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Computation with Pentagons

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Abstract. The paper deals with properties of pentagons in a plane which are related to the area of a pentagon. First the formulas of Gauss and Monge holding for any pentagon in a plane are studied. Both formulas are derived by the theory of automated theorem proving. In the next part the area of cyclic pentagons is investigated. On the base of the Nagy–Rédey theorem and other results, the formula for the area of a cyclic pentagon which is given by its side lengths is rediscovered. This is the analogue of well-known Heron and Brahmagupta formulas for triangles and cyclic quadrilaterals. The method presented here could serve as a tool for solving this problem for cyclic n-gons for a higher n.

Key Words: Area of a cyclic pentagon, Monge formula, Gauss formula, Groebner bases of ideals

MSC 2000: 51M25, 51N20, 52A38

1. Introduction

In the paper we will study some properties of pentagons in a plane which are related to the their areas. These properties of pentagons are generalizations of well-known relations holding for triangles and quadrilaterals. First we will give some notions from geometry of polygons that we need in our investigation.

Suppose that $A_1A_2\dots A_n$ is a polygon with vertices $A_1, A_2, \dots, A_n$ and sides A_1A_2, A_2A_3, $\dots, A_nA_1$. All indices are considered mod n, i.e., $A_{j+n} = A_j$ for all $j = 1, 2, \dots, n$. Computation of the area of a polygon may be carried out in two basic ways.

The first way consists of computing the area of a polygon once knowing the coordinates of its vertices in a given system of coordinates. Then the area of a polygon can be computed by the following theorem [3]

Let $A_i = [x_i, y_i]$, $i = 1, 2, \dots, n$, be coordinates of the vertices of an n-gon $A_1A_2\dots A_n$ in a given Cartesian system of coordinates. Then for the area p of an n-gon $A_1A_2\dots A_n$

$$p = \frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix}. \tag{1}$$

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By (1) we compute the area of an n-gon as the sum of (signed) areas of individual triangles. We can easily check that formula (1) does not depend on the choice of the system of coordinates.

The second way of computing the area of a polygon is based on distances between the vertices of a polygon. The area p of an n-gon $A_1A_2\dots A_n$ can be expressed in terms of all $\binom{n}{2}$ mutual distances between its vertices. We will use the formula (2) which was published by B.Sz. Nagy and L. Rédey [9]. The Nagy–Rédey theorem reads:

Let $d_{ij}=|A_iA_j|^2$ denote a square of the distance of the vertices A_i, A_j. Then the area p of an n-gon $A_1A_2\dots A_n$ is given by

$$16p^2=\sum_{i,j=1}^{n}\begin{vmatrix} d_{i,j} & d_{i,j+1}\\ d_{i+1,j} & d_{i+1,j+1}\end{vmatrix}. \tag{2}$$

A special case of (2) for $n=4$ is known as the formula of Staudt [16]

In a quadrilateral $A_1A_2A_3A_4$ with side lengths $a=|A_1A_2|$, $b=|A_2A_3|$, $c=|A_3A_4|$, $d=|A_4A_1|$ and diagonals $e=|A_1A_3|$, $f=|A_2A_4|$

$$16p^2=4e^2f^2-(a^2-b^2+c^2-d^2)^2 \tag{3}$$

holds.

To prove and discover all formulas in this paper we will use the theory of automated theorem proving based on Gröbner bases computation in ideals. See [5, 11, 13] for details.

2. Gauss and Monge formulas

The following formula belongs to one of less known results of C.F. Gauss (1777–1855) [7, 19]. In accordance with [17] we will call it *Gauss formula*:

Let $\mathcal{P}=A_1A_2A_3A_4A_5$ be an arbitrary plane pentagon and let p_i denotes the area of a vertex triangle $A_{i-1}A_iA_{i+1}$, $i=1,\dots,5$. Then for the area f of a pentagon $\mathcal{P}$ the following relation holds

$$f^2-c_1f+c_2=0, \tag{4}$$

where $c_1=\sum_{i=1}^5 p_i$ and $c_2=\sum_{i=1}^5 p_ip_{i+1}$.

Let us show how to discover and prove the formula (4) by computer. Choose a Cartesian coordinate system such that $A_1=[0,0]$, $A_2=[a,0]$, $A_3=[x,y]$, $A_4=[u,v]$, $A_5=[w,z]$ (Fig. 1).

