Common Normals of Two Tori

Boris Odehnal

Institute of Discrete Mathematics and Geometry, Vienna University of Technology
Wiedner Hauptstraße 8-10/104, A 1040 Wien, Austria
e-mail: boris@geometrie.tuwien.ac.at

Abstract. The present paper deals with the computation of the common normals of two tori. We use line geometry in order to describe these common normals as the solution of a system of algebraic equations. Further the non trivial configurations of two tori with infinitely many common normals are investigated.

Key Words: torus, normal, common normal, line geometry, line complex, congruence of surface normals, ruled surface.

MSC 2000: 51N20, 51M35, 51N35

1. Introduction

The computation of relatively extremal distances between two surfaces \(S_1\) and \(S_2\) in three-dimensional Euclidean space is equivalent to the problem of finding common normals of \(S_1\) and \(S_2\), respectively. It is important in different areas such as robotics (e.g. in collision detection), computer graphics, computer animation, and virtual reality, see [4, 12, 16].

The computation of distances between polyhedral objects is well understood. There are a lot of efficient algorithms for that. It is much more difficult to find the minimum or maximum distance between two curved objects. In general one has to solve a system of polynomial equations, which can be time consuming. In case the surfaces do not allow a convenient analytical representation the surfaces have to be replaced by sufficiently well approximating polyhedra.

In [12] line geometry was used to simplify the computation of distances between ellipsoids. The techniques presented in this paper can be applied to other algebraic surfaces as well. Thus the computation of distances (or equivalently of common normals) of further simple geometric objects as used in geometric modeling can be done this way. A different approach is presented in [4]. Because of the absence of convincing examples and the rather high degrees of involved algebraic equations this approach looks not very promising.

Another simple object which sometimes appears in geometric modeling is a torus. A torus is not a quadratic surface, it is of degree four, but it is rational and so it can be written as rational tensor product surface. There are algorithms to find the minimum and maximum distance between such surfaces. One of these algorithms uses recursive subdivision, see [3].
In the following we show a way to find the common normals of two tori. On these normals the minimum and maximum distance can be found. We use some ideas from line geometry in order to simplify the computations and clarify the number of solutions. For this purpose we make the reader familiar with some basic concepts of line geometry. After that we discuss the general case, i.e., the two tori are in general position relatively to each other.

Then we focus on those cases where two tori have infinitely many common normals. Some of the configurations are trivial, others not.

2. Fundamentals of line geometry

In the following we use the projective closure $\mathbb{P}^3$ and the complex extension of Euclidean three-space as well, if necessary. We describe points $P$ by their Cartesian coordinates $p = (p_x, p_y, p_z)$. Oriented lines $L$ in Euclidean three-space can easily be described by normalized Plücker coordinates $L = (l, \bar{l}) = (L_1, L_2, L_3; L_4, L_5, L_6) \in \mathbb{R}^6$, where $l$ is a unit vector parallel to $L$. Thus $L$ becomes oriented. The vector $\bar{l}$ is called momentum vector of $L$. It is computed by $\bar{l} = p \times l$, if $P$ is any point on $L$. The momentum vector is independent on the choice of $P$ on $L$ and thus the Plücker coordinates of $L$ are also independent of $P$. Obviously

$$\langle l, \bar{l} \rangle = L_1L_4 + L_2L_5 + L_3L_6 = 0$$

holds for the Plücker coordinates of a line. Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product. Conversely any six tuple of real (indeed complex) numbers $L_i$ satisfying (1) are the coordinates of a uniquely determined (oriented) line.

When the direction vector $l$ is scaled by a non vanishing factor the momentum vector is scaled by the same factor, and the Plücker coordinates describe the same line. The Plücker coordinates of a line are homogeneous, i.e., we give up the orientation and drop the norming condition of $l$. So the Plücker coordinates of $L$ can be interpreted as homogeneous coordinates of points in a projective five-space $\mathbb{P}^5$.

The mapping $\gamma : L \mapsto (L_1, L_2, L_3; L_4, L_5, L_6)$ is called Klein mapping, see [6, 13]. It maps lines in (projective) three-space to points in $\mathbb{P}^5$. The Klein mapping is not onto. Only points whose coordinates satisfy (1) appear as $\gamma$-images of lines. Eq. (1) defines a quadratic hypersurface $M_2^4$ called Klein quadric or Plücker quadric. The subscript and superscript denote the algebraic degree and the dimension, respectively. The Klein quadric is a point model for the set of lines in (projective) three-space, see [6, 13].

The mapping $\gamma$ maps manifolds of lines to submanifolds of $M_2^4$. For instance the normals of a regular $C^1$-surface are mapped to a two-dimensional submanifold of $M_2^4$. Line manifolds of dimension one, two, and three are called ruled surfaces, congruences of lines, and complexes of lines, respectively. At least one example of each type will appear in the present paper. The set of lines whose homogeneous Plücker coordinates satisfy a homogeneous algebraic equation of degree $d$ is called algebraic line complex of degree $d$.

