equivalent form, namely, the negation of (4) implies the negation of (1).

So, assume that X is not B-convex. According to [11, Theorem 3.10] there exists a multifunction $F : [0,1] \to \mathrm{b}(X)$ such that $\operatorname{conv} F$ is Riemann integrable, but F is not. That function F takes at all points one the same value $W = \{e_j\}_{j\in\mathbb{N}} \subset X$ (that was built upon the finite representability of ℓ_1 in X and Masur's basic sequences selection technique) and it was demonstrated that $\rho_H(\operatorname{conv} W, S_n) \ge 1/24$ for all n, where S_n were the Minkowski averages $\frac{1}{2^n}(W + \ldots + W)$ of 2^n replicas of W. So, this gives the needed example of a bounded set in X whose Minkowski averages do not converge to its convex hull. □

Here is the promised rate of convergence.

Corollary 3.4. *If X has the convexification property then there are $D, \gamma > 0$ such that $\varphi_X(n) \le Dn^{-\gamma}$.*

Proof. According to the previous theorem, X has an infratype $p > 1$. Then, Theorem 3.3 gives us the desired result with $\gamma = \frac{p-1}{p}$. □

4. SLLN for random sets

First, as promised, we explain the source of the term B-convex. While the strong law of large numbers for real-valued random variables (hence, for finite-dimensional vector-valued functions) holds under general conditions, the case is different for Banach space-valued random variables. Examples of uniformly bounded and independent random variables that share the same expectation, yet do not satisfy the SLLN, can easily be constructed. Anatole Beck (see [5], see also Beck, Giesy and Warren [6]), discovered a necessary and sufficient condition on the Banach space for possessing an SLLN. He called it property (B). The form of property (B) displayed in [5] is equivalent (not identical) to the type condition used earlier in the present paper. Later on, as was mentioned, Pisier established the equivalence of a non trivial p-type and the absence of finite representability of ℓ_1. The B in B-convexity is there to honor Anatole Beck.

The theory and applications of random subsets of a topological space have been the subject of extensive research, see Molchanov [13] and the many references therein. Statistical limit laws, in particular the Strong Law of Large Numbers with respect to the Minkowski addition in a linear space, attracted a lot of attention, see [13, Section 3], see also Hess [8].

When the set-valued random variable takes values in the collection of convex sets, the Minkowski addition and multiplication by a non-negative number, make the collection of bounded convex sets a positive cone in a linear space. The collection of bounded convex sets (recall that we do not distinguish between two sets that share the same closure) with the Hausdorff metric, make this cone a subset of a Banach space. Here we do not allude to a specific embedding, yet mention that several forms of such an embedding have been carried out allowing, for instance, to define the expectation of the random set. It allowed to establish the SLLN in some cases of random convex valued sets. As for random set with general bounded values, non necessarily convex, the following strategy (see Artstein and Vitale [4], or Puri and Ralescu [17], see also [8, Section 9]) has been adopted quite successfully.

The strategy: *First, consider the sequence of random sets defined by taking the convex hulls of the values of the original random sets, and establish an SLLN for it. Second, invoke a convexification argument to establish an SLLN for the original sequence of random sets.*

In order to apply the strategy one may try to employ properties of the space in which the values of the random set are embedded. For instance, when the space of convex sets is B-convex, Beck's theorem could be applied. Unfortunately, for an infinite-dimensional Banach space X, the space in which the bounded convex sets are embedded, is never B-convex, as we show next.

Lemma 4.1. *Given a natural number n, consider a set M consisting of 2^n elements. There are n subsets of M, say $T_1, \dots, T_n$, such that whenever V is a subset of $\{1, \dots, n\}$, then there is an element m_V in M, such that $m_V \in T_j$ whenever $j \in V$ and $m_V \notin T_j$ when $j \notin V$.*

Proof. It is sufficient to present the construction for $M = \{0,1\}^n$. In this case elements of M are vectors of the form $a = (a_1, a_2, \dots, a_n)$ with $a_k \in \{0,1\}$. For each $j \in \{1, .., n\}$ let us define the required T_j as the set of those vectors $(t_1, t_2, ..., t_n) \in M$, for which $t_j = 1$.

Then, for every $V \subset \{1, \dots, n\}$ we can define $m_V = (a_1, a_2, \dots, a_n)$ that we need by the rule

$$a_k = \begin{cases} 1, & \text{if } k \in V \\ 0, & \text{if } k \notin V. \end{cases}$$

□

Here is the promised result. We state and prove it for convex compact subsets. It automatically applies to the space of all convex subsets.

Theorem 4.2. *Let X be an infinite-dimensional Banach space, and let Z be a Banach space in which the cone $K(X)$ of compact subsets of X, with the Hausdorff distance, is isometrically embedded. Then Z is not B-convex.*

Proof. We first verify the claim for X being the Hilbert space ℓ_2. Denote by $R : K(X) \to Z$ the embedding mapping.