For the areas p_i, $i=1,2,\dots,5$, of the vertex triangles and the area f of the pentagon $\mathcal{P}$ we get by Eq. (1)

$$\begin{aligned}
p_1 &= \text{area of } A_5A_1A_2 & &\Leftrightarrow\ h_1\colon\ az-2p_1=0,\\
p_2 &= \text{area of } A_1A_2A_3 & &\Leftrightarrow\ h_2\colon\ ay-2p_2=0,\\
p_3 &= \text{area of } A_2A_3A_4 & &\Leftrightarrow\ h_3\colon\ ay+xv-yu-av-2p_3=0,\\
p_4 &= \text{area of } A_3A_4A_5 & &\Leftrightarrow\ h_4\colon\ xv-yu+uz-vw+wy-xz-2p_4=0,\\
p_5 &= \text{area of } A_4A_5A_1 & &\Leftrightarrow\ h_5\colon\ uz-vw-2p_5=0,\\
f &= \text{area of } A_1A_2A_3A_4A_5 & &\Leftrightarrow\ h_6\colon\ ay+xv-yu+uz-vw-2f=0.
\end{aligned}$$

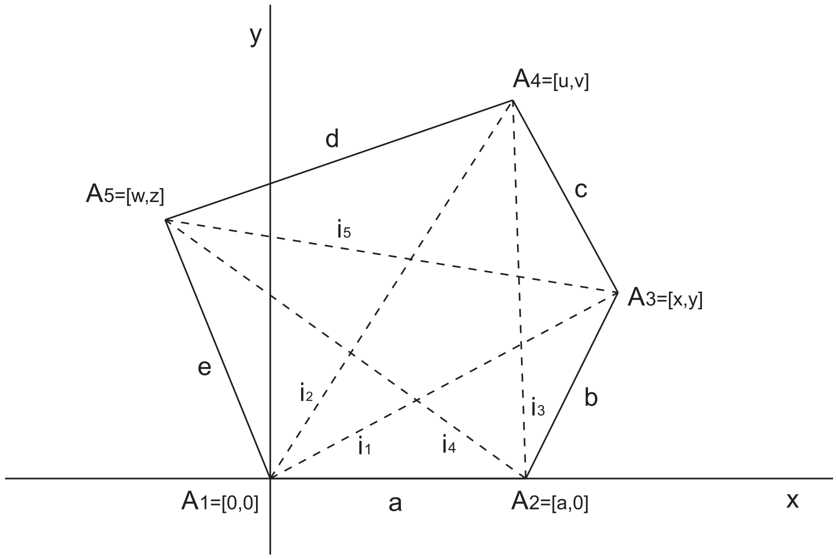

Figure 1: Area of a pentagon $A_1A_2A_3A_4A_5$

The polynomials $h_1, h_2, \ldots, h_6$ in the variables $a, x, y, u, v, w, z, f, p_1, p_2, p_3, p_4, p_5$ generate an ideal $I = (h_1, h_2, \ldots, h_6)$. In the ideal I we eliminate the independent variables a, x, y, u, v, w, z to obtain the elimination ideal which contains polynomials in variables $f, p_1, p_2, p_3, p_4, p_5$. In CoCoA[1] we enter

```
Use R::=Q[axyuvwzfp[1..5]];
I:=Ideal(ay+xv-yu+uz-vw-2f,az-2p[1],ay-2p[2],ay+xv-yu-av-2p[3],
xv-yu+uz-vw+yw-xz-2p[4],uz-vw-2p[5]);
Elim(a..z,I);
```

and get the polynomial which leads to the equation

$$f^2 - f(p_1 + p_2 + p_3 + p_4 + p_5) + (p_1p_2 + p_2p_3 + p_3p_4 + p_4p_5 + p_1p_5) = 0$$

which is the desired Gauss formula (4). The theorem is proved (and rediscovered). □

The Gauss formula (4) is closely connected with the following *Monge formula* [2, 17]

Let $A_1A_2A_3A_4A_5$ be a pentagon. Denote the area of a triangle $A_iA_jA_k$ by p_{ijk}. Then

$$p_{123}p_{145} + p_{125}p_{134} = p_{124}p_{135} \tag{5}$$

holds.

