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Generalized Arbelos in Aliquot Parts: Intersecting case

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Abstract. We will generalize the argument of arbelos in [3] to the case where circles are intersecting. Almost all results are naturally generalized in this case. Also we construct a new family of Archimedean circles as a Corollary.

Keywords: Arbelos, geometry of circles, Archimedean circles

MSC 2007: 51M04, 51M15

1. Preliminaries

Let Γ be a coaxial system of intersecting type and E be a point on the line consisting of centers of circles in Γ. We call the pair (Γ, E) a coaxial system with a fixed point. Throughout this paper we take the line of centers as the x-axis and the line passing through two intersection points of members in Γ as the y-axis whenever we consider the coaxial system Γ. Let L, L' be those intersection points, ε be the circle with center E passing through L and L', e be the x-coordinate of E and f be the radius of ε. Then ε meets the x-axis at two points which we denote by E^+ and E^-. We assume that the x-coordinate of E^+ is larger than that of E^- and the y-coordinate of L is larger than that of L' by renaming if necessary.

Let α be a member of Γ except ε. Then α meets the x-axis at a point between E^+ and E^-, which we denote by A. If $\alpha = \varepsilon$, we regard that $A = E^-$. Now for $\alpha \in \Gamma$ we define $\mu_*(\alpha)$ (with respect to E) as

$$\mu_*(\alpha) = \frac{|AE^-|}{|AE^+|} = \frac{a-e+f}{e+f-a}\ ,$$

where a is the x-coordinate of A. We know immediately that $\mu_*(\varepsilon) = 0$, $\mu_*(\alpha) = \mu_*(\beta)$ if and only if $\alpha = \beta$, $\mu_*(\alpha)$ is non negative and can take any non negative value for some $\alpha \in \Gamma$ since $|AE^-|/|AE^+|$ take any non negative value when A moves between E^- and E^+ (including the case $A = E^-$).

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If α is a circle, it meets the x-axis at a point A' different from A. We define $\mu^*(\alpha)$ as

$$\mu^*(\alpha) = \frac{|A'E^+|}{|A'E^-|}.$$

If α is the unique line in Γ we take $\mu^*(\alpha) = 1$.

Lemma 1 *For any $\alpha \in \Gamma$ we have*

$$\mu_*(\alpha) = \mu_*(\lambda)\mu^*(\alpha),$$

where λ denotes the line in Γ.

Proof: It is trivial if α is the line. Assume α is a circle and a' is the x-coordinate of A'. Then

$$\mu^*(\alpha) = \frac{a' - e - f}{a' - e + f}.$$

Since $aa' + f^2 - e^2 = 0$ we have

$$(e + f - a)a' = (e + f)(a' - e + f) \ ,$$
$$(a - e + f)a' = (-e + f)(a' - e - f) \ .$$

These imply the conclusion. □

2. Incircles

Let (Γ, E) be a coaxial system with a fixed point, and ε, E^+, E^-, e, f, L be as in the previous section. Let γ be a circle with center E and radius g such that $g > f$. A circle $\alpha \in \Gamma$ is tangent to γ if and only if $a' = e + g$ or $a' = e - g$, and this is equivalent to $\mu^*(\alpha) = \frac{g - f}{g + f}$ or $\mu^*(\alpha) = \frac{g + f}{g - f}$. Circle α intersects γ at two points if and only if

$$\frac{g - f}{g + f} < \mu^*(\alpha) < \frac{g + f}{g - f}.$$

Let α, β be members of Γ with $0 < \mu^*(\alpha) < \mu^*(\beta)$ (this is equivalent to $0 < \mu_*(\alpha) < \mu_*(\beta)$). We assume that $\mu^*(\alpha) \le \frac{g + f}{g - f}$ and $\mu^*(\beta) \ge \frac{g - f}{g + f}$. This is the condition that there exist one or two circles tangent to α, β and γ, in the range between ε and γ. If there are two they are symmetric with respect to the x-axis and congruent each other.

We call such circles incircles of α and β in γ. Fig. 1a shows the case that $\mu^*(\alpha) < \mu^*(\beta) < \frac{g - f}{g + f}$. Fig. 1b shows the case that $\mu^*(\alpha) < \frac{g - f}{g + f}$ and $\frac{g - f}{g + f} \le \mu^*(\beta) \le \frac{g + f}{g - f}$, so the incircles exist.