Given a natural number n, choose an orthonormal collection of 2^n vectors $e_1, \dots, e_{2^n}$ in X, namely, each e_i is a unit vector, and they are mutually perpendicular to each other. For $M = \{1, 2, \dots, 2^n\}$ let $T_k \subset M$ be the subsets of indices guaranteed in Lemma 4.1. For each index k define $\Delta_k = \operatorname{conv}\{e_i : i \in T_k\}$. These are n compact subsets of X.

If Z is B-convex, it has infratype $p > 1$, namely, a constant C exists such that

$$\min_{\theta_i=\pm 1} \Big\| \sum_{i=1}^{n} \theta_i R(\Delta_i) \Big\| \le C \left(\sum_{i=1}^{n} \|R(\Delta_i)\|^p \right)^{\frac{1}{p}} = C n^{\frac{1}{p}} \tag{8}$$

(in the last equality we use $\|R(\Delta_i)\| = \rho_H(\Delta_i, \{0\}) = 1$).

Consider now the first term in (8). For a given choice of $\theta_i = \pm 1$ let V be the subset of those $i \in \{1, \dots, n\}$ where $\theta_i = 1$. Denote by s the number of elements of V. Without loss of generality $s \ge \frac{n}{2}$, otherwise we may work with the complement to V instead of V. By Lemma 4.1 there is an index $m \in M$ such that $m \in T_j$ whenever $j \in V$ and $m \notin T_j$ when $j \notin V$. The corresponding unit vector e_m belongs to each Δ_k with $k \in V$. This implies that $se_m \in \sum_{i \in V} \Delta_i$. On the other hand, se_m is perpendicular to all elements of Δ_j when $j \notin V$. It follows that

$$\Big\| \sum_{i=1}^{n} \theta_i R(\Delta_i) \Big\| = \Big\| R\Big(\sum_{i \in V} \Delta_i\Big) - R\Big(\sum_{i \notin V} \Delta_i\Big) \Big\| = \rho_H\Big(\sum_{i \in V} \Delta_i, \sum_{i \notin V} \Delta_i\Big) \ge \|se_m\| \ge \frac{n}{2}.$$

Since the above estimation holds true for every choice of $\theta_i = \pm 1$, we have that

$$\frac{n}{2} \le \min_{\theta_i = \pm 1} \Big\| \sum_{i=1}^{n} \theta_i R(\Delta_i) \Big\|. \tag{9}$$

Combining (9) and (8) we obtain that

$$\frac{n^{1-\frac{1}{p}}}{2} \le C.$$

But the left hand side of the inequality tends to ∞, a contradiction. This contradiction completes the proof for the space ℓ_2.

To verify the claim for any infinite-dimensional Banach space notice that the construction of $\{\Delta_k\}_{k=1}^{n}$ in the above proof uses finite-dimensional subspaces of ℓ_2. According to the Dvoretzky's almost Euclidean sections theorem, every infinite-dimensional Banach space has subspaces of arbitrarily high dimension arbitrarily close to Euclidean spaces. See Kadets and Kadets [10, Chapter 6]. This implies that the result is valid in all infinite-dimensional Banach spaces X. □

The previous result implies that to carry out the first step in the strategy, namely, establishing an SLLN for random convex subsets of an infinite-dimensional space, one should impose limitations on the sequence of random sets. For instance, consider sequences that are identically distributed (as done, e.g., in Artstein and Hansen [3], and in Puri and Ralescu [17], see also Molchanov [13], Hess [8]). The SLLN for identically distributed random variables is well known, see Mourier [14]. The literature offers such possibilities, examining these is, however, beyond the scope of the present note, that focuses on convexification. Hence we proceed to take care of the latter with the following terminology.

Let S_k be a sequence of independent random sets, taking as values convex subsets of the unit ball of a Banach space X. We say that the sequence satisfies the Strong Law of Large Numbers in the topology generated by the Hausdorff distance, if almost surely $\frac{1}{n}(S_1 + \ldots + S_n)$ converges in the Hausdorff distance to a common constant set.

Theorem 4.3. *Let X be B-convex. Let S_k be a sequence of independent random sets, taking values within the unit ball of X. Suppose that the sequence* $\operatorname{conv} S_k$ *satisfies the Strong Law of Large Numbers in the topology generated by the Hausdorff distance. Then almost surely $\frac{1}{n}(S_1 + \ldots + S_n)$ converges in the Hausdorff distance to a constant convex set, identical to the a.s. limit of $\frac{1}{n}(\operatorname{conv} S_1 + \ldots + \operatorname{conv} S_n)$.*

Proof. It is a straightforward implication of our main result Theorem 3.1. □

The B-convexity of X cannot be dropped. As an example one can take the trivial sequence mentioned in the introduction, namely, replicas of $\{0, e_1, e_2, ...\}$ in ℓ_1, interpreted as a random set. The SLLN does not hold for this sequence. The straightforward application of Theorem 3.1, namely Theorem 4.3, generalizes the results listed in [3] and in [17], where the random sets were assumed identically distributed (which guaranteed the SLLN of the convex version) and have compact values. Here we demand only that the values be bounded.

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