Let us prove (5) by computer (Fig. 1). We have:

$$\begin{aligned}
p_{123} &= \text{area of } A_1A_2A_3 &\Leftrightarrow\quad & g_1:\ ay - 2p_{123} = 0,\\
p_{145} &= \text{area of } A_1A_4A_5 &\Leftrightarrow\quad & g_2:\ uz - vw - 2p_{145} = 0,\\
p_{125} &= \text{area of } A_1A_2A_5 &\Leftrightarrow\quad & g_3:\ az - 2p_{125} = 0,\\
p_{134} &= \text{area of } A_1A_3A_4 &\Leftrightarrow\quad & g_4:\ xv - yu - 2p_{134} = 0,\\
p_{124} &= \text{area of } A_1A_2A_4 &\Leftrightarrow\quad & g_5:\ av - 2p_{124} = 0,\\
p_{135} &= \text{area of } A_1A_3A_5 &\Leftrightarrow\quad & g_6:\ xz - yw - 2p_{135} = 0.
\end{aligned}$$

[1]Software CoCoA is freely distributed at the address `http://cocoa.dima.unige.it`

In the ideal $J = (g_1, g_2, \ldots, g_6)$ we eliminate the independent variables a, x, y, u, v, w, z. Entering the polynomials $g_1, \ldots, g_6$ into CoCoA

```
Use R::=Q[axyuvwzp[123..345]];
J:=Ideal(ay-2p[123],uz-vw-2p[145],az-2p[125],xv-yu-2p[134],av-2p[124],
xz-yw-2p[135]);
Elim(a..z,J);
```

we obtain the elimination ideal which is generated by the polynomial

$$p_{123}p_{145} + p_{125}p_{134} - p_{124}p_{135}.$$

The Monge formula (5) is proved (and rediscovered).

Remark 1: Note that the Monge formula (5) involves the areas of those triangles of a pentagon $A_1A_2A_3A_4A_5$ which have a common vertex A_1.

Remark 2: The Monge formula (5) holds for *arbitrary* pentagons, i.e., even for those which intersect itself. This follows from the proof above where we used signed areas of triangles.

Another proof of the Monge formula (5), which can be found in [17], is based on the following algebraic identity (Fig. 2)

Let A_1, A_2, A_3, A_4 be four points on a line in this order. Then

$$|A_1A_2| \cdot |A_3A_4| + |A_2A_3| \cdot |A_1A_4| - |A_1A_3| \cdot |A_2A_4| = 0. \tag{6}$$

To prove (6) realize that $|A_1A_3| = |A_1A_2| + |A_2A_3|$, $|A_2A_4| = |A_2A_3| + |A_3A_4|$ and $|A_1A_4| = |A_1A_2| + |A_2A_3| + |A_3A_4|$. The formula (6) is a special case of the Ptolemy formula for cyclic quadrilaterals.

Figure 2: Visualizing Eq. (6)

It is easy to prove that Gauss, Monge and Ptolemy formulas are equivalent. For instance to show that the Monge formula implies Gauss formula it suffices to put obvious relations

$$p_{134} = f - p_2 - p_5, \quad p_{124} = f - p_3 - p_5, \quad p_{135} = f - p_2 - p_4$$

into the Monge formula (5) using the notation $p_2 = p_{123},\ p_3 = p_{234},\ p_4 = p_{345},\ p_5 = p_{451}$. See [17], where various generalizations of the theorems above are given.

3. Nagy–Rédey formula for a pentagon

Now we will derive by computer the Nagy–Rédey formula (2) for $n = 5$. We will need it in the next section to find the formula for the area of a cyclic pentagon.

Given a pentagon $A_1A_2A_3A_4A_5$ in a plane, denote the side and diagonal lengths by $a = |A_1A_2|$, $b = |A_2A_3|$, $c = |A_3A_4|$, $d = |A_4A_5|$, $e = |A_5A_1|$, $i_1 = |A_1A_3|$, $i_2 = |A_1A_4|$, $i_3 = |A_2A_4|$,

$i_4 = |A_2A_5|$, $i_5 = |A_3A_5|$. Choose a Cartesian system of coordinates so that $A_1 = [0,0]$, $A_2 = [a,0]$, $A_3 = [x,y]$, $A_4 = [u,v]$, $A_5 = [w,z]$ (Fig. 1).