Theorem 1 *Let $\mathcal{C}$ be an incircle of α and β in γ. If $\mu_*(\alpha) < \mu_*(\beta)$, then the radius of $\mathcal{C}$ is*

$$\frac{(g^2 - f^2)(\mu_*(\beta) - \mu_*(\alpha))}{2((g + f)\mu_*(\beta) - (g - f)\mu_*(\alpha))}, \text{ which equals } \frac{(g^2 - f^2)(\mu^*(\beta) - \mu^*(\alpha))}{2((g + f)\mu^*(\beta) - (g - f)\mu^*(\alpha))}.$$

Proof: Let A, A' (resp. B, B') be the intersection points of α (resp. β) and the x-axis such that A (resp. B) is between E^+ and E^-, and a, a' (resp. b, b') be the x-coordinates of A, A' (resp. B, B'), respectively. Note that $a < b$ by the assumption. If α (resp. β) is the line we regard that $a' = \infty$ (resp. $b' = \infty$). Let ℓ be the y-coordinate of L, so we have $f^2 = e^2 + \ell^2$.

First we invert α, β, γ and $\mathcal{C}$ by the circle with center $(b, 0)$ and radius $\sqrt{b^2 + \ell^2}$. Then α and γ are inverted to the circles $\overline{\alpha}$ and $\overline{\gamma}$ with centers on the x-axis and intersecting with the x-axis at

$$m = \frac{(b-a)\ell^2}{ab + \ell^2}, \quad n = \frac{ab + \ell^2}{a - b} \tag{1}$$

and

$$p = \frac{b(e+g) + \ell^2}{e + g - b}, \quad q = \frac{b(e-g) + \ell^2}{e - g - b} \tag{2}$$

respectively. β is inverted to the y-axis. Note that $q < b < p$.

Let $\overline{\mathcal{C}}$ be the inverted image of $\mathcal{C}$ and (x, y) be the center of $\overline{\mathcal{C}}$. Then $\overline{\mathcal{C}}$ touches $\overline{\alpha}$ internally, $\overline{\gamma}$ externally and the y-axis from the right (see Fig. 2). So the radius of $\overline{C}$ is x and we have

$$\left(x - \frac{m+n}{2}\right)^2 + y^2 = \left(\frac{m-n}{2} - x\right)^2, \quad \left(x - \frac{p+q}{2}\right)^2 + y^2 = \left(\frac{p-q}{2} + x\right)^2,$$

and then

$$x = \frac{mn - pq}{2(n-p)}, \tag{3}$$

$$b^2 + y^2 - 2bx = \frac{(b^2 - mn)(n-p) + (mn - pq)(n-b)}{n - p}. \tag{4}$$

Since the incircle $\mathcal{C}$ is the inverted image of $\overline{\mathcal{C}}$, then the radius of $\mathcal{C}$ is

$$\frac{b^2 + \ell^2}{|(x-b)^2 + y^2 - x^2|} \cdot x .$$

$\sqrt{(x-b)^2 + y^2}$ is the distance between the center of $\overline{\mathcal{C}}$ and $(b, 0)$, and is larger than x since the point $(b, 0)$ is outside $\overline{\mathcal{C}}$ by the fact $q < b < p$. Then we have $(x-b)^2 + y^2 - x^2 > 0$ and $|(x-b)^2 + y^2 - x^2| = b^2 + y^2 - 2bx$. By using (1), (2), (3), (4), we know the radius $\mathcal{C}$ is

$$\frac{(\ell^2 + b^2)x}{b^2 + y^2 - 2bx} = \frac{(b-a)(f^2 - g^2)}{2((e-a)(e-b) - (b-a)g - f^2)}.$$

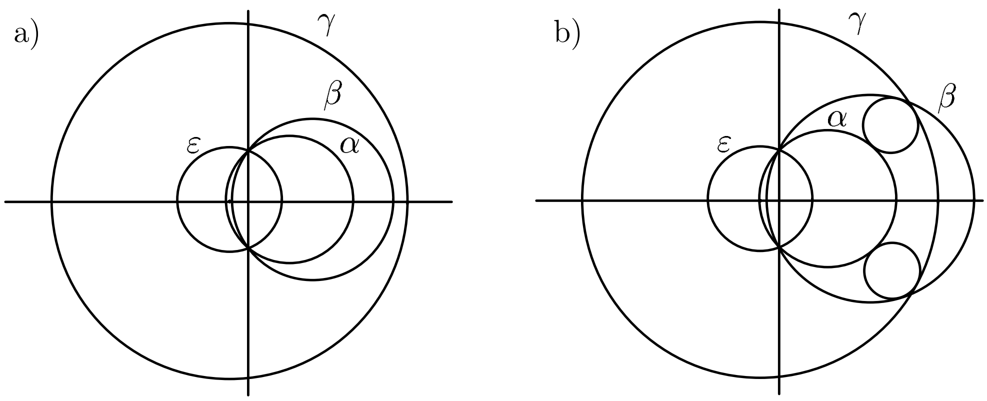

Figure 1: Examples of (a) non-existence and (b) existence of incircles of α and β in γ

