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A Quadratic Transformation Based on a Straight Line and a Conic

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Abstract. The Λ-transformation is quadratic in the projective 3-space and originally based on an irreducible spatial cubic. Here the case is addressed where the cubic splits into a conic and a straight line. Two cases are distinguished depending on whether the conic and the line are disjoint or not. An analytic representation of the Λ-transformation is given and the images of planes and lines are studied in detail.

Keywords: Λ-transformation, quadratic transformation
MSC 2000: 51N15, 51N35

1. Introduction

In 1978 J. Fellmann [1] published a paper on the Λ-transformation based on a twisted curve C^3 of third order and an autocollineation φ of this curve. In the following work we use a straight line t and a non-coplanar conic s. In this sense one can say that this is a special case of [1]. Instead of using an autocollineation, our definition of the Λ-transformation is based on the elements of the *congruence* $K[2,2]$ *of bisecants* of the spatial cubic $t \cup s$ (Fig. 1).

2. Part I

2.1. Definition of the Λ-transformation

In the projective space $\mathbb{P}^3$ over the field of real numbers $\mathbb{R}$ we choose a certain conic s and a non-coplanar straight line t; in Part I we suppose $s \cap t = \emptyset$. At the beginning we prove that the set of bisecants of the spatial cubic $s \cup t$ is a congruence $K[2,2]$ of second order and second class.

Really, remembering that the *order* of a congruence is equal to the number of elements (lines) of the congruence passing through an arbitrary point Q, let us take Q apart from line t. Line t and point Q span a plane α (Fig. 2). This plane intersects the conic s at two points R_1 and R_2. The lines QR_1 and QR_2 intersect line t at the points $\overline{R}_1$ and $\overline{R}_2$, respectively.

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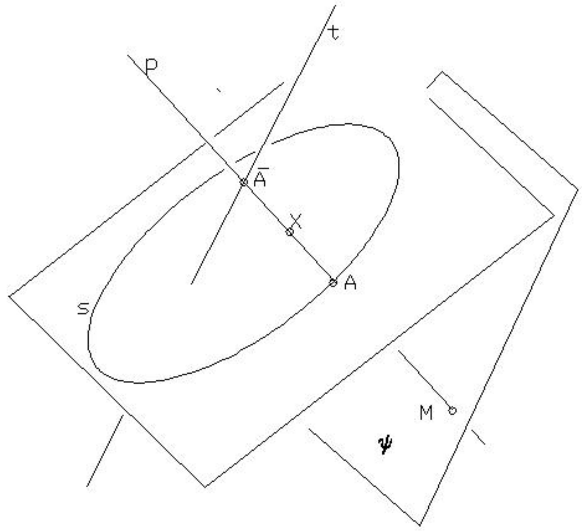

Figure 1: Definition of the Λ-transformation $M \mapsto X$ for $s \cap t = \emptyset$ (Part I)

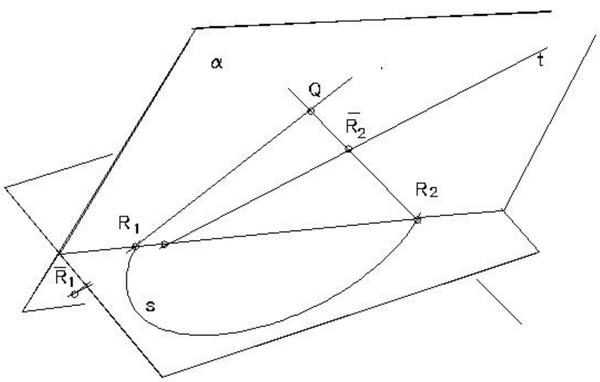

Figure 2: The congruence K of bisecants is of order 2

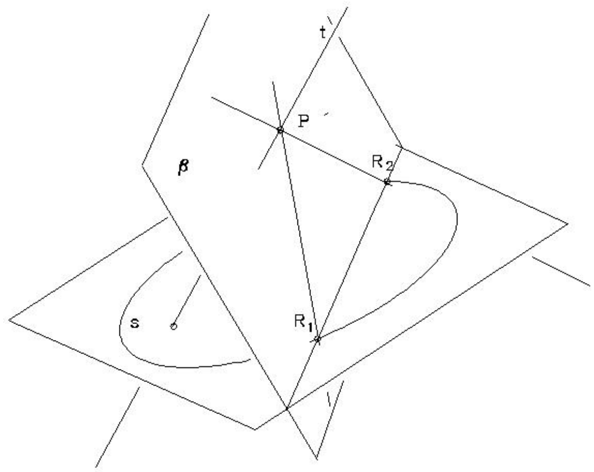

Figure 3: The congruence K of bisecants is of class 2

In the congruence K the lines $R_1\overline{R}_1$ and $R_2\overline{R}_2$ are the only two elements which pass through point Q. Hence, the congruence K is of the second order.

Further, the *class* of a congruence is defined as the number of elements of the congruence lying in an arbitrary plane. So, let β be any plane distinct from the plane of the conic s and not passing through the line t. The two lines joining the points $R_1, R_2 \in s \cap \beta$ with the point $P = t \cap \beta$ are the only two elements in the congruence K which lie in the plane β (Fig. 3). Hence, the congruence K is of second class.

Definition 1 *(Definition of the Λ-transformation):*
In the real projective 3-space $\mathbb{P}^3$ let a conic s and a non-coplanar line t be given. Let $K[2,2]$ denote the congruence of bisecants of the reducible spatial cubic $s \cup t$. In addition, let λ be any given real number, $\lambda \neq 0, 1$.
Then, on any line p intersecting s at A and t at $\overline{A}$, i.e. $p \in K[2,2]$, for given point $M \in p$ the point $X \in p$ is called the image of point M under the Λ-transformation if the cross-ratio of

the four points A, $\overline{A}$, M, and X (see Fig. 1) is equal to the given λ, i.e., if

$$(A\,\overline{A}\,M\,X) = \lambda\,. \tag{1}$$

Varying line p and applying the properties of the cross-ratio of four points (cf. [2]), we get in this way a *single-valued correspondence* $M \mapsto X$ of points $M, X \in \mathbb{P}^3$.