Then

$$\begin{array}{llll}
|A_2A_3| = b & \Leftrightarrow & h_1: & (x-a)^2 + y^2 - b^2 = 0, \\
|A_3A_4| = c & \Leftrightarrow & h_2h_1: & (u-x)^2 + (v-y)^2 - c^2 = 0, \\
|A_4A_5| = d & \Leftrightarrow & h_3h_1: & (w-u)^2 + (z-v)^2 - d^2 = 0, \\
|A_5A_1| = e & \Leftrightarrow & h_4h_1: & w^2 + z^2 - e^2 = 0, \\
|A_1A_3| = i_1 & \Leftrightarrow & h_5h_1: & x^2 + y^2 - i_1^2 = 0, \\
|A_1A_4| = i_2 & \Leftrightarrow & h_6h_1: & u^2 + v^2 - i_2^2 = 0, \\
|A_2A_4| = i_3 & \Leftrightarrow & h_7h_1: & (u-a)^2 + v^2 - i_3^2 = 0, \\
|A_2A_5| = i_4 & \Leftrightarrow & h_8h_1: & (w-a)^2 + z^2 - i_4^2 = 0, \\
|A_3A_5| = i_5 & \Leftrightarrow & h_9h_1: & (x-w)^2 + (y-z)^2 - i_5^2 = 0.
\end{array}$$
$$\text{Area of } A_1A_2A_3A_4A_5 = p \Leftrightarrow h_{10}:\ p - \tfrac{1}{2}(ay + xv - yu + uz - vw) = 0.$$

Elimination of x, y, u, v, w, z in the ideal $I = (h_1, h_2, \ldots, h_{10})$ gives

$$\begin{aligned} 16p^2 = & -(a^4+b^4+c^4+d^4+e^4) + 2(a^2b^2+b^2c^2+c^2d^2+d^2e^2+e^2a^2) \\ & + 2(i_1^2i_3^2+i_2^2i_4^2+i_3^2i_5^2+i_4^2i_1^2+i_5^2i_2^2) - 2(a^2i_5^2+b^2i_2^2+c^2i_4^2+d^2i_1^2+e^2i_3^2) \end{aligned} \tag{7}$$

which is indeed the Nagy–Rédey formula (2) in case of $n = 5$.

4. Area of a cyclic pentagon

In this section we will investigate cyclic pentagons, i.e., those pentagons in a plane which can be inscribed into a circle. The problem is related to the formula of Heron for triangles, and the formula of Brahmagupta for cyclic quadrilaterals. The formula of Heron was likely known to Archimedes, 287–212 B.C., whereas the formula of Brahmagupta comes from sixth century (Brahmagupta — Indian mathematician, 598–c. 665 A.D.). Since that time, despite a great effort of mathematicians, no formula for the area of a cyclic pentagon has appeared until 1994 when American D.P. Robbins published his results [14]. Almost 1400 years the formula for the area of a cyclic pentagon was missing. The main reason for the long time elapse is a big complexity of such formulas. See the latest results [4, 6, 8, 10].

Suppose that $\mathcal{P} = A_1A_2A_3A_4A_5$ is a cyclic pentagon with side lengths $a = |A_1A_2|$, $b = |A_2A_3|$, $c = |A_3A_4|$, $d = |A_4A_5|$, $e = |A_5A_1|$ (Fig. 3).

We will express the area p of a cyclic pentagon $\mathcal{P}$ in terms of its side lengths a, b, c, d, e. To simplify the expressions we use elementary symmetric functions of squares of side lengths a, b, c, d, e:

$$\begin{aligned} k &= \textstyle\sum a^2 = a^2+b^2+c^2+d^2+e^2, \\ l &= \textstyle\sum a^2b^2 = a^2b^2 + a^2c^2 + \cdots + d^2e^2, \\ m &= \textstyle\sum a^2b^2c^2 = a^2b^2c^2 + a^2b^2d^2 + \cdots + c^2d^2e^2, \\ n &= \textstyle\sum a^2b^2c^2d^2 = a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + a^2c^2d^2e^2 + b^2c^2d^2e^2, \\ o &= a^2b^2c^2d^2e^2. \end{aligned} \tag{8}$$

Further denote $q = 16p^2$. The following lemma holds:

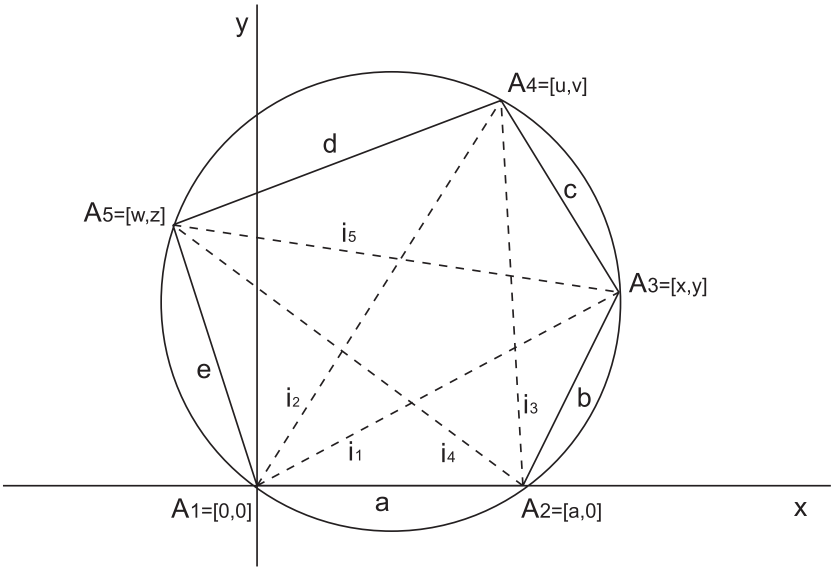

Figure 3: Area of a cyclic pentagon — convex case

Lemma 1 *Let $\mathcal{P} = A_1A_2A_3A_4A_5$ be a convex cyclic pentagon. Then, with the notation as above,*

$$k^2 - 4l + q = 4(abci_2 + bcdi_4 + cdei_1 + deai_3 + eabi_5). \qquad (9)$$

Proof: The statement (9) follows immediately from (7). Applying Ptolemy's theorem to cyclic quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $A_3A_4A_5A_1$, $A_4A_5A_1A_2$, $A_5A_1A_2A_3$, we get

$$i_1i_3 = ac + bi_2,\ i_3i_5 = bd + ci_4,\ i_5i_2 = ce + di_1,\ i_2i_4 = da + ei_3,\ i_4i_1 = eb + ai_5. \qquad (10)$$

Substitution of (10) into the Nagy–Rédey formula (7) gives (9). □

Remark 3. Similarly, for a cyclic quadrilateral with side lengths a, b, c, d, in accordance with the Staudt formula (3), we get

$$k^2 - 4l + q = 8abcd \qquad (11)$$

in the convex case, and

$$k^2 - 4l + q = -8abcd \qquad (12)$$

in the non-convex case, where k, l, m, n are respective elementary symmetric functions of a^2, b^2, c^2, d^2 and $q = 16p^2$. Relations (11), (12) can be written in a compact form as

$$(k^2 - 4l + q)^2 - 64n = 0. \qquad (13)$$

If we denote the left side of (9) as

$$k^2 - 4l + q = s \qquad (14)$$

then (9) is of the form

$$s = 4(abci_2 + bcdi_4 + cdei_1 + deai_3 + eabi_5). \qquad (15)$$

Note that s *does not* depend on the diagonal lengths i_1, i_2, i_3, i_4, i_5. It turns out that to express an (unknown) relation between the area of a cyclic pentagon and its side lengths it suffices to find a relation between s and a, b, c, d, e, where s is given by (15). To do this we need to *eliminate* variables i_1, i_2, i_3, i_4, i_5 from (15). This elimination requires besides the Ptolemy's conditions (10) another conditions.

As we are working with planar pentagons we have to ensure that the cyclic pentagon $\mathcal{P} = A_1A_2A_3A_4A_5$ is planar. This is equivalent to the planarity of cyclic quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $\dots$, $A_5A_1A_2A_3$. We will use the well-known Cayley–Menger determinant which expresses the volume V_n of a simplex $A_1A_2\dots A_{n+1}$ in E^n in terms of all mutual distances between its vertices [1].