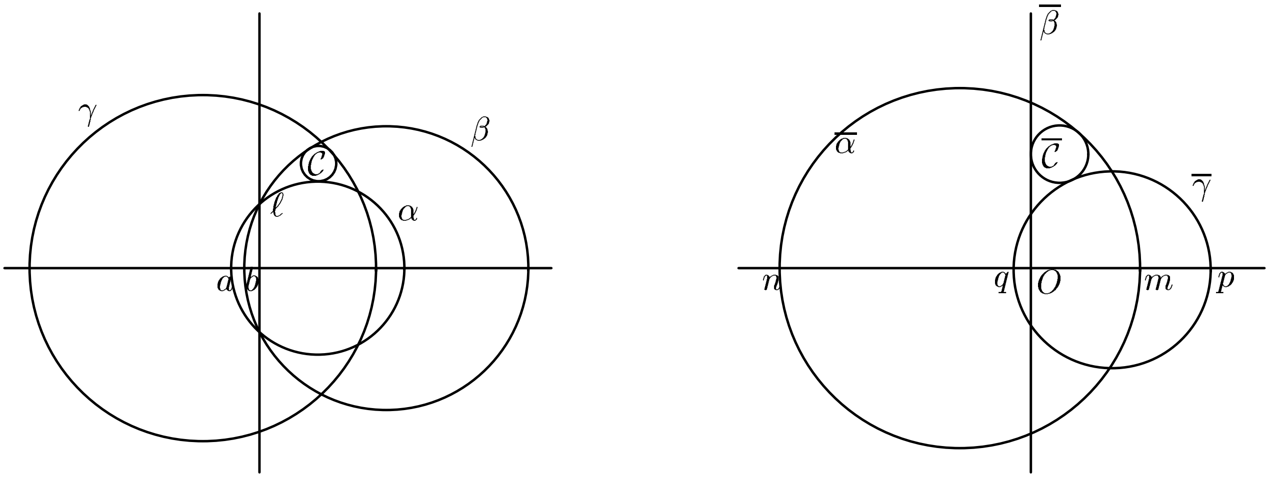

Figure 2: A configuration of α, β, γ and its inversion

On the other hand, we have

$$\frac{(g^2-f^2)(\mu_*(\beta)-\mu_*(\alpha))}{2((g+f)\mu_*(\beta)-(g-f)\mu_*(\alpha))} = \frac{(g^2-f^2)\left(\dfrac{b-e+f}{e+f-b}-\dfrac{a-e+f}{e+f-a}\right)}{2\left((g+f)\dfrac{b-e+f}{e+f-b}-(g-f)\dfrac{a-e+f}{e+f-a}\right)} = \frac{(b-a)(f^2-g^2)}{2((e-a)(e-b)-(b-a)g-f^2)}.$$

Then we have the first result. The second result follows from Lemma 1. □

Corollary 1 *Let α_1, α_2, β_1, β_2 be members of Γ such that $0<\mu_*(\alpha_1)<\mu_*(\beta_1)$ and $0<\mu_*(\alpha_2)<\mu_*(\beta_2)$. Assume there exist incircles of α_i and β_i in γ for $i=1,2$. Then the incircle of α_1 and β_1 in γ are congruent to the incircle of α_2 and β_2 in γ if and only if $\dfrac{\mu_*(\beta_1)}{\mu_*(\alpha_1)}=\dfrac{\mu_*(\beta_2)}{\mu_*(\alpha_2)}$. This is also equivalent to $\dfrac{\mu^*(\beta_1)}{\mu^*(\alpha_1)}=\dfrac{\mu^*(\beta_2)}{\mu^*(\alpha_2)}$.*

Proof: The result follows from Theorem 1 and the fact that the map

$$x \mapsto \frac{(g^2-f^2)(x-1)}{2((g+f)x-(g-f))}$$

is injective. □

Corollary 2 *Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be members of Γ such that*

$$0<\mu^*(\alpha_0)<\mu^*(\alpha_1)<\cdots<\mu^*(\alpha_n), \quad \mu^*(\alpha_1)\geq\frac{g-f}{g+f} \text{ and } \mu^*(\alpha_{n-1})\leq\frac{g+f}{g-f}.$$

Then the incircles of α_{i-1} and α_i in γ $(i=1,2,\ldots,n)$ are all congruent if and only if $\dfrac{\mu^*(\alpha_i)}{\mu^*(\alpha_{i-1})}$ are constant for $i=1,2,\ldots,n$.