3. The set of normals of a torus

A torus $T$ can be generated by rotating a circle $c$ about an axis $A$ contained in $c$'s plane. If $A$ passes through the center of $c$ we obtain a sphere. Each position of $c$ is called a meridian circle of $T$. The path $s$ of the center of $c$ is called spine curve. An example is illustrated in Fig. 1.
Sometimes it makes sense to distinguish between three types of tori: They correspond to the three possible set of intersection points of \( c \) and \( A \). It will turn out that we are able to treat these types simultaneously, so we do not distinguish.

In order to represent the normals of \( T \) by Plücker coordinates we choose a Cartesian coordinate system and let \( A \) be the \( z \)-axis. Further we assume that \((R,0,0)\) with positive \( R \) is the center of \( c \) and \( r > 0 \) is its radius. Since 
\[ T = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u), \]
where \((u,v) \in [0,2\pi]^2\). Eliminating \( u \) and \( v \) from (2) we obtain
\[
(x^2 + y^2 + z^2 - r^2 - R^2)^2 - 4R^2(r^2 - z^2) = 0
\]
as an irreducible equation of \( T \). Obviously \( T \) is an algebraic surface of degree four.

In order to compute the normals of \( T \) given by (2) we compute
\[
\mathbf{n} = \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}.
\]
Normalizing \( n \) and computing the momentum vector, we find
\[
\mathbf{N} = (\cos u \cos v, \cos u \sin v, \sin u; R \sin u \sin v, R \sin u \cos v, 0)
\]
as a parametrization of \( T \)'s congruence \( \mathcal{N} \) of surface normals.

We recall that a normal \( N \) of \( T \) is a normal of a meridian circle \( c \). Thus \( N \) is the normal of \( T \) in at least two diametral points \( P_1 \) and \( P_2 \) on \( c \), see Fig. 1. If either \( P_1 \) or \( P_2 \) lies on the equator of \( T \), then \( N \) is normal in four points of \( T \). Therefore any normal of \( T \) is at least a double normal of \( T \).

Since \( N \) is a normal of \( c \) it contains the center of \( c \) and meets the axis \( A \) of \( T \), see Fig. 1. So the normals of \( T \) along a meridian form a pencil of lines and the normals of \( T \) along a path circle of a point on \( c \) form a cone of revolution. The axis of the cone obviously is \( A \). So the congruence \( \mathcal{N} \) can be decomposed into a one-parameter family of cones of revolution.

Figure 1: Torus as a surface of revolution
sharing the axes and into a one-parameter family of pencils of lines all of whose vertices lie on a circle \( s \) in planes through \( A \) (orthogonal to \( s \)). Fig. 1 shows a torus with a cone and a pencil of normals.

The lines of \( \mathcal{N} \) belong to a *singular linear line complex* \( \mathcal{A} \), i.e., the set of lines intersecting a fixed straight line called the *axis* of the complex (cf. [6]). Here the axis \( A \) of \( s \) is the axis of \( \mathcal{A} \). The Plücker coordinates of lines in \( \mathcal{A} \) satisfy

\[
N_6 = 0
\]

as can be seen from (3).

Now we eliminate the parameters \( u \) and \( v \) from (3). Besides of (1), (4) and the inhomogeneous norming condition \( N_1^2 + N_2^2 + N_3^2 = 1 \) we find

\[
N_4^2 + N_5^2 = R^2 N_3^2.
\]

This is the equation of the *quadratic complex* \( \mathcal{Q} \) of lines which intersect the spine curve \( s \).

Eqs. (4) and (5) describe the congruence \( \mathcal{N} \) of normals of \( T \).

Later we are interested in the normals of two tori, so we have to ask for the algebraic equations describing these congruences for tori in a more general position.

It is well known (see e.g. [6, 13]) that Euclidean motions induce automorphic collineations of \( M_4^2 \). We assume \( \beta : \mathbb{R}^3 \to \mathbb{R}^3 \) with \( p \mapsto p' = Bp + b \), where \( B^{-1} = B^T \) and \( b = \text{const.} \in \mathbb{R}^3 \), is a Euclidean motion. The mapping \( \beta \) transforms a torus into another one and it induces a transformation of the linear line complex (4) and the quadratic complex (5) too.

In order to write down the equations of the induced mapping we extend \( \mathbb{R}^3 \) projectively and thus \( \beta \) reads

\[
\beta : \begin{bmatrix} 1 \\ p' \end{bmatrix} = \begin{bmatrix} 1 & 0^T \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ p \end{bmatrix}.
\]

The induced automorphic collineation of \( M_4^2 \) is now described by the \( 6 \times 6 \) matrix

\[
\mathcal{B} = \begin{bmatrix} B & 0 \\ b \wedge B & B \end{bmatrix},
\]

where \( b \wedge B \) is the \( 3 \times 3 \)-matrix of the linear mapping \( x \mapsto b \times (Bx) \). Plücker coordinates \( N = (N_1, N_2, N_3; N_4, N_5, N_6) \) and the equations of linear line complexes \( C \) transform according to

\[
N' = \mathcal{B}N \quad \text{and} \quad C' = \mathcal{M}^{-1} \mathcal{B}^{-1} \mathcal{M} C,
\]

where \( \mathcal{M} \) is the matrix of the quadratic form (1). With \( \mathcal{Q} = \text{diag} (0, 0, -R^2; 1, 1, 0) \) we can write down the equation of the transformed quadratic complex as

\[
N^T \mathcal{B}^{-T} \mathcal{Q} \mathcal{B}^{-1} N = 0.
\]

