Geometry of Regular Heptagons

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Abstract. This paper explores some new geometric properties of regular heptagons. We add to the list of results from the BANKOFF-GARFUNKEL famous paper on regular heptagons 30 years ago enlisting the help from computers. Our idea is to look at the central points (like incenters, centroids, circumcenters and orthocenters) of certain triangles in the regular heptagon to find new related regular heptagons which have simple constructions with ruler and compass from the original heptagon. In the proofs we use complex numbers and the software Maple V. The eleven figures are made with the Geometer’s Sketchpad.

Key Words: regular heptagon, heptagonal triangle
MSC 2000: 51N20, 51M04, 14A25, 14Q05

1. Introduction

Leon BANKOFF and Jack GARFUNKEL in the reference [1] thirty years ago gave the review of several results on regular heptagons and on the associated heptagonal triangle (see Fig. 1). In [3], [4] and [5] some of these initial theorems in [1] have been improved. In this paper our goal is to continue these investigations. We use again complex numbers to discover new relationships in these geometric configurations.

In order to simplify our statements we use the following notation. The midpoint of points $X$ and $Y$ is $[X; Y]$ while $X \parallel \ell$ and $X \perp \ell$ are the parallel and the perpendicular to the line $\ell$ through the point $X$.

Let $\Theta = ABCDEFG$ be a regular heptagon inscribed into the circle $k$ with the center $O$ and the radius $R$. We now define fourteen regular heptagons associated to $\Theta$. It suffices to describe only their first vertex because the other vertices are obtained by rotations about the point $O$. The first vertices are shown in Fig. 1 and are defined as follows:

$$A_m = [O; A], \quad A = [O; A_2], \quad A' = [A; B], \quad A^d = AC \cap BG,$$

$$A^* = BC \cap AG, \quad A^*_m = [A_m; B_m], \quad A^*_2 = [A_2; B_2], \quad A^d_m = A_mC_m \cap B_mG_m,$$

$$A^d_2 = A_2C_2 \cap B_2G_2, \quad A^*_m = B_mC_m \cap A_mG_m, \quad A^*_2 = B_2C_2 \cap A_2G_2,$$

and let $A^*, A^*_m, A^*_2$ be the midpoints of the shorter arcs $AB, A_mB_m, A_2B_2$.

For different points $X$ and $Y$ and a real number $r > 0$ let $\gamma(X; Y)$ and $\gamma(X; r)$ denote circles with the center at $X$ which goes through $Y$ and with the radius $r$, respectively, while $\varepsilon(XY)$ is the complement of the segment $XY$ in the line $XY$.

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2. New theorems

We begin with an improvement of the observations attributed to THÉBAULT on the pages 10 and 11 of [1]. The parts (1)–(4) are from there while (5)–(10) are new. When investigating regular heptagons it seems natural to look for various regular heptagons associated to it. In our first theorem we discovered four such heptagons.

Theorem 1

1. The segment $A'F_m$ is the diagonal of the square built on the inradius of $\Theta$.
2. Extend $A'B$ over $B$ to the point $W$ so that $|A'W| = |A'B^*|$. The segment $WO$ is the diagonal of the square constructed on half the side of the equilateral triangle inscribed to the circle $k$.
3. The circle with the center at $W$ which is orthogonal to $k$ has $|A'B_m^*|$ as radius.
4. Let $T = AB \cap (F_m \perp OF)$, $S = [O; T]$, and $m = \gamma(S; O)$. The points $A'$ and $F_m$ are on the circle $m$ and the line $A'B_m^*$ is its tangent (Fig. 2).
5. Let $L = (m \cap OB) \setminus \{O\}$, $H = m \cap \varepsilon(OA)$, $I = m \cap \varepsilon(OD)$, $J = m \cap \varepsilon(OG)$, $K = m \cap \varepsilon(OC)$. Then $F_mLA'HJK$ is a regular heptagon with the length of sides $|A'B_m^*|$.
6. Let $H' = m \cap \varepsilon(A'B')$, $U' = m \cap \varepsilon(A'C')$, $L' = (m \cap A'D') \setminus \{A'\}$, $Q' = (m \cap A'E') \setminus \{A'\}$, $K' = m \cap \varepsilon(A'F')$, $J' = m \cap \varepsilon(A'G')$. Then $TH'U'LQ'K'J'$ is a regular heptagon.
7. The midpoints $I'', H'', U'', L'', Q'', K''$, $J''$ of shorter arcs $TI$, $H'H$, $U'A'$, $L'L$, $Q'F_m$, $K'K$, $J'J$ are vertices of a regular heptagon whose sides are parallel with sides of $DEFGABC$.
8. Let $n = \gamma(A'; B_m^*)$. The circles $m$ and $n$ intersect in the points $H$ and $L$.
9. Let $M = n \cap A'F_m$, $N = n \cap A'J$, $P = n \cap \varepsilon(A'L)$, $Q = n \cap \varepsilon(A'K)$, $U = n \cap \varepsilon(A'I)$. Then $B_m^*MNPQU$ is a regular heptagon.
Figure 2: The circles $m$ and $n$ with the regular heptagons inscribed into them. 