Let $|A_iA_j| = a_{ij}$ be the distances between vertices of a simplex $A_1A_2\dots A_{n+1}$ in E^n. Abbreviating a_{ij}^2 by d_{ij} we get for the volume V_n of a simplex $A_1A_2\dots A_{n+1}$ the expression

$$(-1)^{n+1}2^n(n!)^2V_n^2 = D_n = \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & d_{12} & d_{13} & \dots & d_{1,n+1} \\ 1 & d_{21} & 0 & d_{23} & \dots & d_{2,n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & d_{n+1,1} & d_{n+1,2} & \dots & \dots & 0 \end{vmatrix}. \qquad (16)$$

The determinant D_n on the right of (16) is called Cayley–Menger determinant.

For a simplex $A_1A_2A_3A_4$ we get by (16)

$$288V_{1234}^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & i_1^2 & i_2^2 \\ 1 & a^2 & 0 & b^2 & i_3^2 \\ 1 & i_1^2 & b^2 & 0 & c^2 \\ 1 & i_2^2 & i_3^2 & c^2 & 0 \end{vmatrix}. \qquad (17)$$

Now we are to eliminate the variables i_1, i_2, i_3, i_4, i_5 in the system of algebraic equations which consists of the Ptolemy conditions (10), the equation (15) and the Cayley–Menger conditions $V_{1234} = V_{2345} = V_{3451} = V_{4512} = V_{5123} = 0$ which express the planarity of individual quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $\dots$, $A_5A_1A_2A_3$. In this case the elimination failed because of the complexity of Cayley–Menger conditions given by the determinant on the right of (17) and its analogies. Instead of it we express the relation (17) in the form [15, 18]

$$Q^2 + 144V^2 = P \cdot K, \qquad (18)$$

where

$$P = ac + bi_2 - i_1i_3, \qquad (19)$$

$$Q = i_3(bc + ai_2) - i_1(ab + ci_2),^2 \qquad (20)$$

and

$$K = ac(-a^2-c^2+b^2+i_2^2+i_1^2+i_3^2)+bd(a^2+c^2-b^2-i_2^2+i_1^2+i_3^2)-i_1i_3(a^2+c^2+b^2+i_2^2-i_1^2-i_3^2). \qquad (21)$$

The quadrilateral $A_1A_2A_3A_4$ is assumed to be planar and cyclic, hence $V = 0$ and $P = 0$. From (18) we see that instead of the condition $V = 0$ we can consider the *simpler* condition

[2] In a convex quadrilateral the condition $Q = 0$ is equivalent to $P = 0$, see [12, 15].

$Q = 0$. Thus applying (18) to quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $\ldots$, $A_5A_1A_2A_3$ we obtain the conditions

$$\begin{aligned} i_1(ab+ci_2) &= i_3(bc+ai_2),\ i_3(bc+di_4)=i_5(cd+bi_4),\ i_5(cd+ei_1)=i_2(de+ci_1), \\ i_2(de+ai_3) &= i_4(ea+di_3),\ i_4(ea+bi_5)=i_1(ab+ei_5). \end{aligned} \tag{22}$$

Now the elimination of variables i_1, i_2, i_3, i_4, i_5 in the system (10), (15), (22) is feasible. In CoCoA we enter

```
Use R::=Q[abcdei[1..5]s];
I:=Ideal(ac+bi[2]-i[1]i[3],bd+ci[4]-i[3]i[5],ce+di[1]-i[5]i[2],
da+ei[3]-i[2]i[4],eb+ai[5]-i[4]i[1],i[1](ab+ci[2])-i[3](bc+ai[2]),
i[3](bc+di[4])-i[5](cd+bi[4]),i[5](cd+ei[1])-i[2](de+ci[1]),i[2](de+ai[3])
-i[4](ea+di[3]),i[4](ea+bi[5])-i[1](ab+ei[5]),
4(abci[2]+bcdi[4]+cdei[1]+deai[3]+eabi[5])-s);
Elim(i[1]..i[5],I);
```

In 1 m and 22 s (on Intel Core Duo E8500 3.16GHz/3.5GB RAM) we get the polynomial in variables a, b, c, d, e, s with 827 terms. This polynomial is symmetric in variables a^2, b^2, c^2, d^2, e^2. The use of the elementary symmetric functions (8) and the following elimination of a, b, c, d, e gives a polynomial H in k, l, m, n, o, s with 37 terms which leads to the equation $H = 0$. The polynomial H, ordered by the variable o, is as follows:

$$\begin{aligned} H \ := \ & -442368(k^2-4l-s)^2o^2+ \\ & 256(ks+8m)(k^2s^2-576k^2n-128kms-36ls^2-9s^3-512m^2+2304ln+576ns)o \\ & +(s^2-64n)^2(64k^2n+16kms+4ls^2+s^3+64m^2-256ln-64ns)=0\,. \end{aligned} \tag{23}$$