In the whole paper the reasoning is done analytically by use of homogenous projective coordinates in $\mathbb{P}^3$. Without loss of generality, we can choose a simplex P_0, P_1, P_2, P_3 as the *frame of reference* for the projective coordinates such that the line t is identical with the edge P_2P_3 and the conic s is located in the plane $P_0P_1P_3$. We may assume that s is tangent at points P_0, P_1 to the edges P_0P_3, P_1P_3, respectively (Fig. 4). We specify the conic s in the pencil $x_2 = 0$, $x_3^2 - kx_0x_1 = 0$, $k \in \mathbb{R}$, by setting

$$s: \quad x_3^2 - x_0x_1 = 0, \quad x_2 = 0. \tag{2}$$

The line t is determined by the system of equations

$$t: \quad x_0 = x_1 = 0. \tag{3}$$

In order to obtain equations of the Λ-transformation we must determine at least one line $p \subset K[2,2]$, i.e., passing through an arbitrary point M and intersecting the conic s and the line t simultaneously:

The line t and the point M span a plane φ. The equation of this plane can be written in the form

$$\det \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ m_0 & m_1 & m_2 & m_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 0\,.$$

i.e., $m_1x_0 - m_0x_1 = 0$ with arbitrary x_2, x_3.

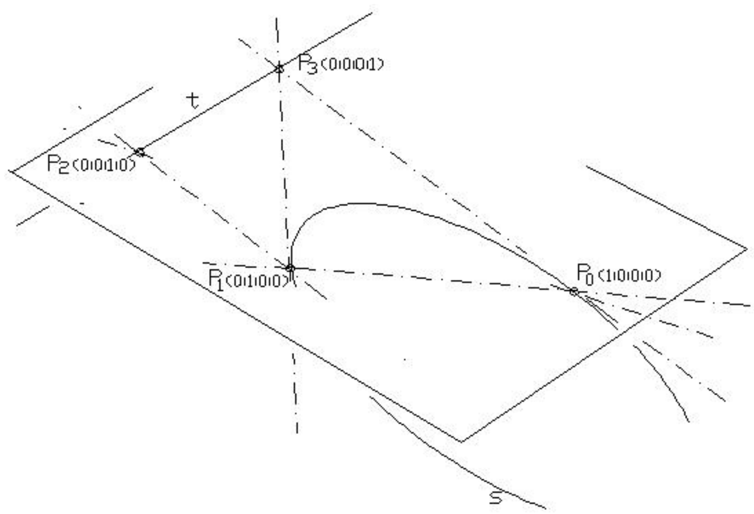

Figure 4: The simplex $P_0 \ldots, P_3$ of reference

The plane φ and the face $P_0P_1P_3$ share a line k determined by the two equations

$$m_1x_0 - m_0x_1 = x_2 = 0\,, \quad (x_3 \text{ arbitrary}).$$

For the computation of the common points of conic s and line k we solve the system of equations

$$\begin{aligned} x_3^2 - x_0x_1 &= 0 \\ m_1x_0 - m_0x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

and get

$$x_0 = \pm\sqrt{\frac{m_0}{m_1}}\,x_3\,, \quad x_1 = \pm\frac{m_1}{m_0}\sqrt{\frac{m_0}{m_1}}\,x_3\,, \quad x_2 = 0\,.$$

Since x_3 has an arbitrary value, we can assume $x_3 = \sqrt{m_1/m_0}$. Finally, we obtain coordinates of points A_1, A_2 in the form

$$A_1 = (m_0 : m_1 : 0 : \sqrt{m_0m_1}\,), \quad A_2 = (-m_0 : -m_1 : 0 : \sqrt{m_0m_1}\,). \tag{4}$$

In the following discussion we use point A_1 only. We will return to A_2 later.

From [2] we know that the coordinates of any third point on the line MA_1 can be written as a linear combination of coordinates of M and A_1. So, we have

$$\begin{aligned} \rho x_0 &= p_1m_0 + p_2m_0 \\ \rho x_1 &= p_1m_1 + p_2m_1 \\ \rho x_2 &= p_1m_2 + p_20 \\ \rho x_3 &= p_1m_3 + p_2\sqrt{m_0m_1}\,. \end{aligned} \tag{5}$$

Using systems (5) and (3) we get the coordinates of point $\overline{A}_1$, the common point of lines t and MA_1, in the form $\left(0 : 0 : m_2 : m_3 - \sqrt{m_0m_1}\right)$.

For three distinct points, A_1, $\overline{A}_1$, M on line p there is a local system of projective coordinates such that the local coordinates of these points are $(1 : 0)$, $(0 : 1)$, $(1 : 1)$, respectively. Hence, the coordinates of any fourth point X on line p can be written as

$$\begin{aligned} \rho x_0 &= \mu_1m_0 + \mu_20 \\ \rho x_1 &= \mu_1m_1 + \mu_20 \\ \rho x_2 &= \mu_10 + \mu_2m_2 \\ \rho x_3 &= \mu_1\sqrt{m_0m_1} + \mu_2(m_3 - \sqrt{m_0m_1}\,). \end{aligned} \tag{6}$$

It is well known (see [2]) that the cross ratio equals $(A_1\overline{A}_1MX) = \mu_1/\mu_2$. So, recalling condition (1), we get finally $\lambda = \mu_1/\mu_2$, and the equations of the Λ-transformation are

$$\begin{aligned} \rho x_0 &= \lambda m_0 \\ \rho x_1 &= \lambda m_1 \\ \rho x_2 &= m_2 \\ \rho x_3 &= (\lambda - 1)\sqrt{m_0m_1} + m_3\,. \end{aligned} \tag{7}$$

When deriving the above equations we took A_1 and $\overline{A}_1$ as the basic points. It is easy to show that for A_2 and $\overline{A}_2$ the system (7) turns into the form

$$\begin{aligned} \rho x_0 &= \lambda m_0 \\ \rho x_1 &= \lambda m_1 \\ \rho x_2 &= m_2 \\ \rho x_3 &= (1 - \lambda)\sqrt{m_0m_1} + m_3\,. \end{aligned} \tag{8}$$

Obviously, the two points X, X' obtained from equations (7) and (8) are distinct.