3. Generalized arbelos in aliquot parts

Let α, β, γ be three circles such that α and β intersect at two points L and L', γ touches both of α and β with α and β being inside γ, and centers of three circles are collinear. If we see these figures only in a half plane divided by the line passing through three centers, this is a kind of generalization of Arbelos, so, we call this configuration of three circles a generalized arbelos of intersecting type. In this paper we call just a generalized arbelos in short.

Let (Γ, E) be a coaxial system with a fixed point, where Γ is generated by α and β, and E is the center of γ. Note that the value $\mu_*(\alpha')$ and $\mu^*(\alpha')$ for $\alpha' \in \Gamma$ are independent from the radius of γ. Assume $\mu^*(\alpha) < \mu^*(\beta)$ by renaming if necessary. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be members in Γ such that $\alpha_0 = \alpha$, $\alpha_n = \beta$ and $\mu^*(\alpha_0) < \mu^*(\alpha_1) < \cdots < \mu^*(\alpha_n)$ (see Fig. 3).

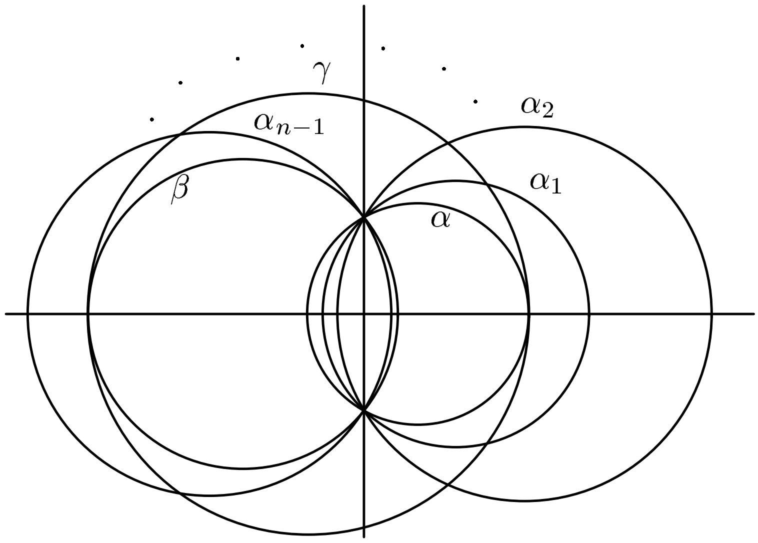

Figure 3: Members in Γ with $\mu^*(\alpha_0) < \mu^*(\alpha_1) < \cdots < \mu^*(\alpha_n)$

We call the configuration of $\{\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_n = \beta, \gamma\}$ a *generalized arbelos in n-aliquot parts*, if all incircles of α_{i-1} and α_i in γ $(i = 1, 2, \ldots, n)$ are congruent. We also call congruent incircles *Archimedean circles in n-aliquot parts*.

Theorem 2 *$\{\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta, \gamma\}$ with $\mu^*(\alpha_0) < \mu^*(\alpha_1) < \cdots < \mu^*(\alpha_n)$ is a generalized arbelos in n-aliquot parts if and only if the sequence*

$$\mu^*(\alpha) = \mu^*(\alpha_0),\ \mu^*(\alpha_1),\ \ldots,\ \mu^*(\alpha_n) = \mu^*(\beta)$$

is a geometric sequence with common ratio $\left(\dfrac{g+f}{g-f}\right)^{2/n}$. And the radius of the Archimedean circle in n-aliquot parts is

$$\frac{(g^2 - f^2)\left((g+f)^{\frac{2}{n}} - (g-f)^{\frac{2}{n}}\right)}{2\left((g+f)^{\frac{2}{n}+1} - (g-f)^{\frac{2}{n}+1}\right)},$$

where g is the radius of γ and f is the distance between L and E.