\(^1\) \( \mathcal{Q} \) has the characteristic \([(22)2]\), which means that \( s \) is its singular surface. More on the classification of quadratic complexes can be found in [15].
4. Common normals of two tori

Let $T_1$ and $T_2$ be two tori with axes $A_1$, $A_2$ and pairs of radii $(R_1, r_1)$ and $(R_2, r_2)$, respectively. They may be centered at $C_i \in A_i$, $i \in \{1,2\}$. The normals of $T_i$ intersect both the axes $A_i$ and the spine curves $s_i$, $i = 1,2$. Thus the common normals of $T_1$ and $T_2$ belong to two singular linear line complexes $\mathcal{A}_i$ with axis $A_i$ and two quadratic complexes $\mathcal{Q}_i$ with $s_i$ for their singular curves.

In order to find common normals of two tori one has to write down the equations of the involved complexes. The underlying Cartesian coordinate system can always be chosen such that the linear and quadratic complex containing the congruence of surface normals $\mathcal{N}_1$ of $T_1$ have the equations (4) and (5). The equations of the complexes corresponding to $\mathcal{N}_2$ can always be obtained by transforming the ones determining $T_1$ and according to (7).

Taking into account that the coordinates of a common normal $N$ of $T_i$ have to satisfy (1) the solution of a system of two linear homogeneous and three quadratic homogeneous equations in the unknowns $N_1$, $N_2$, $N_3$, $N_4$, $N_5$, and $N_6$ has to be solved. Now we have the following:

**Theorem 1** Two tori $T_1$ and $T_2$ in general position have eight common normals. The number of common normals is independent on the meridian radii.

**Proof:** We use Bézout’s theorem. Since the meridian radius does not appear in any of the equations (1), (4), and (5) the solutions of the above mentioned system of algebraic equations is independent on the meridian radii. \hfill \Box

Note that each common normal $N$ of $T_1$ and $T_2$ is a double normal of both $T_1$ and $T_2$. Thus $N$ is four times a common normal of $T_1$ and $T_2$.

**Example:** Let $A_1 = (0,0,1;0,0,0)$ and $A_2 = (-24,0,7;28,0,96)$ be the axes of $T_1$ and $T_2$, respectively. The centers of $T_1$ and $T_2$ shall be given by $(0,0,-1)$ and $(-24/25,4,7/25)$ and let further $R_1 = 6$ and $R_2 = 7$ be radii of the respective spine curves. The thus determined tori have (independent on $r_i$) eight real common normals with Plücker coordinates:

\[
\begin{align*}
(1.1539, & \quad 2.5805, \quad 0.1302; \quad 1.8672, \quad -0.8350, \quad 0), \\
(-1.2300, & \quad 16.9597, \quad 1.8424; \quad 5.9345, \quad 0.4304, \quad 0), \\
(-3.8744, & \quad -3.1911, \quad 7.1665; \quad 24.1458, \quad -29.3160, \quad 0), \\
(-6.0066, & \quad -6.3847, \quad -2.1113; \quad -15.5507, \quad 14.8343, \quad 0), \\
(-0.7265, & \quad -8.4488, \quad 3.8430; \quad 14.5246, \quad -1.2489, \quad 0), \\
(-0.8284, & \quad 3.8595, \quad -2.5857; \quad -11.3091, \quad -2.4273, \quad 0), \\
(2.5020, & \quad -6.6736, \quad 5.9281; \quad 26.6312, \quad 9.9841, \quad 0), \\
(2.2539, & \quad -1.4237, \quad -5.0933; \quad -17.7435, \quad -28.0909, \quad 0).
\end{align*}
\]

Fig. 3 shows $T_1$ (yellow), $T_2$ (orange), and the eight common normals (blue). In order to illustrate that normals intersect both axes and spine curves, we give Fig. 2. In this example the maximum number of eight real solutions is achieved.

Computations are done with Maple 9.5. The images displayed in Figs. 2, 3, 5, 6, and 7 are created with POV-Ray. \hfill \Diamond

As a consequence of Theorem 1 we can state:

**Corollary 2** Let $s_1$ and $s_2$ be circles in Euclidean three-space $\mathbb{R}^3$. There exists a two-parameter family of pairs $(T_1, T_2)$ of tori with spine curves $s_1, s_2$ such that each pair of the two-parameter family has the same eight common normals.
Figure 2: Eight common normals of two tori (blue): Each of them intersects both spine curves and both axes (yellow: $A_1$ and $s_1$, orange $A_2$, $s_2$)

Figure 3: Tori with eight common normals
The radii of the respective meridian curves can serve as the coordinates in this two-parameter family of tori.