2.2. Properties of the Λ-transformation

Let us start with the investigation of fixed points under the transformation (7): Any point of the conic s described by eq. (2) has coordinates of the form $A = \big(a_0 : a_1 : 0 : \sqrt{a_0 a_1}\big)$. Its image is the point $A' = \big(\lambda a_0 : \lambda a_1 : 0 : (\lambda - 1)\sqrt{a_0 a_1} + \sqrt{a_0 a_1}\big)$. Hence A equals A', and all points of the conic s are fixed points under the Λ-transformation. Similarly, any point $B = (0 : 0 : b_2 : b_3)$ lying on line t is mapped onto $B' = (0 : 0 : b_2 : (\lambda - 1)\sqrt{0 \cdot 0} + b_3)$. So we obtain $B = B'$, and line t is pointwise fixed under the Λ-transformation.

When investigating the Λ-transformation, the main problem is to get an answer to the following question: What is the locus of points X when point M describes a certain plane ψ? Let us choose a plane ψ described by

$$\psi\colon\ u_0 x_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 = 0. \tag{9}$$

In order to receive the answer to the above question, we must add to the system (7) the condition that point $M = (m_0 : \ldots : m_3)$ lies in plane ψ, and then eliminate from the obtained system the coordinates $m_0, \ldots, m_3$ of the varying point M. So, we have the system of equations

$$\begin{aligned} \rho x_0 &= \lambda m_0 \\ \rho x_1 &= \lambda m_1 \\ \rho x_2 &= m_2 \\ \rho x_3 &= (\lambda - 1)\sqrt{m_0 m_1} + m_3 \\ 0 &= m_0 x_0 + m_1 x_1 + m_2 x_2 + m_3 x_3\,. \end{aligned}$$

From this system of equations we deduce

$$m_0 = \frac{\rho x_0}{\lambda}, \qquad m_1 = \frac{\rho x_1}{\lambda}, \qquad m_2 = \rho x_2,$$
$$m_3 = \rho x_3 - (\lambda - 1)\sqrt{m_0 m_1} = \rho x_3 - \frac{\rho(\lambda - 1)}{\lambda}\sqrt{x_0 x_1}\,.$$

After substituting the first three expressions in the last equation we get

$$\Phi(x_0, \ldots, x_3) :\equiv (u_0 x_0 + u_1 x_1 + \lambda u_2 x_2 + \lambda u_3 x_3)^2 - u_3^2(\lambda - 1)^2 x_0 x_1 = 0. \tag{10}$$

If we use the system (8) instead of (7), we obtain the same expression (10). It means that the two distinct points X and X' (described by (7) or (8), respectively) satisfy the same equation (10). Hence, we get the basic theorem of Part I

Corollary 1 *The image of any plane $\psi \subset \mathbb{P}^3$ under the Λ-transformation is a quadric Φ, described by the equation (10).*

Note that we obtained all the above results under the tacid assumption that in the pair of points $(A_1, \overline{A}_1)$ the first one is always on the conic s and the second on the line t. In the reversed order, it is well known (cf. [2]) that the value of the cross ratio changes to $1/\lambda$. Then, the quadric Φ would be replaced by the quadric

$$\Phi_1(x_0, \ldots, x_3) :\equiv (\lambda u_0 x_0 + \lambda u_1 x_1 + u_2 x_2 + u_3 x_3)^2 - u_3^2(1 - \lambda)^2 x_0 x_1 = 0\,.$$

The two quadrics Φ and Φ_1 are essentially distinct. Equality $\Phi = \Phi_1$ occurs iff $1/\lambda = \lambda$, i.e., when $\lambda = -1$.

2.3. Investigation of the quadric Φ

In order to determine the character of the quadric Φ we apply the standard method [3] of checking the sign of certain expressions built from the coefficients a_{ik} of the quadratic form. According to [3], we define the following two matrices

$$V := \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \qquad W := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

and the expression (trace) $W_1 := a_{11} + a_{22} + a_{33}$. In our case we have

$$V_\Phi = \begin{bmatrix} u_0^2 & \frac{1}{2}\left(2u_0u_1 - (\lambda-1)^2u_3^2\right) & \lambda u_0u_2 & \lambda u_0u_3 \\ \frac{1}{2}\left(2u_0u_1 - (\lambda-1)^2u_3^2\right) & u_1^2 & \lambda u_1u_2 & \lambda u_1u_3 \\ \lambda u_0u_2 & \lambda u_1u_2 & \lambda^2u_2^2 & \lambda^2u_2u_3 \\ \lambda u_0u_3 & \lambda u_1u_3 & \lambda^2u_2u_3 & \lambda^2u_3^2 \end{bmatrix},$$

$$W_\Phi = \begin{bmatrix} u_0^2 & \frac{1}{2}\left(2u_0u_1 - (\lambda-1)^2u_3^2\right) & \lambda u_0u_2 \\ \frac{1}{2}\left(2u_0u_1 - (\lambda-1)^2u_3^2\right) & u_1^2 & \lambda u_1u_2 \\ \lambda u_0u_2 & \lambda u_1u_2 & \lambda^2u_2^2 \end{bmatrix},$$

and

$$W_{1\Phi} = u_0^2 + u_1^2 + \lambda^2u_2^2 .$$

Since Rank V_Φ = Rank $W_\Phi = 3$ and $\det W_\Phi = -\frac{1}{4}\left(\lambda^2(\lambda-1)^2u_2^2u_3^4\right)$, so $\det W_\Phi \cdot W_{1\Phi} \le 0$, and by virtue of [3] we have

Theorem 1 *In the generic case the quadric Φ defined by eq. (10) is a cone of the second order.*

2.4. Position of the cone Φ

Let us consider the location of the cone Φ with respect to the basis simplex, the conic s and the line t. Does the conic s lie on the cone Φ?