(10) The circle with the center at $W$ which is orthogonal to $n$ has $|A'A|$ as radius.

Proof: (1) In this proof we shall assume that the complex coordinates (or the affixes) of the vertices of the heptagon $ABCDEFG$ are

$$
F = 1, \quad G = f^2, \quad A = f^4, \quad B = f^6, \quad C = f^8, \quad D = f^{10}, \quad E = f^{12},
$$

where $f$ is the 14th root of unity. From $A_0 = f^6 + f^4$ and $F_m = \frac{1}{2}$ follows that $2|A'O|^2 - |A'F_m|^2$ is equal to $\frac{(f^4 + f^2 - 1)p_+p_-}{4}$, where

$$
p_- = f^6 - f^5 + f^4 - f^3 + f^2 - f + 1 \quad \text{and} \quad p_+ = f^6 + f^5 + f^4 + f^3 + f^2 + f + 1.
$$

But, $f^{14} - 1$ is $(f^2 - 1)p_-p_+$ and $p_+ = 1 + 2i(1 + 2\cos\frac{\pi}{7})\sin\frac{2\pi}{7} \neq 0$. We see that $p_- = 0$ so that $|A'F_m| = (R\cos\frac{\pi}{7})\sqrt{2}$.

(2) The point $W$ is on $AB$ and $\gamma(A'; B^*)$ if $f^4W + \overline{W} - f^{10} - f^8 = 0$ and

$$
2W\overline{W} - (f^{10} + f^8)W - (f^6 + f^4)\overline{W} + f^{13} + f^{11} + f^3 + f - 2 = 0.
$$

By solving this system we get

$$
W = \frac{f^6 + f^4 - f^3\sqrt{f(f-1)^2(f^2 + f + 2)(2f^2 + f + 1)}}{2}.
$$
It is now easy to check that \( \frac{3}{2} - |WO|^2 = \frac{(f_{11} - 3f_8 - 3f_7 + 3f + 3)p_+}{2} = 0. \)

(3) One of the intersections of \( k \) and the circle with diameter \( WO \) is the point \( Z \) with affix

\[
\frac{2 + \sqrt{2}(f_{13} - f_{12} + f_{11} - f^2 + f - 2)}{f^2f^3 + f + \sqrt{f(f - 1)^2(f^2 + f + 2)(2f^2 + f + 1)}}.
\]

The difference \( |WZ|^2 - |A'B_m|^2 \) contains \( p_- \) as a factor.

(4) The equations of \( AB \) and \( F_m \perp OF \) are \( f^6z + f^2\overline{z} = 1 - f^{10} \) and \( z + \overline{z} = 1 \). Hence, \( S = \frac{f^{10} + f^2 - 1}{2f^2(1 - f^4)}. \) Since both \( |SO|^2 - |A'S|^2 \) and \( |SO|^2 - |F_mS|^2 \) contain \( p_- \) as a factor we infer that \( A' \) and \( F_m \) are on \( m \). Also, the lines \( A'B_m^* \) and \( A'S \) are perpendicular.