Before starting the discussion of particular cases we take a different point of view. We do already know that a normal \( N \) of \( T_1 \) meets the axis \( A_1 \) and the spine curve \( s_1 \). If \( N \) is a normal of \( T_2 \) at the same time it also intersects \( A_2 \) and \( s_2 \). Thus we are looking for lines \( N \) intersecting all \( A_1, A_2, s_1 \) and \( s_2 \). It is well known (see e.g. [5]) that the lines intersecting two lines \( A_1, A_2 \) and a circle \( s_1 \) are the set of rulings on an algebraic ruled surface \( \Phi_1 \) of degree four and Sturm type 7.\(^2\) One example is displayed in Fig. 6, and Fig. 5 shows a discrete version.

Therefore the computation or construction of common normals can be done by intersecting \( \Phi_1 \) with \( s_2 \). Since \( \Phi_1 \) is of degree four its intersection curve \( \phi_1 \) with the plane \( \pi_1 \) carrying \( s_1 \) is (in general) an algebraic plane curve of degree four.

Applying Bézout’s theorem we can expect up to eight points lying on both \( s_2 \) and \( \phi_1 \), where multiplicities and the complex extension of \( \pi_1 \) have to be taken into account. The common normals of \( T_1 \) and \( T_2 \) are passing through the previously mentioned eight points.

Analogously we could take the algebraic ruled surface \( \Phi_2 \) consisting of all lines intersecting \( A_1, A_2 \) and \( s_1 \).

The computation of common normals is obviously equivalent to the computation of common normals of the spine curves. In [16] the extreme distances between two spatial circles are computed. Constructive and algebraic methods are used in order to find the minimum and maximum distance between points and circle, line and circle, plane and circle, and finally, circle and circle. The authors even tried a line geometric approach. Unfortunately, they did not see that the computation of common normals of two tori is done automatically.

5. Special configurations

In this section we ask for configurations of two tori where common normals and extremal distances can be given explicitly. We also look for configurations of \( T_i \) such that they have infinitely many common normals. We use the preparations and considerations from Section 4 and compute the conditions for two quartic ruled surfaces \( \Phi_i \) to coincide.

5.1. Skew axes

First we assume skew axes \( A_1, A_2 \) with the enclosed angle \( 2\varphi < \pi \) and distance \( 2d > 0 \). Then a Cartesian coordinate system can always be chosen such that the \( z \)-axis is the common perpendicle of \( A_i \) and their Plücker coordinates are

\[
A_1 = (c_\varphi, s_\varphi, 0; -ds_\varphi, dc_\varphi, 0) \quad \text{and} \quad A_2 = (c_\varphi, -s_\varphi, 0; -ds_\varphi, -dc_\varphi, 0),
\]  

where \( c_\varphi := \cos \varphi \) and \( s_\varphi := \sin \varphi \), see Fig. 4. Note that \( A_1 \) and \( A_2 \) are oriented.

With \( C_i \) we denote the center of \( T_i \). The distance of \( C_i \) to the \( z \)-axis shall be denoted by

\(^2\)Algebraic ruled surfaces of degree four and Sturm type 7 are characterized by having two skew straight lines as their double curve, here these lines are \( A_1 \) and \( A_2 \).
The equation of the quartic ruled surface $\Phi_1$ with directrices $A_1$, $A_2$ and $s_1$ is given by

$$\Phi_1: 2(d^2 - (m_1^2 + R_1^2)s_\varphi^2c_\varphi^2)(-s_\varphi c_\varphi xy(z^2 + d^2) + d(s_\varphi^2 x^2 + c_\varphi^2 y^2)z)$$

$$+ s_\varphi^2 c_\varphi^2 (-m_1^2(z^2 - d^2)^2 + 2d^2 x^2 y^2) - d^2(x^4 s_\varphi^4 + y^4 c_\varphi^4)$$

$$(R_1^2 c_\varphi^4 - m_1^2 s_\varphi^2 c_\varphi^2 - d^2)(s_\varphi^2 x^2 z^2 + d^2 c_\varphi^2 y^2) + (R_1^2 s_\varphi^4 - m_1^2 s_\varphi^2 c_\varphi^2 - d^2)(c_\varphi^2 y^2 z^2 + d^2 s_\varphi^2 x^2)$$

$$+ 2ds_\varphi c_\varphi (2d^2 + 2m_1^2 s_\varphi^2 c_\varphi^2 - R_1^2(s_\varphi^4 + c_\varphi^4))xyz$$

$$+ 2d_1 s_\varphi c_\varphi (s_\varphi x + c_\varphi y)(z + d)((z - d)^2 + (s_\varphi x - c_\varphi y)^2) = 0$$

The equation of $\Phi_2$ is obtained by replacing $d$, $\varphi$, $R_1$, $m_1$ in (11) by $-d$, $-\varphi$, $R_2$, $m_2$, respectively. Again we observe that the existence of common normals is independent on the meridian radii $r_i$ since they do not show up in the equations of $\Phi_1$.