Setting $x_2 = 0$ in eq. (10) we get a conic section

$$(u_0x_0 + u_1x_1 + \lambda u_3x_3)^2 - (\lambda-1)^2u_3^2x_0x_1 = 0. \tag{11}$$

We see that the conic (11) differs from s. Hence,

Corollary 2 *The cone Φ does not pass through the conic s.*

Is line s a generator of the cone Φ? Line t is defined as the edge P_2P_3 of our simplex. Setting $x_0 = 0$ and $x_1 = 0$ in (10), we get

$$(u_0x_0 + \lambda u_2x_2 + \lambda u_3x_3)^2 = 0\,, \tag{12}$$

$$(u_1x_1 + \lambda u_2x_2 + \lambda u_3x_3)^2 = 0\,, \tag{13}$$

respectively. As we can see we have obtained two twofold covered straight lines. Hence

Corollary 3 *The line t does not lie on the cone Φ. The cone touches the two faces $x_0 = 0$, $x_1 = 0$ of the simplex along lines (12) and (13), respectively (see Fig. 5).*

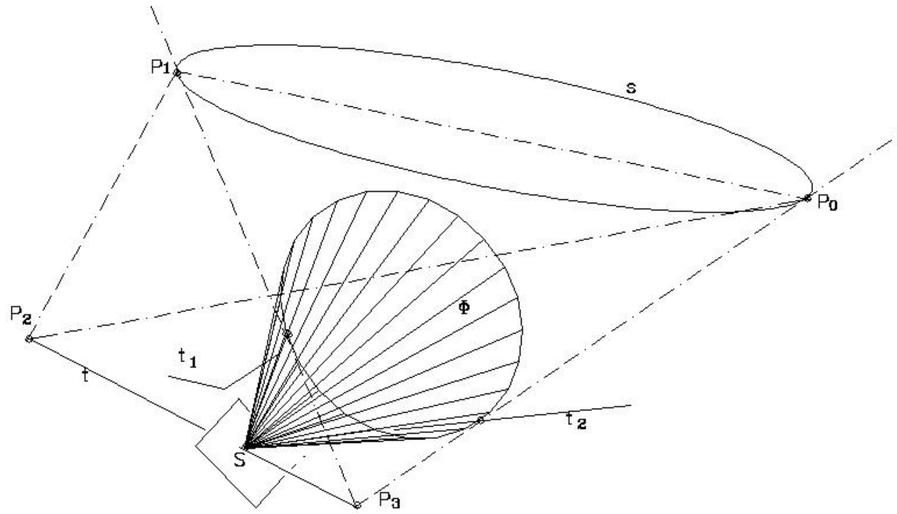

Figure 5: Position of the cone Φ with respect to the simplex $P_0, \ldots, P_3$ of reference

To determine the vertex S of the cone, it is sufficient to set $x_0 = 0$ in (13) and $x_1 = 0$ in (12). This results in two identical formulas of the form $\lambda u_2 x_2 + \lambda u_3 x_3 = 0$, and the coordinates of the vertex are

$$S = (0 : \ 0 : \ -u_3 : \ u_2)\,.$$

It is easy to see that these coordinates satisfy eq. (9) of the plane ψ. So, we have

Corollary 4 *The vertex S of cone Φ is the common point of the line t and the given plane ψ.*

2.5. Degeneration of the cone Φ

If plane ψ passes through the vertex $P_3 = (0 : 0 : 0 : 1)$ of the basis simplex, i.e., if we put $u_3 = 0$ in eq. (9), then equation (10) turns into the form

$$\Phi(x_0, \ldots, x_3) \equiv (u_0 x_0 + u_1 x_1 + \lambda u_2 x_2)^2 = 0, \quad x_3 \text{ arbitrary.}$$

This is a certain twofold covered plane, and it passes through the vertex $P_3 = (0 : 0 : 0 : 1)$, too. If we assume that ψ passes through vertex $P_2 = (0 : 0 : 1 : 0)$, i.e., if $u_2 = 0$, then eq. (10) takes the form

$$\Phi(x_0, \ldots, x_3) \equiv (u_0 x_0 + u_1 x_1 + \lambda u_3 x_3)^2 - u_3^2(\lambda - 1)^2 x_0 x_1 = 0, \quad x_2 \text{ arbitrary,}$$

and this is still a cone of second order.

2.6. Image of a straight line q

Let us specify a straight line q by its parametric representation

$$\rho m_i = a_i t + b_i, \quad i = 0, \ldots, 3\,. \tag{14}$$

After substituting these coordinates $(m_0 : \dots : m_3)$ into the systems (7) and (8), we get the images of the straight line q described by the two systems

$$\begin{array}{rcl} \rho x_0 &=& \lambda(a_0 t + b_0) \\ \rho x_1 &=& \lambda(a_1 t + b_1) \\ \rho x_2 &=& a_2 t + b_2 \\ \rho x_3 &=& (\lambda - 1)\sqrt{(a_0 t + b_0)(a_1 t + b_1)} + \\ && + a_3 t + b_3\,, \end{array} \qquad \begin{array}{rcl} \rho x_0 &=& \lambda(a_0 t + b_0) \\ \rho x_1 &=& \lambda(a_1 t + b_1) \\ \rho x_2 &=& a_2 t + b_2 \\ \rho x_3 &=& (1 - \lambda)\sqrt{(a_0 t + b_0)(a_1 t + b_1)} + \\ && + a_3 t + b_3\,. \end{array} \tag{15}$$

These two systems describe two different arcs of the same conic s_1. Hence, we have

Theorem 2 *The image of a straight line $q \subset \mathbb{P}^3$ under the Λ-transformation is in the generic case a single conic s_1 defined by the systems (15) of parametric equations.*