(5) Note that \( f^{2k}(F_m - S) + S \) for \( k = 1, 2, 3, 4, 6 \) lie on lines \( OC, OG, OD, OA, OB \) while for \( k = 5 \) it agrees with \( A' \). Hence, these are the vertices of \( F_mKJIHA'L \). Moreover, \( |A'L| = |A'B_m^*| \).

(6) Now \( f^{2k}(T - S) + S \) for \( k = 1, \ldots, 6 \) lie on lines \( A'B', A'C', A'D', A'E', A'F', A'G' \) so that these are the last six vertices of the regular heptagon \( TH'U'LQ'K'J' \).

(7) This part is more complicated even on a computer so that we only outline main steps. First find the equation of \( m \) and of the line \( l \) joining \( S \) with the midpoint of the segment \( IT \). One of the points in the intersection \( m \cap l \) is \( I'' \). Then we rotate six times through the angle \( \frac{2\pi}{3} \) to get points \( H'', U'', L'', Q'', K'', \) and \( J'' \). Finally, we check that (only one) corresponding sides of \( I'H''U''L''Q''K''J'' \) and \( DEFGABC \) are parallel.

(8) The equations of the circles \( m \) and \( n \) (Fig. 2) are

\[
2(f^4 - 1)z\overline{z} + f^4(f^6 + f^4 - 1)z - (f^{10} + f^8 - 1)\overline{z} = 0,
\]

and

\[
4z\overline{z} - 2f^4(f^2 + 1)(f^4 z + \overline{z}) + f^{13} + f^{11} + f^3 + f - 1 = 0.
\]

Their intersections have rather complicated affixes but after some clever manipulation with square roots one can show that they represent points \( H \) and \( L \).

(9) We rotate the point \( B_m^* \) about the point \( A' \) six times through the angle \( \frac{2\pi}{7} \). The third point coincides with the point \( H \) while the others are on lines \( A'F_m, A'J, A'L, A'K, \) and \( A'I \), so that \( B_m^*MNHPQU \) is indeed a regular heptagon.

(10) Similar to the proof of (3). \( \square \)

**Theorem 2** Let the lines \( BE \) and \( BG \) intersect the line \( AD \) in points \( M \) and \( N \). Let \( U, V, \) and \( W \) denote circumcenters of the triangles \( BDM, BMN, \) and \( ABN \) (Fig. 3).

(1) \( W = A^* \) and \( U \) is the reflection of \( O \) at the line \( BD \).

(2) \( V \) is the reflection of \( W \) at the line \( BG \) and the midpoint of the shorter arc \( BM \) on the circumcircle of \( BDM \).

(3) \( |UV| = R \) and \( |UW| = |UH| = R\sqrt{2} \), where \( H \) is the intersection of the lines \( AO \) and \( GV \). Also, \( |VW| = |OV| = |AH| \).

(4) If \( X = OA \cap (B \perp DC), \ Y = OD \cap (X \perp CB), \) and \( Z = OC \cap (G \perp GA), \) then \( OVVWXYGZ \) is a regular heptagon whose sides are perpendicular to the corresponding sides of \( FEDCBAG. \)
(5) If $I = GV \cap (U \perp CB)$, $J = DU \cap (I \perp BA)$, $K = OB \cap (J \perp GA)$, $L = AF^* \cap (K \perp FG)$, and $P = AF^* \cap (D \perp BG)$, then $VUJKLP$ is a regular heptagon whose sides are perpendicular to the corresponding sides of $DCBAGFE$.

Proof: (1) The affixes of the points $M, N, U, V, W$, and $H$ are

$$f^{10} - f^8 + f^6, \quad \frac{f^{10} + f^6 + f^4}{f^4 + f^2 + 1}, \quad \frac{f^{10} + f^6}{f^4 + f^2 + 1}, \quad \frac{f^{10} + f^8 + f^6 + f^4}{f^4 + f^2 + 1}, \quad \frac{f^2(f^4 + 1)^2}{2f^4 + f^2 + 2},$$

respectively. Since $|WA^*|^2 = \left(\frac{f^6}{f^4 + f^2 + 1}\right)^2$, we infer that $W = A^*$. It is easy to find the reflection of $O$ in the line $BD$ and check that it coincides with the point $U$.