Proof: We know that $\{\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta, \gamma\}$ is a generalized arbelos in n-aliquot parts if and only if $\mu^*(\alpha_0), \mu^*(\alpha_1), \ldots, \mu^*(\alpha_n)$ is a geometric sequence by Corollary 2. Since

$\mu^*(\alpha_0) = \mu^*(\alpha) = \dfrac{g-f}{g+f}$ and $\mu^*(\alpha_n) = \mu^*(\beta) = \dfrac{g+f}{g-f}$, the common ratio is $\left(\dfrac{g+f}{g-f}\right)^{1/n}$. Then by Theorem 1 the radius of the Archimedean circle in n-aliquot parts is

$$\frac{(g^2-f^2)(\mu^*(\alpha_i)-\mu^*(\alpha_{i-1}))}{2((g+f)\mu^*(\alpha_i)-(g-f)\mu^*(\alpha_{i-1}))} = \frac{(g^2-f^2)\left(\left(\frac{g+f}{g-f}\right)^{2/n}-1\right)}{2\left((g+f)\left(\frac{g+f}{g-f}\right)^{2/n}-(g-f)\right)}$$
$$= \frac{(g^2-f^2)\left((g+f)^{\frac{2}{n}}-(g-f)^{\frac{2}{n}}\right)}{2\left((g+f)^{\frac{2}{n}+1}-(g-f)^{\frac{2}{n}+1}\right)}. \qquad \square$$

Corollary 3 *If n is a even positive integer, then $\alpha_{n/2}$ is the line passing through L and L'.*

Proof: Since $\mu^*(\alpha_{n/2}) = \dfrac{g-f}{g+f}\cdot\left(\left(\dfrac{g+f}{g-f}\right)^{2/n}\right)^{n/2} = 1$, $\alpha_{n/2}$ is the line in Γ. $\square$

Corollary 4 *If n, m, l are positive integers with $n = ml$ and $\{\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta\}$ is a generalized arbelos in n-aliquot parts with $\mu^*(\alpha_0) < \mu^*(\alpha_1) < \cdots < \mu^*(\alpha_n)$. Then $\{\alpha = \alpha_0, \alpha_l, \alpha_{2l}, \ldots, \alpha_{ml} = \beta, \gamma\}$ is a generalized arbelos in m-aliquot parts.*

Remarks: 1) By the definition $\mu^*(\alpha_i)$ is independent from the radius of γ, so, if $\{\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta\}$ is a generalized arbelos in n-aliquot parts, incircles of α_{i-1} and α_i in γ' are all congruent for any circle γ' which meets all α_i and is concentric to γ.
2) If we write $g+f = 2u$ and $g-f = 2v$, then the radius of the Archimedean circle in n-aliquot parts is written as

$$\frac{uv\left(u^{\frac{2}{n}}-v^{\frac{2}{n}}\right)}{u^{\frac{2}{n}+1}-v^{\frac{2}{n}+1}}.$$

When we consider the ordinary arbelos, $\{u,\ v\}$ are the radii of $\{\alpha,\ \beta\}$. So the above Theorem is a generalization of the result in [3].

Corollary 5 *In the above notations, α_i intersects the x-axis between E^+ and E^- at the points*

$$\left(\frac{(e+u-v)(e-u+v)\left(u^{\frac{2i-n}{n}}-v^{\frac{2i-n}{n}}\right)}{(e-u+v)u^{\frac{2i-n}{n}}-(e+u-v)v^{\frac{2i-n}{n}}},\ 0\right).$$

Proof: If a_i is the x-coordinate of the intersection point of α_i and the x-axis between E^+ and E^-, then $\mu_*(\alpha_i) = \dfrac{a_i-e+f}{e+f-a_i}$ by definition. On the other hand, we have

$$\mu_*(\alpha_i) = \mu_*(\lambda)\mu^*(\alpha_i) = \mu_*(\lambda)\mu^*(\alpha_0)\left(\frac{u}{v}\right)^{\frac{2i}{n}} = \frac{-e+f}{e+f}\cdot\frac{v}{u}\cdot\left(\frac{u}{v}\right)^{\frac{2i}{n}} = \frac{-e+f}{e+f}\cdot\left(\frac{u}{v}\right)^{\frac{2i-n}{n}}.$$

These imply the conclusion. $\square$

4. Embedded patterns of arbelos

Let n be a positive integer and $\{\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta, \gamma\}$ be a generalized arbelos in n-aliquot parts with $\mu^*(\alpha) = \mu^*(\alpha_0) < \mu^*(\alpha_1) < \cdots < \mu^*(\alpha_n) = \mu^*(\beta)$. Then there exists a circle γ' which is tangent to all Archimedean circles externally. It is clearly concentric to γ. (If $n = 1$ we take the circle concentric to γ and tangent to the incircle of α and β in γ.) We use the letters L, L', E, E^+, E^-, ε and (Γ, E) in the same meaning as in the previous sections. Let g , g' be the radii of γ , γ' respectively and f be the distance between E and L. Then,