Now we are able to discuss the configurations of $T_1$ and $T_2$, where they have infinitely many common normals. This situation appears exactly if and only if the ruled surfaces $\Phi_1$ and $\Phi_2$ coincide. Then the set of lines intersecting $A_1$, $A_2$ and $s_1$ is the same as the set of lines intersecting $A_1$, $A_2$ and $s_2$.

Analogously we could look for conditions to the coefficients of Eq. (11) of $\Phi_1$ such that $s_2$ is included, i.e., $\Phi_1$ and $s_2$ have infinitely many common intersection points.

For that end we have to compare the coefficients of the monomials $x^iy^jz^l$ in the equations (11) of $\Phi_1$ and $\Phi_2$. The resulting equations are relations in $R_i$, $m_i$, $d$ and $\varphi$ describing the relative position of $T_1$ and $T_2$.

We assume $d \neq 0$ and by the above assumptions neither $s_\varphi = 0$ nor $c_\varphi = 0$. Therefore $A_1$ and $A_2$ are not parallel. Skipping the trivial relations we arrive at the following seven equations

$$m_2 + m_1 = 0, \quad (12)$$
$$m_2 - m_1 = 0, \quad (13)$$
$$s_\varphi^2(m_2^2 - m_1^2) + (R_1^2 - R_2^2)c_\varphi^2 = 0, \quad (14)$$
$$c_\varphi^2(m_2^2 - m_1^2) + (R_1^2 - R_2^2)s_\varphi^2 = 0, \quad (15)$$
$$2d^2 - (R_1^2 + R_2^2 + m_1^2 + m_2^2)s_\varphi^2 c_\varphi^2 = 0, \quad (16)$$
$$2c_\varphi^2(m_1^2 - m_2^2) + (R_1^2 - R_2^2)s_\varphi^2 = 0. \quad (17)$$
Eqs. (12) and (13) imply \( m_1 = m_2 = 0 \). From Eq. (14) we conclude \( R_1 = R_2 \) or \( R_1 = -R_2 \). The latter equation does not matter, since the radii of spine curves are always assumed to be positive and Eqs. (15) and (17) are satisfied automatically. Eq. (16) now reads \( d^2 - R_1^2 s_\varphi^2 c_\varphi^2 = 0 \) and is equivalent to

\[
2d = \pm R_1 \sin 2\varphi. \tag{18}
\]

Thus we can state the following theorem:

**Theorem 3** The configuration of two tori with skew axes \( (d \neq 0, 2\varphi \neq 0, \pi) \) and infinitely many common normals is characterized by the relations

\[
m_1 = m_2 = 0, \quad R_1 = R_2 \quad \text{and} \quad 2d = \pm R_1 \sin 2\varphi.
\]

Fig. 6 shows axes and spine curves of tori as described in Theorem 3. In Fig. 5 a pair of tori with sixty of infinitely many common normals is shown.

When \( T_1 \) and \( T_2 \) are in a configuration described in Theorem 3, then the equation (11) of the quartic ruled surface simplifies to

\[
(s_x^2 x^2 - c_x^2 y^2)^2 + (s_\varphi^2 - c_\varphi^2) ((x^2 - y^2)z^2 + R^2 (c_\varphi^2 y^2 - s_\varphi^2 x^2) - 2Rxyz(c_\varphi^2 - s_\varphi^2)) = 0. \tag{19}
\]

An example of this type of quartic ruled surface can be seen in Fig. 6.

Let us consider the special case where \( 2d = R_1 \) or equivalently \( 2\varphi = \pi/2 \) holds. Consequently \( A_1 \) and \( A_2 \) are orthogonal. Then the equation of the algebraic ruled surface \( \Phi = \Phi_1 = \Phi_2 \) of degree four becomes \( (x - y)^2 (x + y)^2 = 0 \). Obviously \( \Phi \) splits into a pair of pencils of lines each of multiplicity two.

Eq. (18) allows a reformulation of Theorem 3. Assume that the axis \( A_1 \) and the spine curve’s radius \( R_1 \) of \( T_1 \) are fixed. Then we can characterize the set of axis of all tori \( T_2 \) with \( R_2 = R_1 \) which share a one-parameter family of normals:

**Theorem 4** Let \( A_1 \) be the axis and \( R \) be the radius of the spine curve of a torus \( T_1 \). Up to rotations about \( A_2 \) the axes \( A_2 \) of tori \( T_2 \) with spine curve radius \( R \) sharing infinitely many common normals with a torus \( T_1 \) are rulings of a quartic ruled surface of Sturm type 7.

**Proof:** Assume \( A_1 = (t, 0, 0) \). Thus the axis \( A_2 \) of a torus \( T_2 \) that has infinitely many common normals with \( T_1 \) is given by \( A_2(\varphi, t) = (tc_\varphi, ts_\varphi, 1/2Rs_\varphi) \), with \((t, \varphi) \in \mathbb{R} \times [0, 2\pi]\). This parametrizes a quartic ruled surface of Sturm type 7. The rulings corresponding to \( \varphi = 0, \pi, 2\pi \) have to be excluded. \( \square \)

Further we can state the following theorem:

**Theorem 5** For a given torus \( T \) there exists a 4-parameter manifold \( \mathcal{T} \) of tori each of them having infinitely many common normals with \( T \). The distance of and the angle enclosed by the axes of \( T \) and \( T' \in \mathcal{T} \) are related by (18).