(2) Since the reflection of a complex number $x$ in the line determined by different complex numbers $y$ and $z$ is $\frac{y(x - z) + z(y - x)}{y - z}$, by direct substitution of affixes, we see that $V$ is a reflection of $W$ in the line $BG$.

(3) Since $|UV|^2 = |OW|^2$, we get $|UV| = 1 = R$. Also, since $|UW|^2 = \frac{(2f^{10} + 3f^8 - 2f^4 - 3f^2 - 2)_{p, p}}{(f^4 + f^2 + 1)^2}$ and $|UW|^2 = 2$ factors as $\frac{(2f^{10} + 3f^8 - 2f^4 - 3f^2 - 2)_{p, p}}{(f^4 + f^2 + 1)^2}$, it follows that $|UW| = \sqrt{2} = R\sqrt{2}$. In a similar fashion we can also prove that $|UH| = \sqrt{2} = R\sqrt{2}$ and that $|VW| = |OV| = |AH|$.

(4) Let $T = \frac{f^{12} + f^8}{f^6 - 1}$ denote the circumcenter of the triangle $OVW$. Then $f^{2k}(O - T) + T$ for $k = 1, \ldots, 6$ is $Z, G, Y, X, W, V$. Hence, $OVWXYGZ$ is a regular heptagon. Since $OV$ is perpendicular to $FE$, its sides are perpendicular to the corresponding sides of $FEDCBAG$. 

Figure 3: The triangle on circumcenters $U, V, W$ of the triangles $BDM, BMN$, and $ABN$ and two regular heptagons with sides perpendicular to sides of $ABCDEFG$. 

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(5) Similar to the proof of (4).

There is a similar result for centers of the nine-point circles instead of circumcenters.

**Theorem 3** Let the lines $BE$ and $BG$ intersect the line $AD$ in points $M$ and $N$. Let $A_0$, $B_0$, $C_0$, $D_0$, $E_0$, $F_0$, $G_0$ and $N_0$ be midpoints of the segments $AM$, $BM$, $CM$, $DM$, $EM$, $FM$, $GM$ and $BN$. Let $X$, $Y$ and $Z$ be centers of the nine-point circles $d_9$, $m_9$ and $a_9$ of the triangles $BDM$, $BMN$ and $ABN$ (Fig. 4).

1. The point $X$ is the midpoint of the segment $MO$ and the point $Y$ is the midpoint of the shorter arc $A_0B_0$ of the nine-point circle $d_9$ of the triangle $BDM$.

2. The point $Z$ is the reflection of $Y$ at the line $A_0N_0$. Also, $|XY| = \frac{R}{2}$ and $|XZ| = |OZ| = \frac{R\sqrt{2}}{2}$.

3. The polyhedron $A_0B_0C_0D_0E_0F_0G_0$ is a regular heptagon inscribed into $d_9$ which is the image of $ABCDEFG$ in the homothety $h(M, \frac{1}{2})$.

4. Let $A_0STB_0H N_0K$ and $N_0 A'N'PQA_0L$ are regular heptagons inscribed into $m_9$ and $a_9$, related by the homothety $h([Y; Z], -1)$ and homothetic with the heptagon $CBAGFED$. 

Figure 4: The triangle on centers $X$, $Y$, $Z$ of the nine-point circles of triangles $BDM$, $BMN$, and $ABN$ with $|XZ| = \frac{R\sqrt{2}}{2}$
Figure 5: Enlarged rectangular part of Fig. 4 with two regular heptagons homothetic with the heptagon \( CBAGFED \)