The two arcs meet each other when the parameter t is satisfying the condition $(a_0 t + b_0)(c_1 t + b_1) = 0$, i.e., for $t_1 = -b_0/a_0$ or $t_2 = -b_1/a_1$. Hence, the common points of the two arcs are

$$\begin{array}{rcl} T_0 &=& (0 : \lambda(b_1 a_0 - a b_0) : (b_2 a_0 - a b_0) : (b_3 a_0 - a_3 b_0)) \text{ and} \\ T_1 &=& (\lambda(b_0 a_1 - a_1 b_0) : 0 : (b_2 a_1 - b_1 a_2) : (b_3 a_1 - a_3 b_1))\,. \end{array}$$

These points T_0 and T_1 are the common points of the conic s_1 with the faces $x_0 = 0$ and $x_1 = 0$. So, we get

Corollary 5 *The conic s_1 touches the faces $x_0 = 0$ and $x_1 = 0$ of the basis simplex at the points T_0 and T_1, respectively.*

Finally, let us determine equation of the plane passing through the conic s_1:
If we choose any three arbitrary and distinct points on the conic, e.g., T_0, T_1 and (for $t = 0$) point $R = \big(\lambda b_0 : \lambda b_1 : b_2 : (b_3 - (\lambda - 1)\sqrt{b_0 b_1}\,)\big)$, then one can express the equation of the plane in the form

$$\det \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & \lambda(b_1 a_0 - a_1 b_0) & b_2 a_0 - a_2 b_0 & b_3 a_0 - a_3 b_0 \\ \lambda(b_0 a_1 - b_1 a_0) & 0 & b_2 a_1 - a_2 b_1 & b_3 a_1 - a_3 b_1 \\ \lambda b_0 & \lambda b_1 & b_2 & b_3 - (\lambda - 1)\sqrt{b_0 b_1} \end{bmatrix} = 0\,,$$

or after the evaluation

$$(b_2 a_1 - a_2 b_1) x_0 - (b_2 a_0 - a_2 b_0) x_1 - \lambda(b_0 a_1 - a_0 b_1) x_2 = 0, \quad x_3 \text{ arbitrary.} \tag{16}$$

In this way we obtain

Corollary 6 *The plane of the conic s_1 depends on λ, a_i and b_i, $i = 0, \dots, 3$, and passes through the point $P_3 = (0 : 0 : 0 : 1)$.*

3. Part II

It is known that an irreducible spatial curve C^3 of third order does not possess any singular point. If such a point exists, then the curve C^3 must break up into a certain conic and a straight line cutting the conic at a single point, and this is the case we are investigating now.

3.1. Definition of the Λ-transformation

Let a curve C^3 with a double point be given (Fig. 6). Like in Part I we can prove that the set of all bisecants of such a curve C^3 forms a congruence $K[2,2]$ of second order and second class.

Without loss of generality we may specify the simplex $P_0,\dots,P_3$ in such a way that the conic s passes through vertices P_0, P_1, P_3, and line t is the edge P_2P_3 of the simplex. If we assume that the point $E = (1:1:0:1)$ lies on conic s (Fig 7), then the pencil of conics passing through the four points P_0, P_1, P_3, E can be written in the form

$$x_2 = 0, \quad kx_0x_1 + x_0x_3 - (k+1)x_1x_3 = 0, \quad k \in \mathbb{R}.$$

We select the conic with $k = 1$. Hence the equations of s read

$$s\colon \quad x_2 = 0, \qquad x_0x_1 + x_0x_3 - 2x_1x_3 = 0. \tag{17}$$

As in Part I, line t is defined by the system (3) of equations.

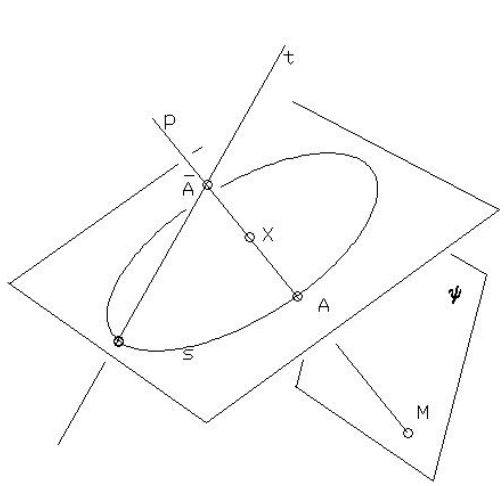

Figure 6: Definition of the Λ-transformation $M \mapsto X$ under $s \cap t \neq \emptyset$ (Part II)

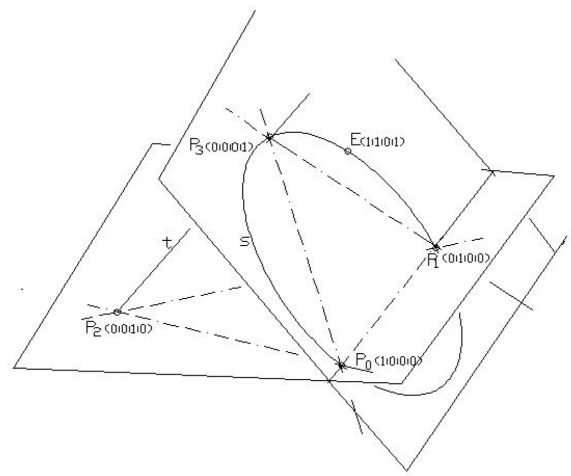

Figure 7: The simplex $P_0 \dots, P_3$ of reference in Part II

In order to determine a line $p_j \in K[2,2]$, passing through an arbitrary point $M(m_0:\ldots:m_3)$, where $M \neq P_i$, $i = 0,\dots,3$, and $M \notin t$, we take the plane $\delta = MP_2P_3$ (Fig. 8). Its equation is

$$m_1x_0 - m_0x_1 = 0, \qquad x_2, x_3 \text{ arbitrary.} \tag{18}$$

We define line p as the common line of δ and the face $P_1P_2P_3$ of the basis simplex. Hence, line p is determined by the two equations

$$m_1x_0 - m_0x_1 = x_2 = 0, \quad x_3 \text{ arbitrary.}$$

From the first equation we have $x_0 = m_0x_1/m_1$, $m_1 \neq 0$. Substituting this expression in the equation (17) of s, we obtain the coordinates of the common points of plane δ and conic s. They satisfy the system

$$x_1(m_0x_1 + m_0x_3 - 2m_1x_3) = x_2 = 0\,.$$

For $x_1 = 0$ we get the following solution: $x_0 = x_1 = x_2 = 0$, x_3 arbitrary. In other words, we get the point $A_1 = P_3 = (0:0:0:1)$, the first common point of conic s and plane δ.