**Proof:** Consider a given torus \( T \) with given axis \( A \) and spine curve radius \( R \). If we prescribe the distance \( 2d \neq 0 \) of the axis \( A' \) of \( T' \in \mathcal{T} \) the angle \( 2\varphi \) between \( A \) and \( A' \) is determined according to Eq. (18). So the distance of the axes can be chosen freely and the angle is determined uniquely. The radii \( r \) and \( r' \) of the respective meridian curves can also be chosen freely. The configuration of \( T \) and \( T' \) can be rotated about \( T' \)'s axis without changing the number of common normals, which gives the fourth parameter. \( \square \)
Figure 5: Tori with infinitely many common normals

Figure 6: Left: Axes (blue) and spine curves (yellow). Right: Quartic ruled surface (green) whose generators intersect both the axes (blue) and the spine curves (yellow)
The generators of the quartic ruled surface \( \Phi \) defined by (19) carrying the continuum of common normals of \( T_1 \) and \( T_2 \) meet the spine curves \( s_1 \) and \( s_2 \) orthogonally. This is equivalent to \( \langle s_1 - s_2, s_1 \rangle - \langle s_1 - s_2, s_2 \rangle = 0 \). Integration yields \( \langle s_1 - s_2, s_1 - s_2 \rangle = \text{const.} \) and consequently on each common normal the two spine curves enclose a segment of the same length. Thus we have

**Theorem 6** In the particular position of Theorem 3 the spine curve \( s_2 \) is entirely contained in a torus with spine curve \( s_1 \), and vice versa.

Thus \( s_2 \) is a Villarceau-circle of a torus \( T_1 \) with spine curve \( s_1 \). For Villarceau sections and generalizations the reader may be referred to [1, 11, 14]. On the other hand we can shrink \( T_1 \) such that it degenerates to \( s_1 \) and simultaneously we can blow up \( T_2 \) such that \( s_1 \) becomes one of its Villarceau-circles. Both cases can be seen as borderline cases of tori with infinitely many common normals. This is illustrated in Fig. 7.

![Figure 7: Normals of a torus along a Villarceau-circle](image)

The right choice of radii of meridian curves of \( T_1 \) and \( T_2 \), respectively, leads to tori in line contact. In this case the meridian radii \( r_1 \) and \( r_2 \) sum up to \( 2d \).

Since the curve of intersection of two tori is of degree eight and both surfaces share the absolute conic\(^3\) the curve of contact is of degree four. In general it is not a circle with multiplicity two since \( \Phi \) given by (19) contains only the two circles \( s_1 \) and \( s_2 \), respectively. The common curve of the two touching tori becomes a Villarceau-circle only in the limiting case where one torus shrinks to a circle.

\(^3\)This conic in the ideal plane \( x_0 = 0 \) of the projectively closed Euclidean three-space obeys the equation \( x_1^2 + x_2^2 + x_3^2 = 0 \).
5.2. Coplanar axes

5.2.1. Axes are not parallel

In case of coplanar axes $A_1$ and $A_2$ we find four common normals of $T_1$ and $T_2$ in the plane $\mu = A_1 \vee A_2$. Their Plücker coordinates can be given explicitly. We use the abbreviations $\Delta_m := m_2 - m_1$, $\Sigma_m := m_1 + m_2$, $\Delta_R := R_2 - R_1$, and $\Sigma_R := R_1 + R_2$ and find

\[
\begin{align*}
(\Delta_m s_\varphi + \Sigma_R s_\varphi, & \quad \Delta_R c_\varphi - \Sigma_m c_\varphi, \quad 0; 0; 0), \quad (m_1 R_2 - m_2 R_1)c_{2\varphi} - (m_1 m_2 + R_1 R_2)s_{2\varphi}, \\
(\Delta_m s_\varphi - \Delta_R s_\varphi, & \quad -\Sigma_R c_\varphi - \Sigma_m c_\varphi, \quad 0; 0, 0), \quad -(m_1 R_2 + m_2 R_1)c_{2\varphi} - (m_1 m_2 - R_1 R_2)s_{2\varphi}, \\
(\Delta_m s_\varphi + \Delta_R s_\varphi, & \quad \Sigma_R c_\varphi - \Sigma_m c_\varphi, \quad 0; 0, 0), \quad (m_1 R_2 + m_2 R_1)c_{2\varphi} - (m_1 m_2 - R_1 R_2)s_{2\varphi}, \\
(\Delta_m s_\varphi - \Sigma_R s_\varphi, & \quad -\Delta_R c_\varphi - \Sigma_m c_\varphi, \quad 0; 0, 0), \quad -(m_1 R_2 - m_2 R_1)c_{2\varphi} - (m_1 m_2 + R_1 R_2)s_{2\varphi},
\end{align*}
\]