Lemma 2
$$\left(\frac{g'+f}{g'-f}\right)^n = \left(\frac{g+f}{g-f}\right)^{n+2}.$$

Proof: Let r be the radius of Archimedean circle in n-aliquot parts and d be the common ratio of the sequence $\mu_*(\alpha) = \mu_*(\alpha_0)$, $\mu_*(\alpha_1)$, $\ldots$, $\mu_*(\alpha_n) = \mu_*(\beta)$. Then

$$2r = \frac{(g^2-f^2)(d-1)}{(g+f)d-(g-f)}$$

by Theorem 1, and we have

$$g'-f = g-2r-f = \frac{2(g-f)f}{(g+f)d-(g-f)} ,$$
$$g'+f = g-2r+f = \frac{2(g+f)fd}{(g+f)d-(g-f)} .$$

Since $d = \left(\frac{g+f}{g-f}\right)^{\frac{2}{n}}$ by Theorem 2,

$$\left(\frac{g'+f}{g'-f}\right)^n = \left(\frac{d(g+f)}{g-f}\right)^n = \left(\frac{g-f}{g+f}\right)^{n+2}. \qquad \square$$

γ' meets the x-axis at two points one of which is inside α and the other is inside β. We denote the member in Γ passing through the former point by α' and the one passing through the latter point by β'.

Theorem 3 $\{\alpha', \alpha_0, \alpha_1, \ldots, \alpha_n, \beta', \gamma'\}$ *is a generalized arbelos in* $(n+2)$*-aliquot parts.*

Proof: By Lemma 2, the sequence

$$\mu^*(\alpha_0),\ \mu^*(\alpha_1),\ \ldots,\ \mu^*(\alpha_n)$$

is a geometric sequence of ratio $\left(\frac{g+f}{g-f}\right)^{\frac{2}{n}} = \left(\frac{g'+f}{g'-f}\right)^{\frac{2}{n+2}}$,

$$\frac{\mu^*(\alpha_0)}{\mu^*(\alpha')} = \frac{\left(\frac{g-f}{g+f}\right)}{\left(\frac{g'-f}{g'+f}\right)} = \left(\frac{g'+f}{g'-f}\right)^{\frac{2}{n+2}},$$

and

$$\frac{\mu^*(\beta')}{\mu^*(\alpha_n)} = \frac{\left(\dfrac{g'+f}{g'-f}\right)}{\left(\dfrac{g+f}{g-f}\right)} = \left(\frac{g'+f}{g'-f}\right)^{\frac{2}{n+2}}.$$

These imply the conclusion. □

Now according to [3] we construct two types of embedded patterns of generalized arbelos inductively starting from a fixed generalized arbelos $\{\alpha, \beta, \gamma\}$. As in the previous sections, let g be the radius of γ, f be the distance between the center of γ and the intersection point of α and β, $g+f=2u$ and $g-f=2v$.

First we consider the odd case. Let $\alpha_{-1}=\alpha$, $\alpha_1=\beta$ and $\gamma_1=\gamma$. If there exists a generalized arbelos in $(2n-1)$-aliquot parts $\{\alpha_{-n}, \alpha_{-(n-1)}, \dots, \alpha_{-1}, \alpha_1, \dots, \alpha_n, \gamma_{2n-1}\}$, we construct the generalized arbelos in $(2n+1)$-aliquot parts as follows. Let γ_{2n+1} be the circle tangent externally to all Archimedean circles of the above arbelos. If $n=1$ we take the circle concentric to γ. If $n>1$ γ_{2n+1} is automatically concentric to γ. γ_{2n+1} meets the x-axis at two points one of which is inside α and the other is inside β. We denote the member in Γ passing through the former point by $\alpha_{-(n+1)}$ and the one passing through the latter point by α_{n+1}.

By Theorem 3 $\{\alpha_{-(n+1)}, \alpha_{-n}, \dots, \alpha_{-1}, \alpha_1, \dots, \alpha_{n+1}, \gamma_{2n+1}\}$ is a generalized arbelos in $(2n+1)$-aliquot parts. We call

$$\{\dots, \alpha_{-n}, \alpha_{-(n-1)}, \dots, \alpha_{-1}, \alpha_1, \dots, \alpha_n, \dots, \gamma_1, \gamma_3, \dots, \gamma_{2n-1}, \dots\}$$

the *embedded pattern* of generalized arbelos of odd type (see Fig. 4).

Let δ_{2n-1} be one of the Archimedean circles in $\{\alpha_{-n}, \alpha_{-(n-1)}, \dots, \alpha_{-1}, \alpha_1, \dots, \alpha_n, , \gamma_{2n-1}\}$.