Proof: (1) The affixes of the points \( M \) and \( N \) have been given in the proof of the previous theorem. It follows that the midpoints \( A_0, B_0, N_0 \) are

\[
\frac{f^{10} - f^8 + f^6 + f^4}{2}, \quad \frac{f^{10} - f^8 + 2f^6}{2}, \quad \frac{2f^{10} + f^8 + 2f^6 + f^4}{2(f^4 + f^2 + 1)}
\]

while \( X, Y, Z \) are

\[
\frac{f^{10} - f^8 + f^6}{2}, \quad \frac{2f^{10} + f^8 + 2f^6 + f^4 + 1}{2(f^4 + f^2 + 1)}, \quad \frac{f^{10} + f^8 + 2f^6 + f^4}{2(f^4 + f^2 + 1)}
\]

respectively. By direct inspection we see that \( X \) is the midpoint of \( MO \) and that \( Y \) is the midpoint of the shorter arc \( A_0B_0 \) of the nine-point circle of \( BDM \) because \( |XY| = \frac{1}{2} \) and \( |YA_0|^2 = |YB_0|^2 \).

(2) Just as easy is to check that \( Z \) is the reflection of \( Y \) at the line \( A_0N_0 \). Finally, since

\[
|XZ|^2 = \frac{f^{10} + 2f^8 + f^6 + f^4 - 2f^{10} - f^{12}}{2(f^8 + 2f^6 + 3f^4 + 2f^2 + 1)}
\]

and both \( |XZ|^2 - \frac{1}{2} \) and \( |OZ|^2 - \frac{1}{2} \) contain the polynomial \( p_- \) as a factor we get that \( |XZ| = |OZ| = \frac{\sqrt{2}}{2} \).

(3) This part is obvious.
Similarly, since homothetic to Longchamps points of the triangles are orthologic. It is also easy to verify that claims about homotheties.

Theorem 5

\[
\begin{align*}
\frac{f^{10} + f^8 + 3f^6 + f^4}{2(f^4 + f^2 + 1)} , & \quad \frac{f^{10} + 3f^8 + 3f^6 + 3f^4 + f^2 + 1}{2(f^{12} + f^4 + 2f^2 + 2)} , \\
\frac{f^{12} + 2f^{10} + 2f^8 + 3f^6 + 3f^4 + f^2}{2(f^{12} + f^4 + 2f^2 + 2)} , & \quad \frac{f^{10} + f^8 + 2f^6 + 2f^4}{2(f^4 + f^2 + 1)} , \\
\frac{f^6 + f^2}{2} , & \quad \frac{2f^{10} - f^8 + f^6 + f^4}{2}, \\
\frac{2f^{10} + 2f^6 + f^4 + 1}{2(f^4 + f^2 + 1)}, & \quad \frac{f^{10} + 8 + 2f^6 + 2f^4 + 1}{2(f^4 + f^2 + 1)}, \\
\frac{5f^{12} + f^8 - 3f^6 - 2f^2 + 6}{2(3f^{10} + 4f^6 + 3f^6 - 2f^2 - 2)} , & \quad \frac{-2f^{12} - f^8 + f^6 - f^2 - 1}{2(f^{12} - f^8 - 2f^6 - f^4 + 1)}.
\end{align*}
\]

respectively. Since \( f^{2k}(A_9 - Y) + Y \) for \( k = 1, \ldots, 6 \) is \( K, N_9, H, B_9, T, S \), we infer that \( A_9STB_9HN_9K \) is a regular heptagon inscribed into \( m_9 \). Since \( A_9S \) is parallel to \( BC \), it is homothetic to \( CBAGFED \) (Fig. 5).

Similarly, since \( f^{2k}(N_9 - Z) + Z \) for \( k = 1, \ldots, 6 \) is \( L, A_9, Q, P, N', A' \), we get that \( N_9A'N'PQA_9L \) is a regular heptagon inscribed into \( a_9 \). Since \( A'N' \) is parallel to \( AB \), it is homothetic to \( CBAGFED \).

Of course, the triangles \( UVW \) and \( XYZ \) from the previous two theorems are closely related as the following result clearly shows.

Recall that triangles \( ABC \) and \( XYZ \) are orthologic provided the perpendiculars at vertices of \( ABC \) onto sides \( YZ, ZX, \) and \( XY \) of \( XYZ \) are concurrent. It is well-known that the relation of orthology for triangles is reflexive and symmetric.