The equations $m_0x_1 + m_0x_3 - 2m_1x_3 = 0$ and $x_2 = 0$ result in

$$\frac{x_1}{x_3} = \frac{2m_1 - m_0}{m_0},$$

and finally we get the second common point

$$A_2 = (m_0(2m_1 - m_0) : \ m_1(2m_1 - m_0) : \ 0 : \ m_0m_1).$$

In this way we obtain the two bisecants $p_1, p_2 \in K[2,2]$ passing through the arbitrary point M, i.e., the lines MA_1 and MA_2. In the sequel we use line MA_2 only. We will return to line MA_1 later.

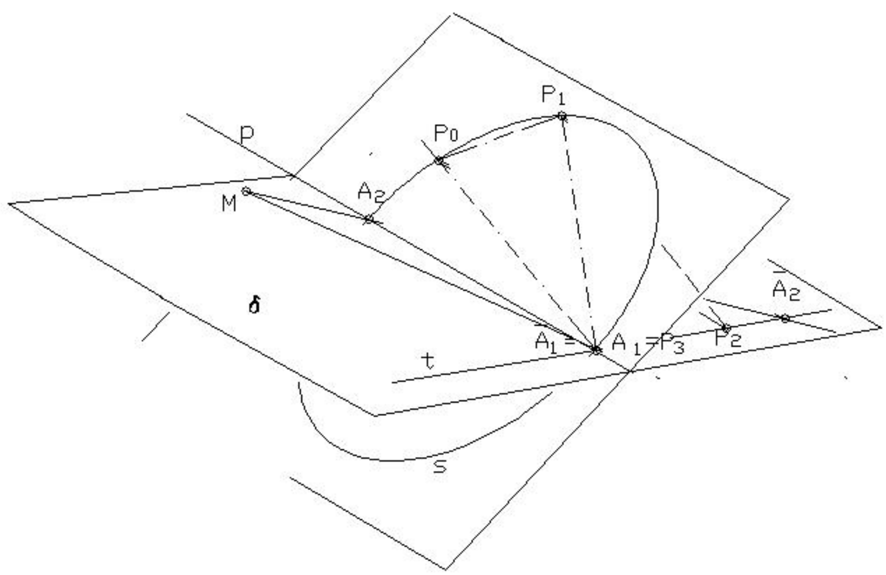

Figure 8: Construction of lines $p_1, p_2 \in K[2,2]$ through point M

Now we compute point $\overline{A}_2$, the common point of the lines t and MA_2: This point on MA_2 can be written as a linear combination of the coordinates of M and A_2. Hence

$$\begin{aligned}
\rho x_0 &= p_1m_0 + p_2m_0(2m_1 - m_0) \\
\rho x_1 &= p_1m_1 + p_2m_1(2m_1 - m_0) \\
\rho x_2 &= p_1m_2 + p_2 0 \\
\rho x_3 &= p_1m_3 + p_2m_0m_1 .
\end{aligned}$$

Line t is defined by the system (3). So the coordinates of point $\overline{A}_2$ must fulfil the following conditions:

$$0 = p_1m_0 + p_2m_0(2m_1 - m_0) = p_1m_1 + p_2m_1(2m_1 - m_0),$$

i.e., $0 = p_1 + p_2(2m_1 - m_0)$ and finally $p_1 = -2m_1 + m_0$, $p_2 = 1$. Substituting the last expressions in the system of equations, we get the coordinates of point $\overline{A}_2$

$$\overline{A}_2 = (0 : \ 0 : \ (2m_1 - m_0)m_2 : \ (2m_1 - m_0)m_3 - m_0m_1). \tag{19}$$

For three distinct points M, A_2, $\overline{A}_2$ on a line a local system of projective coordinates on this line can be introduced [2] such that $A_2 = (1:0)$, $\overline{A}_2 = (0:1)$, $M = (1:1)$. In our case the coordinates of any fourth point X on line $MA_2\overline{A}_2$ can be written in the form

$$\begin{aligned} \rho x_0 &= \mu_0 m_0(2m_1 - m_0) + \mu_1 0 \\ \rho x_1 &= \mu_0 m_1(2m_1 - m_0) + \mu_1 0 \\ \rho x_2 &= \mu_0 0 + \mu_1 m_2(2m_1 - m_0) \\ \rho x_3 &= \mu_0 m_0 m_1 + \mu_1[m_3(2m_1 - m_0) - m_0 m_1]. \end{aligned}$$

According to condition (1) we have $\mu_0/\mu_1 = \lambda$. Finally the equations of the Λ-transformation in Part II are

$$\begin{aligned} \rho x_0 &= \lambda m_0(2m_1 - m_0) \\ \rho x_1 &= \lambda m_1(2m_1 - m_0) \\ \rho x_2 &= m_2(2m_1 - m_0) \\ \rho x_3 &= (\lambda - 1)m_0 m_1 + m_3(2m_1 - m_0). \end{aligned} \tag{20}$$

Now, it is the right place to return to point A_1. It is clear (Fig. 9) that $A_1 = P_3 = \overline{A}_1$ for any position of point M. From the properties of a cross-ratio [2] we know that if $A_1 = \overline{A}_1$ then $(A_1\overline{A}_1MX) = 1$, and this contradicts the assumption in Definition 1. This implies that we can omit in our investigations the line MA_1 and use the line MA_2 only.