Substitution of $k^2 - 4l - s = -q$, $s^2 - 64n = A$, $ks + 8m = B$ and $128o = C$ into (23) leads to the more compact formula [14]

$$qA^3 + A^2B^2 - 18qABC - 16B^3C - 27q^2C^2 = 0. \tag{24}$$

The formula (24), which expresses the area of cyclic pentagon in terms of it side lengths, can be considered as the generalization of the Heron and Brahmagupta formulas.

Thus, almost 1400 years elapsed since Brahmagupta formula appeared. The reasons, why it lasted so long, are obvious — a big complexity of such a formula. It is difficult to imagine to discover (24) without the use of computers.

Remark 4. Consider that $o = 0$ in (23), i.e., at least one of the side lengths a, b, c, d, e of a pentagon equals 0. Putting this into (23) we get the equation $(s^2 - 64n)^2 = 0$ which is in accordance with the relation (13) for the area of a cyclic quadrilateral.

Remark 5. The polynomial on the left hand side in (23) and others which express the area of a cyclic polygon in terms of its side lengths are called *Heron polynomials* [8, 6, 10].

Remark 6. The polynomial H in (23) is of 7th degree in $q = 16p^2$ which means that there exist at most *seven* cyclic pentagons with given side lengths and *different* circumradii.

In the previous part we supposed that a cyclic pentagon is convex. Now we will show, that the formula (24) holds also in a non-convex case.

Let us suppose that a cyclic pentagon is of the form as in Fig. 4. Then instead of (10) and (22) we have (Fig. 4)

$$i_1i_3 = ac - bi_2,\ i_3i_5 = -bd + ci_4,\ i_5i_2 = ce + di_1,\ i_2i_4 = da + ei_3,\ i_4i_1 = -eb + ai_5, \tag{25}$$

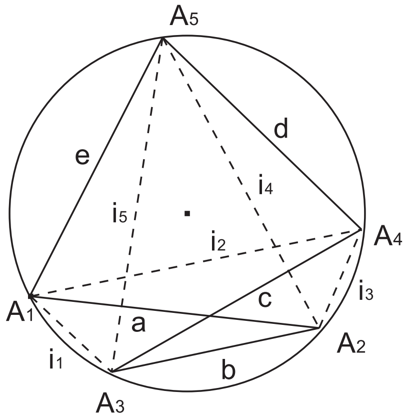

Figure 4: Area of a cyclic pentagon — non-convex case

and

$$\begin{aligned} a(i_1b+i_2i_3) &= c(i_3b+i_1i_2),\ c(i_3b+di_5)=i_4(i_3d+bi_5),\ i_2(i_1c+de)=i_5(i_1e+cd), \\ i_2(i_3a+de_3) &= i_4(i_3d+ce),\ a(i_1b+i_4e)=i_5(i_1e+bi_4). \end{aligned} \tag{26}$$

The formula (15) changes into

$$s = 4(-abci_2 - bcdi_4 + cdei_1 + deai_3 - eabi_5). \tag{27}$$

We see that conditions (25), (26), (27) differ from (10), (22), (15) only in the sign of the variable b. If we write $-b$ in (25), (26), (27) instead of b then we get (10), (22), (15). This means that after the elimination of variables i_1, i_2, i_3, i_4, i_5 in the ideal which is generated by the polynomials from the conditions (25), (26), (27) we get the same polynomial H as in (23) taking into account that in H only b^2 and its various powers occur.

Similarly we proceed in another cases.

Conclusion

Gauss and Monge formulas together with the formula for the area of a cyclic pentagon were presented. To describe and simplify the geometric situation we used besides Ptolemy conditions another conditions which come from the formula (18). After the translation geometric properties into the system of algebraic equations we used elimination based on Gröbner bases computation.

Up to date the formula for the area of a cyclic heptagon and octagon appeared [8]. These formulas are very complex. For instance the expansion of the formula for a cyclic heptagon in terms of its symmetric functions has almost one million coefficients.

The technique which was used in [8] differs from the technique used here. Perhaps the method presented in this paper could serve as a possible tool for solving the problem for cyclic n-gons for $n > 8$.

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