**Theorem 4** Let \( U'V'W' \) and \( X'Y'Z' \) be reflections of \( UVW \) and \( XYZ \) at the line \( AD \). Let \( K \) be the intersection of the lines \( AF \) and \( DG \). Then any two among the triangles \( UVW, XYZ, U'V'W' \) and \( X'Y'Z' \) are orthologic. The triangles \( UVW \) and \( U'V'W' \) are images under the homotheties \( h(K, 2) \) and \( h(B, 2) \) of the triangles \( X'Y'Z' \) and \( XYZ \) (Fig. 6).

**Proof:** Since the points \( U', V', W', X', Y', Z' \) have the affixes

\[
\begin{align*}
\frac{f^{10} - f^8}{f^4 + f^2 + 1} , & \quad \frac{f^{10} + f^6 + f^4 + 1}{f^4 + f^2 + 1}, \\
\frac{f^6 + f^4}{f^4 + f^2 + 1}, & \quad \frac{2f^{10} - f^8 + f^6 + f^4}{2}, \\
\frac{f^{10} + 2f^6 + f^4 + 1}{2(f^4 + f^2 + 1)} , & \quad \frac{f^{10} + 8 + 2f^6 + 2f^4 + 1}{2(f^4 + f^2 + 1)},
\end{align*}
\]

respectively, it is easy to check using Theorem 5 in [2] that the triangles \( UVW \) and \( XYZ \) are orthologic. It is also easy to verify that \( X', Y', Z' \) are midpoints of the segments \( KU, KV, KW \) and that \( X, Y, Z \) are midpoints of the segments \( BU', BV', BW' \) which proves the claims about homotheties.

**Theorem 5** Let \( ABCDEFG \) be a regular heptagon inscribed to a circle of radius \( R \). Let the lines \( BE \) and \( BG \) intersect the line \( AD \) in points \( M \) and \( N \). If \( H \) and \( K \) are the de Longchamps points of the triangles \( BDM \) and \( ABN \), then \( |HK| = R\sqrt{11} \).

**Proof:** Since \( H = 2 f^{10} + f^8 + 2 f^6 \) and \( K = \frac{2f^{10} + 2f^8 + 6 + 2f^4}{f^4 + f^2 + 1} \), we get that \( |HK|^2 = 11 \) is

\[
\frac{-5p + p}{f^8 + 2f^6 + 3f^4 + 2f^2 + 1}.
\]

\( \square \)
Theorem 6 Let the line $BE$ in the regular heptagon $ABCDEFG$ intersect the line $AD$ in the point $M$. Let $H$, $I$, $J$, $K$ be centroids of the triangles $BDM$, $DEM$, $EGM$, $BGM$. Then $HIJK$ is a rhombus whose side is $\frac{2R}{3} \cos \frac{\pi}{14}$ and whose area is $\frac{2}{9}$ of the area of the quadrangle $BDEG$. The angles $\angle IHK$ and $\angle KJI$ are equal to $\frac{4\pi}{7}$ so that the regular heptagons build on $IH$, $HK$ and on $IJ$, $JK$ share one side (Fig. 7).

Proof: Since $M = f^{10} - f^8 + f^6$, we easily find that

$$H = \frac{2f^{10} - f^8 + 2f^6}{3}, \quad I = \frac{f^{12} + 2f^{10} - f^8 + f^6}{3}, \quad J = \frac{f^{12} + f^{10} - f^8 + f^6 + f^2}{3},$$

and $K = \frac{f^{10} - f^8 + 2f^6 + f^2}{3}$.

Now, we compute $|HI|^2 - |IJ|^2$, $|HI|^2 - |JK|^2$, and $|HI|^2 - |KH|^2$ to discover that they are all zero (this is immediate for the second difference while for the first and the third it follows from the fact that both contain $p_-$ as a factor). Hence, the quadrangle $HIJK$ is the rhombus. The claim about the side and the angles is a consequence of easily verified equalities $|IJ|^2 = \frac{4}{3} \cos^2 \frac{\pi}{14}$ and $|HI|^2 = \cos^2 \frac{2\pi}{7}$, where $P$ is the center of the rhombus. Finally, since the triangles $HIJ$, $BEG$ and $BDE$ have areas $\frac{1}{36}(f^{12} + 3f^{10} - 3f^4 - f^2)$, $\frac{1}{4}(2f^{10} + f^8 - f^6 - 2f^4)$, and $\frac{1}{4}(f^{12} + f^{10} - f^8 + f^6 - f^4 - f^2)$, respectively, it follows that $\frac{|HIJK|}{|BDEG|} = \frac{2}{9}$.