3.2. Properties of the Λ-transformation

It is important to observe that the transformation defined by the system (20) is a birational Cremona transformation [4], more exactly, a special kind of a quadratic transformation. Let us start with the determination of the fixed points of the transformation:

We write the system (20) in the form

$$\begin{aligned} \rho x_0 &= \lambda m_0 \\ \rho x_1 &= \lambda m_1 \\ \rho x_2 &= m_2 \\ \rho x_3 &= \frac{(\lambda - 1)m_0 m_1}{2m_1 - m_0} + m_3\,. \end{aligned}$$

and note that for any point $D \in t$ we have

$$D = (0:0:a:b) \;\mapsto\; D' = (0:0:a:b),$$

and for any $B \in s$

$$B = (c:d:0:-cd/(c-2d)) \;\mapsto\; B' = (c:d:0:-cd/(c-2d))\,.$$

Corollary 7 *The line t and the conic s are pointwise fixed under the Λ-transformation.*

The main problem of this chapter is to give an answer to the question: What is the locus of points X when point M varies in a plane ψ?

If we assume that the equation of plane ψ is again in the form (9), then after adding to the system (20) the condition that point $M = (m_0 : \ldots : m_3)$ lies in the plane ψ, we get

$$\begin{aligned} \rho x_0 &= \lambda m_0(2m_1 - m_0) \\ \rho x_1 &= \lambda m_1(2m_1 - m_0) \\ \rho x_2 &= m_2(2m_1 - m_0) \\ \rho x_3 &= (\lambda - 1)m_0 m_1 + m_3(2m_1 - m_0) \\ 0 &= u_0 m_0 + u_1 m_1 + u_2 m_2 + u_3 m_3\,. \end{aligned}$$

We eliminate from this system the coordinates $m_0, \dots, m_3$ of the varying point M and obtain

$$\Phi(x_0, \dots, x_3) :\equiv (2x_1 - x_0)(u_0x_0 + u_1x_1 + \lambda u_2x_2 + \lambda u_3x_3) - (\lambda - 1)u_3x_0x_1 = 0. \qquad (21)$$

This equation describes the locus of points X. Hence, we got

Corollary 8 *In the generic case the image of a plane ψ (eq. (9)) under the Λ-transformation is the quadric Φ with equation (21).*

3.3. Investigation of the quadric Φ

As we remember from Part I, line t does not belong to the quadric. Now, we have quite a different situation: It is easy to observe that any point with coordinates $(0:0:a:b)$ on this line satisfies the equation (21). This means

Corollary 9 *Line t lies on the quadric Φ, i.e., the quadric Φ is a ruled quadric.*

In a similar way as in Part I we get

Corollary 10 *The conic s does not lie on the quadric Φ.*

Following the way of reasoning from Part I, let us investigate the character of the quadric Φ: The two matrices V_Φ and W_Φ are of the form

$$V_\Phi = \begin{bmatrix} u_0 & -\frac{1}{2}(2u_0 - u_1 - (\lambda-1)u_3) & \frac{1}{2}\lambda u_2 & \frac{1}{2}\lambda u_3 \\ -\frac{1}{2}(2u_0 - u_1 - (\lambda-1)u_3) & -2u_1 & -\lambda u_2 & -\lambda u_3 \\ \frac{1}{2}\lambda u_2 & -\lambda u_2 & 0 & 0 \\ \frac{1}{2}\lambda u_3 & -\lambda u_3 & 0 & 0 \end{bmatrix},$$

$$W_\Phi = \begin{bmatrix} u_0 & -\frac{1}{2}(2u_0 - u_1 - (\lambda-1)u_3) & \frac{1}{2}\lambda u_2 \\ -\frac{1}{2}(2u_0 - u_1 - (\lambda-1)u_3) & -2u_1 & -\lambda u_2 \\ \frac{1}{2}\lambda u_2 & -\lambda u_2 & 0 \end{bmatrix}.$$

After some calculations we get $\operatorname{Rank} V_\Phi = \operatorname{Rank} W_\Phi = 3$. By virtue of [3] and remembering that the space $\mathbb{P}^3$ is over the field of real numbers, we obtain

Theorem 3 *In the generic case the quadric Φ with equation (21) is a cone of the second order (Fig. 9)*

3.4. Location of the quadric Φ

As we have proved above, line t is one of the generators of the cone Φ. Let us intersect the cone with the faces $P_0P_2P_3$, and $P_1P_2P_3$ of the simplex. It means that we must set in eq. (21) either $x_1 = 0$ or $x_0 = 0$. Hence, we get $x_0(u_0x_0 + \lambda u_2x_2 + \lambda u_3x_3) = 0$ and $x_1(u_1x_1 + \lambda u_2x_2 + \lambda u_3x_3) = 0$, respectively. In both cases we get the line t and two generators t_1: $u_0x_0 + \lambda u_2x_2 + \lambda u_3x_3 = 0$ and t_2: $u_1x_1 + \lambda u_2x_2 + \lambda u_3x_3 = 0$ of the cone.

In order to determine the vertex of the cone, we compute the common point of the three lines t, t_1 and t_2. This common point is $S = (0:0:-u_3:u_2)$, and its coordinates satisfy the equation (9) of the plane ψ. So, we get

Corollary 11 *The vertex S of the cone Φ is the common point of plane ψ and line t.*

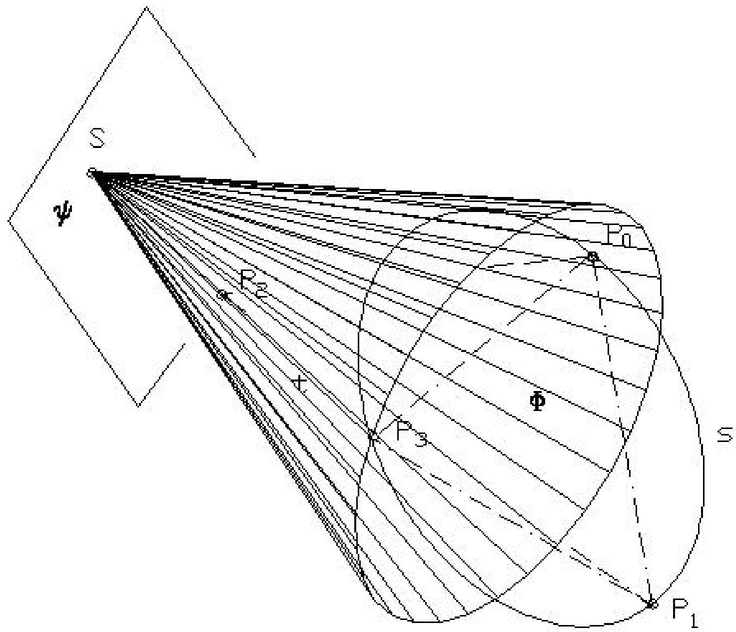

Figure 9: The image Φ of a plane ψ under the Λ-transformation

We proved above that in the generic case the image of a plane is a cone Φ, and the line t is one of its generators. An interesting question arises: Do all cones, images of different planes, simply pass through line t or touch each other along this line?