Theorem 4 *The radius of* γ_{2n-1} *is*

$$\frac{(u^{2n-1}+v^{2n-1})(u-v)}{u^{2n-1}-v^{2n-1}}, \text{ that of } \delta_{2n-1} \text{ is } \frac{u^{2n-1}v^{2n-1}(u-v)^2(u+v)}{(u^{2n-1}-v^{2n-1})(u^{2n+1}-v^{2n+1})}.$$

Proof: Let g_{2n-1} be the radius of γ_{2n-1} and d_{2n-1} be the radius of δ_{2n-1}. By Lemma 2 we have

$$\left(\frac{g_{2n-1}+f}{g_{2n-1}-f}\right)^{\frac{1}{2n-1}} = \left(\frac{g_{2n-3}+f}{g_{2n-3}-f}\right)^{\frac{1}{2n-3}} = \cdots = \frac{g_1+f}{g_1-f} = \frac{u}{v}.$$

Then

$$\frac{g_{2n-1}+f}{g_{2n-1}-f} = \left(\frac{u}{v}\right)^{2n-1}$$

and

$$g_{2n-1} = \frac{(u^{2n-1}+v^{2n-1})(u-v)}{u^{2n-1}-v^{2n-1}} \qquad (5)$$

$$d_{2n-1} = \frac{1}{2}(g_{2n-1}-g_{2n+1}) = \frac{u^{2n-1}v^{2n-1}(u-v)(u^2-v^2)}{(u^{2n-1}-v^{2n-1})(u^{2n+1}-v^{2n+1})}.$$

□

As in the odd case, we can construct the embedded pattern of generalized arbelos of even type $\{\dots, \beta_{-n}, \beta_{-(n-1)}, \dots, \beta_{-1}, \beta_0, \beta_1, \dots, \beta_n, \dots, \gamma_2, \gamma_4, \dots, \gamma_{2n}, \dots\}$ inductively by starting from a generalized arbelos of 2-aliquot parts $\{\beta_{-1}, \beta_0, \beta_1, \gamma_2\}$, where $\beta_{-1} = \alpha$, β_0 is the line passing through the intersection points of α and β, $\beta_1 = \beta$ and $\gamma_2 = \gamma$.

Let δ_{2n} be one of the Archimedean circles in $\{\beta_{-n}, \beta_{-(n-1)}, \dots, \beta_{-1}, \beta_0, \beta_1, \dots, \beta_n, \gamma_{2n}\}$. In the same way as in the proof of Theorem 4 we have

Theorem 5 *The radius of γ_{2n} is*

$$\frac{(u^n + v^n)(u - v)}{u^n - v^n}, \text{ that of } \delta_{2n} \text{ is } \frac{u^n v^n (u - v)^2}{(u^n - v^n)(u^{n+1} - v^{n+1})}.$$

Corollary 6 $\gamma_{2(2n-1)} = \gamma_{2n-1}$, $\alpha_{-n} = \beta_{-(2n-1)}$, $\alpha_n = \beta_{2n-1}$.

Fig. 4 shows the odd pattern together with the even pattern reflected in the x-axis.

5. A new family of Archimedean circles

In this section, we consider an ordinary arbelos α, β, γ. Following [1] and [2], we call a circle congruent to the Archimedean twin circles an Archimedean circle. Assuming that α and β are not congruent we construct a family of Archimedean circles.

In the assumption that α and β are not congruent, there exists a circle concentric to γ and passing through the tangent point of α and β. We denote this circle by ε. Let L be a