Remark (Adrian Oldknow): The statement about $\frac{|HIJK|}{|BDEG|} = \frac{2}{9}$ has actually nothing to do with heptagons — it is just a particular case of the very easily proved result that if $M$
is any internal point of any convex quadrangle $ABCD$, then the centroids of the triangles $AMB$, $BMC$, $CMD$ and $DMA$ form a parallelogram with sides $\frac{1}{3}|AC|$ and $\frac{1}{3}|BD|$ and area $\frac{2}{9}|ABCD|$.

**Theorem 7** Let $X$, $Y$, $Z$ be the centroids of the triangles $BDE$, $BEG$, $ABG$ in the regular heptagon $\Theta = ABCDEFG$. Let $S$ be the intersection of the lines joining $E$ and $G$ with the midpoints $H$ and $K$ of the segments $BD$ and $AB$. Let $\lambda = \sqrt{u/v}$ where $v = 9 - 18 \cos \frac{2\pi}{7}$ and $u = 2 \cos \frac{3\pi}{7} - 2 \cos \frac{2\pi}{7} + 4 \cos \frac{\pi}{7}$. Let $P$ and $Q$ be intersections of the lines $EY$ and $FZ$ with the lines $FX$ and $GY$.

1. If $h_1$ is the homothety $h(S, \lambda)$, then $XYZ = h_1(EFG)$ (Fig. 8).
2. If $h_2$ and $h_3$ are the homotheties $h(P, -\lambda)$ and $h(Q, -\lambda)$, then $\Theta_2 = h_2(\Theta)$ and $\Theta_3 = h_3(\Theta)$ are regular heptagons built on segments $XY$ and $YZ$.
3. If $U$, $M$ and $N$ are the centers of $\Theta_1$, $\Theta_2$ and $\Theta_3$ and $\Phi$ and $\Psi$ denote the regular heptagons built on the the segments $MN$ and $XZ$ and containing the vertex $B$, then the center of $\Phi$ is $U$ and $\Psi$ is obtained from $\Phi$ by the translation for the vector $UY$.
4. The point $S$ lies on the side of $\Psi$.

**Proof:** The centroids are

$$X = \frac{f^{12} + f^{10} + f^6}{3}, \quad Y = \frac{f^{12} + f^6 + f^2}{3}, \quad Z = \frac{f^6 + f^4 + f^2}{3}.$$
The lines $EX$, $FY$ and $GZ$ concur at the point $S = EH \cap GK$ with the affix 
$$\frac{2f^{12} + 2f^{10} + f^8 + f^6 + 1}{2f^8 + f^6 + f^4 + f^2 + 2}.$$ 
The conditions for the corresponding sidelines of the triangles $XYZ$ and $EFG$ to be parallel are satisfied because they are zero (as desired) or are zero because they contain the polynomial $p_-$ as a factor. The quotient $\lambda = \frac{|EF|}{|XY|}$ is equal to the square root of 
$$-\frac{2f^{12} + 2f^{10} + f^8 + f^6 + 1}{9(f^8 + f^4 + 1)}.$$ 
Let $M = \frac{f^{12} - f^8 - f^4}{3}$ be the intersection of the perpendicular bisector of $XY$ with the parallel through $X$ to the line $FO$. The point $M + f^2(X - M)$ lies on the line $GP$. Replacing $f^2$ with other even powers of $f$ we check that $\Theta_2 = h_2(\Theta)$ is the regular heptagon built on the segment $XY$. Let $U = \frac{f^{12} + f^{10} + f^8 + 2f^6 + f^4 + 1}{3}$ be the intersection of the perpendicular bisector of $XZ$ with the parallel through $M$ to the line $BO$. Then $U$ is the center of the regular heptagon built on $MN$ which contains the point $B$. We can now easily check that $MX$, $NZ$ and $YU$ are parallel segments of the same length. For the last claim, we translate points $U + f^6(M - U)$ and $U + f^8(M - U)$ for $U - Y$ and check that the point $S$ lies on the segment joining these translated points.}