which are always real and connect the two intersection points $P_1, P_2 \in s_1$ and $Q_1, Q_2 \in s_2$ of either $s_i$ and $\mu$ with each other. The remaining four common normals pass through the intersection point $A_1 \cap A_2$. They are the common generators of two cones of revolution given by

\[
\begin{align*}
K_1 : \quad & (R_1^2 c_{\varphi}^2 - m_1^2 s_{\varphi}^2)(x^2 + y^2) - m_1^2 c_{\varphi}^2 z^2 + 2s_\varphi c_\varphi x y (m_1^2 + R_1^2) = 0, \\
K_2 : \quad & (R_2^2 c_{\varphi}^2 - m_2^2 s_{\varphi}^2)(x^2 + y^2) - m_2^2 c_{\varphi}^2 z^2 - 2s_\varphi c_\varphi x y (m_2^2 + R_2^2) = 0,
\end{align*}
\]

which connect $A_1 \cap A_2$ with the spine curves $s_1$ and $s_2$, respectively. Since $K_1$ and $K_2$ are concentric cones of revolution they share a pair of conjugate complex lines and a further pair of lines. The latter pair can be a pair of real lines, a single double line, or a pair of conjugate complex lines. Therefore we can expect to find four or six real common normals of $T_i$ depending on the number of common real generators of $K_i$, respectively. In case of the single double line $K_1$ and $K_2$ are in line contact along this line. Thus one of the four common normals of $T_1$ and $T_2$ is of multiplicity three.

If the spine curves $s_i$ intersect $\mu$ in a common point, $s_1$ and $s_2$ are in first order contact. In this case the above mentioned cones touch along a generator. Then the two tori $T_i$ have a pencil of common normals and one common normal $N$ not contained in the pencil. One line in the pencil of common normals is the line of contact of $K_1$ and $K_2$ and has thus multiplicity three. In this case the radii of the spine curves have to satisfy $m_1^2 + R_1^2 = m_2^2$. It is not necessary that $T_i$ share a meridian to have a pencil of common normals.

Finally we observe that in case $m_1 = m_2 = 0$ there are four different real common normals with multiplicity one. The common perpendicule of $A_1$ and $A_2$ appears as one common normal with multiplicity four. In this case we have five common normals.

5.2.2. Axes are parallel

Now we assume $A_1$ and $A_2$ are parallel. In this case one has to distinguish two subcases:

1. The spine curves lie in different planes: It is a special situation of the above mentioned case of coplanar axes. We choose a Cartesian coordinate system such that $T_1$’s center coincides with the origin and $A_1 = (1, 0, 0; 0, d, 0)$, $A_2 = (1, 0, 0; 0, -d, 0)$. $T_1$ may be centered at $(0, 0, d)$ and $T_2$ may be centered at $(m, 0, -d)$. The Plücker coordinates of the four common normals which are always real are given by

\[
\begin{align*}
(m, 0, \Sigma_R - 2d; 0, m(d - R_1), 0), \quad & (m, 0, -\Sigma_R - 2d; 0, m(d + R_1), 0), \\
(m, 0, \Delta_R - 2d; 0, m(d + R_1), 0), \quad & (m, 0, -\Delta_R - 2d; 0, m(d - R_1), 0).
\end{align*}
\]

\[\text{(20)}\]
Table 1: Number of common normals of two tori $T_1$ and $T_2$ depending on distance $2d$ and angle $2\varphi < \pi$ of axes, radii $R_1$, $R_2$ of spine curves and position $m_1$, $m_2$ of centers

<table>
<thead>
<tr>
<th>$2d$</th>
<th>$2\varphi$</th>
<th>$R_1$, $R_2$, $m_1$, $m_2$</th>
<th>$#$ real solutions</th>
<th>shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
<td>$2R_1 \sin 2\varphi$, $m_i = 0$</td>
<td>$\infty^1$</td>
<td>quartic ruled surface</td>
</tr>
<tr>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
<td>$2d = R_i \sin 2\varphi$, $m_i = 0$</td>
<td>$2\infty^1$</td>
<td>two two-fold pencils</td>
</tr>
<tr>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
<td>$R_1^2 + m_1^2 \neq R_2^2 + m_2^2$</td>
<td>$4, 6$</td>
<td></td>
</tr>
<tr>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
<td>$R_1^2 + m_1^2 = R_2^2 + m_2^2$</td>
<td>$\infty^1$</td>
<td>pencil + line$^5$</td>
</tr>
<tr>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
<td>$m_1 = m_2 = 0$</td>
<td>$5$</td>
<td></td>
</tr>
<tr>
<td>$\neq 0$, $\pi$</td>
<td>$m \neq 0$, $</td>
<td>R_1 \pm R_2</td>
<td>\neq 2d$</td>
<td>$4, 6$</td>
</tr>
<tr>
<td>$\neq 0$, $\pi$</td>
<td>$m \neq 0$, $</td>
<td>R_1 \pm R_2</td>
<td>= 2d$</td>
<td>$5$</td>
</tr>
<tr>
<td>$\neq 0$, $\pi$</td>
<td>$m = 0$, $</td>
<td>R_1 \pm R_2</td>
<td>\neq 2d$</td>
<td>$1, 3$</td>
</tr>
<tr>
<td>$\neq 0$, $\pi$</td>
<td>$m = 0$, $</td>
<td>R_1 \pm R_2</td>
<td>= 2d$</td>
<td>$\infty^1$</td>
</tr>
</tbody>
</table>