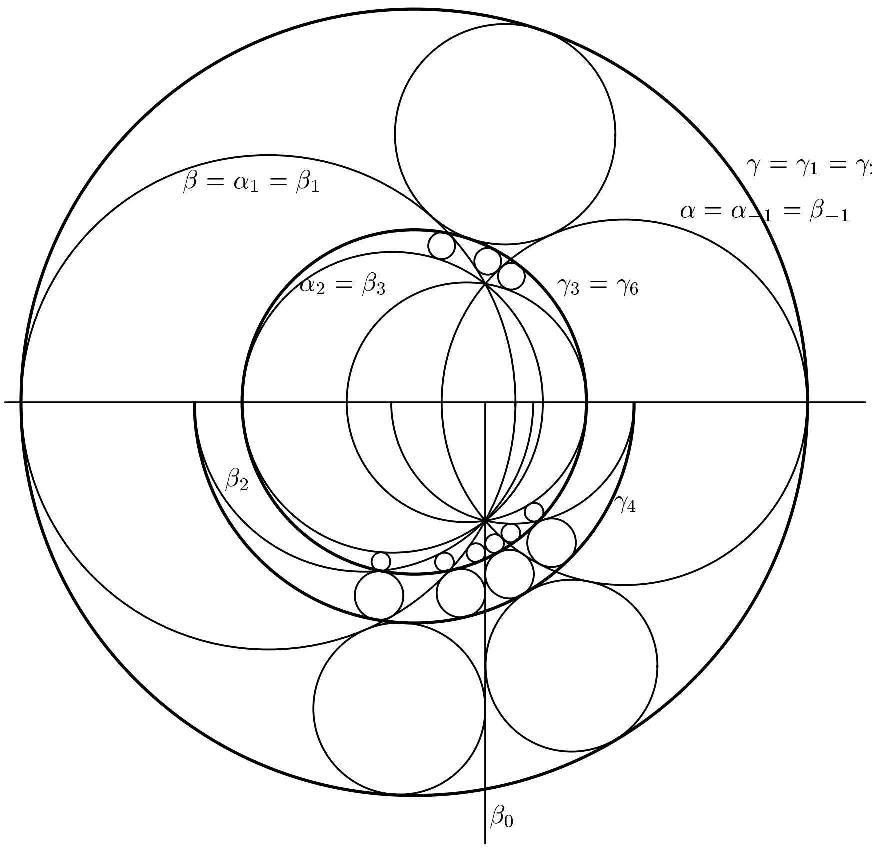

Figure 4: Embedded patterns of odd and even type reflected in the x-axis

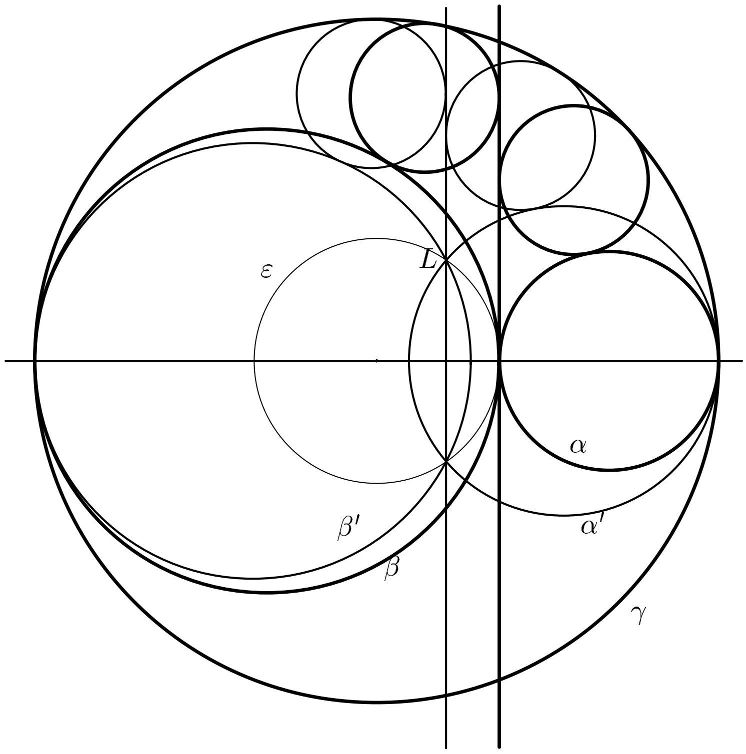

Figure 5: A infinite family of Archimedean circles

point moving freely on the circle ε and L' be the reflecting point of L with respect to the line through the centers of α, β and γ. If α' and β' are two circles tangent to γ and intersecting each other at L and L', three circles α', β', γ make a generalized arbelos of intersecting type. By Theorem 2, the radius of the Archimedean circle in this generalized arbelos in two aliquot parts is determined by two numbers, that is, the radius of γ and the distance between the center of γ and L. But these two numbers are independent from the choice of L on ε. This means that the Archimedean circles in the generalized arbelos in two aliquot parts are always congruent wherever L is on ε. If L is the tangent point of α and β, these are ordinary Archimedean circles.

So, any Archimedean circle in the generalized arbelos in two aliquot parts determined by L on ε is always an Archimedean circle in the first situation $\{\alpha,\ \beta,\ \gamma\}$ (see Fig. 5), and we get infinite many Archimedean circles according as L moves on the circle ε.

These argument holds for any arbelos in n-aliquot part. So, we can construct a family of Archimedean circles in n-aliquot parts similarly.

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