**Theorem 8** Let $U$, $V$, $W$ be the orthocenters of the triangles $BDE$, $BEG$, $ABG$ in the regular heptagon $\Theta = ABCDEFG$. Let $T$ be the intersection of the perpendiculars at $E$ and $G$ to the lines $BD$ and $AB$. Let 

$$\mu = \frac{\sqrt{u}}{\sqrt{w}} \quad \text{with} \quad u = 2\cos\frac{3\pi}{7} - 2\cos\frac{2\pi}{7} + 4\cos\frac{\pi}{7} \quad \text{and} \quad w = 1 - 2\cos\frac{3\pi}{7}. $$

Let $P$ be the intersection of the perpendicular bisectors $p$ and $q$ of $UV$ and $VW$. Let $M$ and $N$ be intersections of $p$ and $q$ with perpendiculars to $UW$ at $U$ and $W$. Let $H$ and $K$ be the intersections of the lines $FU$ and $FW$ with the lines $GO$ and $EV$. 

*Figure 8: The triangle $XYZ$ on centroids of triangles $BDE$, $BEG$, $ABG$ is homothetic with the triangle $EFG$*
Figure 9: The triangle $UVW$ on orthocenters of triangles $BDE$, $BEG$, $ABG$ is homothetic with the triangle $EFG$

(1) If $h_4$ is the homothety $h(T, \mu)$, then $UVW = h_4(EFG)$ (Fig. 9).

(2) If $h_5$ and $h_6$ are the homotheties $h(H, -\mu)$ and $h(K, -\mu)$, then $\Theta_5 = h_5(\Theta)$ and $\Theta_6 = h_6(\Theta)$ are regular heptagons built on segments $UV$ and $VW$ with centers $M$ and $N$.

(3) If $\Phi$ denotes the regular heptagon built on the segment $MN$ containing the vertex $B$ and $\Psi = h_4(\Theta)$, then $P$ is a common center of $\Phi$ and $\Psi$.

**Proof:** The orthocenters are $U = f^{12} + f^{10} + f^6$, $V = f^{12} + f^6 + f^2$, $W = f^6 + f^4 + f^2$. The claims of the theorem are now easily verified in the same way as in the proof of the previous theorem.

The triangles $XYZ$ and $UVW$ from the previous two theorems are themselves homothetic as the following result shows.

**Theorem 9** The line $ST$ goes through the center $O$ and the triangle $UVW$ on orthocenters is the image under the homothety $h(O, 3)$ of the triangle $XYZ$ on centroids (Fig. 10).

**Proof:** Since

\[ S = \frac{2f^{12} + f^{10} + f^6 + 2f^2 + 1}{1 - f^2 - f^6 - f^8 - f^{12}} \quad \text{and} \quad T = \frac{2f^{12} + 2f^{10} + f^8 + f^6 + 1}{2f^8 + f^6 + f^4 + f^2 + 2} \]

the free term of the equation of the line $ST$ is zero so that the center $O$ (the origin) lies on $ST$. The second claim about the triangles $UVW$ and $XYZ$ is clearly true because the complex
Figure 10: The triangles $UVW$ on orthocenters and $XYZ$ on centroids of triangles $BDE$, $BEG$, $ABG$ are homothetic.

Figure 11: The triangles $UVW$ on the de Longchamps points and $XYZ$ on the centers of the nine-point circles of triangles $BDE$, $BEG$, $ABG$ are related in homothety $h(O, -2)$. 
coordinates of the vertices of the second triangle are one third of the complex coordinates of the vertices of the first triangle.

There are analogous results for the centers of the nine-point circles and for the de Longchamps points of the triangles $BDE$, $BEG$, $ABG$. We shall not give precise formulations of these theorems leaving this task as an exercise to the reader. In Fig. 11 the isosceles triangles on the de Longchamps points and on the centers of the nine-point circles are shown together.

References


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