The remaining four common normals appear as common generators of two cylinders $K_1$ and $K_2$ of revolution. Like in the previous case the cylinders share a pair of conjugate complex lines, and a further pair of lines all of them being common normals of $T_1$ and $T_2$. The latter pair is a pair of real lines, a single double line, or a pair of conjugate complex lines, if and only if

$$D := (R_1^2 - R_2^2)^2 - 8d^2(R_1^2 + R_2^2 - 2d^2)$$

is greater, equal, or less than zero. In the present case we can find four or six real common normals of $T_1$ and $T_2$, respectively, if $D \neq 0$. In analogy to the previous case we can also find four common normals $T_i$, where one of them has multiplicity 3. This occurs exactly if $D = 0$, i.e., the cylinders touch along this common generator.

2. The carrier planes of the spine curves coincide: The four common normals from (21) become one common normal with multiplicity four. In case of $D > 0$ we have three real

---

$^4$In case of four common normals there can be one line with multiplicity three, if the above-mentioned cones touch along a generator.

$^5$The pencil contains a line with multiplicity three. The further line is not contained in the pencil.

$^6$One line is of multiplicity three.

$^7$See Footnote 6.

$^8$There is one common normal of multiplicity four.

$^9$See Footnotes 5, 8.
common normals one of them having multiplicity four. The case \( D < 0 \) leads to one real common normal. In case of \( D = 0 \) the spine curves are in first order contact similar to the case above and there is a pencil of common normals of \( T_i \) containing one normal of multiplicity four.

The cases where pencils of common normals appear can be characterized by

\[
|R_1 - R_2| = 2d \quad \text{or} \quad |R_1 + R_2| = 2d.
\]

5.3. Coinciding axes

Now we have \( A_1 = A_2 \) and denote the distance of the centers of \( T_i \) by \( m \). There are four different cases:

1. \( m \neq 0 \) and \( R_1 \neq R_2 \): The four common normals of \( T_1 \) and \( T_2 \) in any of the common meridian planes of the tori. So there exist two cones of revolution whose generators are common normals of either \( T_i \), i.e., there are two one-parameter families of common normals.

2. \( m \neq 0 \) and \( R_1 = R_2 \): This case is a special case of the previous one. One of the cones of revolution becomes a cylinder of revolution.

3. \( m = 0 \) and \( R_1 \neq R_2 \): Consider one of the common meridian planes. The four common normals of \( T_i \) in this plane coincide and become thus one common normal with multiplicity four. Since this is the fact for all the common meridian planes, the two tori have a pencil of common normals. This pencil is of multiplicity four. Each of these lines is a diameter of the concentric spine curves.

4. \( m = 0 \) and \( R_1 = R_2 \): The two tori share the congruence of surface normals. In case of \( r_1 = r_2 \), i.e., \( T_1 = T_2 \) this is also true.

The results are summarized in Table 1.

6. Conclusion and future research

Line geometry and the methods used in the present paper can obviously be used to compute the common normals of arbitrary algebraic surfaces. The discussion of coinciding ruled surfaces traced out by the normals of two tori was successful since it was of relatively low degree. Dealing with surfaces of higher order the algebraic congruences of their normals become much more complicated.

Unfortunately, (algebraic) surfaces in general do not have two spine curves like the torus. The only algebraic surfaces with two spine curves are Dupin cyclides, including the torus and cylinder and cone of revolution. These types of surfaces do not differ from the viewpoint of Laguerre-Geometrie, see [7, 8]. They are used in CAGD in order to model pipes and connections of pipes with different radii, see [7, 8, 9, 10].

Dupin cyclides are pipe surfaces with two focal conics (ellipse + hyperbola or parabola + parabola) for their spine curves. The normals of a dupin cyclide \( D_1 \) meet the ellipse and hyperbola (or the two parabolas). Common normals of two Dupin cyclides \( D_1 \) and \( D_2 \) meet the two focal conics of both \( D_1 \) and \( D_2 \). Thus in general one can expect to find up to 16 common normals of \( D_1 \) and \( D_2 \), respectively.

The discussion of configurations of \( D_1 \) and \( D_2 \) with infinitely many common normals would start at a list of trivial cases. The non-trivial cases appear if the set of rulings of two algebraic ruled surfaces \( \Phi_1 \) and \( \Phi_2 \) of degree eight coincide. \( \Phi_1 \) is the ruled surface determined
by the directrices of $D_1$ and one of the directrices of $D_2$. The surface $D_2$ is determined by the remaining director curves.

Acknowledgements

I have to thank the reviewers for their valuable comments and Martin Peternell for comments on an earlier version of this paper.

References


Received April 7, 2005; final form August 1, 2005