When an algebraic surface with equation $\Omega(x_0, \ldots, x_3) = 0$ is given, then the plane tangent to this surface at its point $(z_0, \ldots, z_3)$ has the equation

$$\frac{\partial \Omega}{\partial x_0}(z_0, \ldots, z_3)\, x_0 + \ \ldots \ + \frac{\partial \Omega}{\partial x_3}(z_0, \ldots, z_3)\, x_3 = 0\,.$$

Let us determine the plane tangent to our cone Φ at the point $P_3 = (0 : 0 : 0 : 1)$. The derivatives are

$$\begin{aligned}
\frac{\partial \Omega}{\partial x_0}(x_0, \ldots, x_3) &= 2u_0x_0 + [u_1 + (\lambda-1)u_3 - 2u_0]x_1 + \lambda u_2x_2 + \lambda u_3x_3,\\
\frac{\partial \Omega}{\partial x_1}(x_0, \ldots, x_3) &= [u_1 + (\lambda-1)u_3 - 2u_0]x_0 - 4u_1x_1 - 2\lambda u_2x_2 - 2\lambda u_3x_3,\\
\frac{\partial \Omega}{\partial x_2}(x_0, \ldots, x_3) &= \lambda u_2(x_0 - 2x_1),\\
\frac{\partial \Omega}{\partial x_3}(x_0, \ldots, x_3) &= \lambda u_3(x_0 - 2x).
\end{aligned}$$

Finally, the equation of the plane tangent to the cone Φ at P_3 is

$$x_0 - 2x_1 = 0\,. \tag{22}$$

This equation is independent from the coefficients u_i, $i = 0, \ldots, 3$. So we have

Corollary 12 *All cones which are images of planes under the Λ-transformation touch each other along the line t.*

3.5. Degeneration of the cone Φ

Similarly to Part I it follows from eq. (21) that if $u_3 = 0$, i.e., if plane ψ passes through point $P_3 = (0:0:0:1)$, the cone breaks up into two planes

$$2x_1 - x_0 = 0, \quad x_2, x_3 \text{ arbitrary, and}$$
$$u_0x_0 + u_1x_1 + \lambda u_2x_2 = 0, \quad x_3 \text{ arbitrary.}$$

Hence we got

Corollary 13 *If the plane ψ passes through the vertex $P_3 = (0:0:0:1)$, then its image consists of the two planes $2x_1 - x_0 = 0$ and $u_0x_0 + u_1x_1 + \lambda u_2x_2 = 0$.*
The first plane is independent from the coefficients of the plane ψ and passes through the line t. The second plane passes through the vertex $P_3 = (0:0:0:1)$.

3.6. Image of a straight line p

Let the line p be given by its parametric representation

$$\rho m_i = a_it + b_i\,, \quad i = 0, \ldots, 3$$

identical to (14). Under the Λ-transformation (20) we obtain the conic

$$\begin{aligned}
\rho x_0 &= \lambda(a_0t + b_0)[2(a_1t + b_1) - a_0t - b_0] \\
\rho x_1 &= \lambda(a_1t + b_1)[2(a_1t + b_1) - a_0t - b_0] \\
\rho x_2 &= \lambda(a_2t + b_2)[2(a_1 + b_1) - a_0t - b_0] \\
\rho x_3 &= (\lambda - 1)(a_0t + b_0)(a_1t + b_1) + (a_3t + b_3)[2(a_1t + b_1) - a_0t - b_0].
\end{aligned} \tag{23}$$

Hence, we can formulate

Theorem 4 *The image of a straight line p is a conic section s_2 with the parametric representation (23).*

Let us intersect the conic s_2 with the face $x_0 = 0$. The condition

$$(a_0t + b_0)[2(a_1t + b_1) - a_0t - b_0] = 0$$

results in $(a_0t + b_0) = 0$. Hence, $t_1 = -b_0/a_0$, $a_0 \neq 0$, or $2(a_1t + b_1) - a_0t - b_0 = 0$, $t_2 = (b_0 - 2b_1)/(2a_1 - a_0)$. The points corresponding to these values of t are

$$T_1 = (0 : \ \lambda(a_0b_1 - a_1b_0) : \ (a_0b_2 - a_2b_0) : \ (a_0b_3 - a_3b_0) \quad \text{and} \quad T_2 = (0:0:0:1) = P_3\,.$$

Further, we find the common points of the conic s_2 and the face $x_1 = 0$:
Setting $x_1 = 0$ in (23) we get $t_3 = -b_1/a_1$ and $t_4 = (b_0 - 2b_1)/(2a_1 - a_0)$. Because of $t_2 = t_4$ we obtain $T_2 = T_4 = P_3$. Point T_3 has the coordinates

$$T_3 = (\lambda(a_1b_0 - a_0b_1) : \ 0 : \ (a_1b_2 - a_2b_1) : \ (a_1b_3 - a_3b_1))\,.$$

So, we can formulate the last

Corollary 14 *All conics, which are images of straight lines, intersect the line t at point P_3.*

Finally, using the three points T_1, T_3, P_3, one can compute the equation of the plane in which the conic s_2 is located. We get

$$(b_2a_1 - a_2b_1)x_0 - (b_2a_0 - a_2b_0)x_1 - \lambda(b_0a_1 - a_0b_1)x_2 = 0, \quad x_3 \text{ arbitrary.}$$

Note that this equation is identical with equation (16).

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