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About Some Mappings Defined by a Classical Desargues Configuration

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Dedicated to Professor Hans Havlicek on the occasion of his 60th birthday.

Abstract. A Desargues configuration is a planar figure consisting of ten undistinguished points and ten lines with the well-known meaning of a geometric-algebraic axiom for interpretation as an image of a three-dimensional figure. We consider the ten homologies defined by such a configuration. We give an analytic proof for the fact that such a configuration defines a unique polarity too. The (labelled) Desargues figure, i.e., two perspective triangles together with the perspectivity center and axis, can be extended to perspective n-gons, which allow higher dimensional generalizations of the classical interpretation of a Desargues figure in space. Furthermore we consider multi-perspective triangles and an iteration process defined by a pair of not-perspective triangles.

Key Words: Desargues configuration, homology, correlation, polarity, crossratio

MSC 2010: 51A20, 51A30, 51A05

1. Introduction

A $(10_3, 10_3)$-configuration of ten distinct points and lines of a (classical) projective plane π, each element of one set incident with *exactly* three elements of the other one, is called a (planar, non-degenerate) *configuration of Desargues* (Fig. 1). The usual approach to such a Desargues configuration starts with two Z-perspective triangles $(A, B, C), (A', B', C')$, whereby corresponding sides of these triangles meet in the additional points P, Q, R. We call such Z-perspective triangles as being in "Desargues position". The well-known Theorem of Desargues states, that these additional points are collinear with a line z. We distinguish such a labelled Desargues figure (Fig. 2) from the not labelled Desargues configuration. Here we do not consider the fundamental meaning of a Desargues configuration for projective planes as geometric equivalent to a field as coordinate structure of the plane (see, e.g., [3, 8]). The interpretation of a labelled Desargues figure as projection image of an object in space gives rise to higher dimensional generalizations (cf. [10]).

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The aim of this paper is to investigate (linear) mappings defined by a Desargues configuration or figure and to present a short list of seemingly open questions in connection with such mappings. (For the concept *linear mapping* see, e.g., [4, 7].)

In Section 2 we consider mappings of type (Z, z)-homology $\chi : (A, B, C) \to (A', B', C')$, related to a Desargues configuration. There are 10 essentially different such (Z, z)-homologies, and their characteristic cross ratios depend on three parameters. We find a Peczar-like identity for those cross ratios (cf. [1]). One can also connect correlations $\kappa\colon\ \pi \to \pi^*$ of the point set of π into its line set (i.e., the point set of the dual plane π^*) with a Desargues configuration. It turns out that there exists only one such correlation, namely a polarity, which allows a non-Euclidean interpretation of the given Desargues configuration (cf. [9, 2, 5]). We give an analytic proof of this fact in Section 3. In Section 4 we deal with the question, if two non-ordered point triplets allow more than one completion to a Desargues configuration Δ.

After some remarks on higher-dimensional interpretations in Section 5 we put attention to a seemingly open problem in Section 6. Considering an iterative process starting with two triangles (A, B, C), (A', B', C'), not in Desargues position, the lines connecting corresponding points and the intersection points of corresponding lines give rise to a new pair of triangles. The described process can be iterated and the question arises, whether the process is attractive ending up with a limit figure in Desargues position (cf. [11]). But one can also ask for start positions, where the described process returns back to the start figure after a finite number of steps.

It is convenient to assume π to be a real projective plane $\pi(\mathbb{R})$, but obviously many of the posed questions will have answers also in projective planes over an arbitrary commutative field $\mathbb{F}$, as long as the characteristic of $\mathbb{F}$ differs from 2.

2. The homologies defined by a Desargues configuration

In a Desargues configuration Δ each of the 10 points can be chosen as center point Z of a homology χ which permutes the points on the three lines through Z. Then the one line z connecting the three points not on lines through Z must act as axis of the homology χ. The six remaining points of Δ can be labelled such that we get Z-perspective triangles (A, B, C), $(A', B', C') = (A, B, C)^\chi$. Each such labelling starting with a point Z refers to a Desargues figure to the given configuration. To any chosen (Z, z) there exist two homologies, which are just inverse to each other, meaning that the characteristic cross ratio (CR-value) $\mathrm{CR}(Z, z, A, A')$ of one is the reciprocal value of the other. Keeping the axis z fixed and permuting the three points Z, A, A' on line ZA we end up with the well-known 6 CR-values

$$c,\ \frac{1}{c},\ 1-c,\ \frac{1}{1-c},\ 1-\frac{1}{c},\ \frac{c}{c-1}. \tag{1}$$

We call two CR-values "not essentially different", if they belong to the same set of values (1). Obviously the 10 pairs of possible homologies χ will not lead to 10 essentially different and independent CR-values and the question arises, how many CR-values are really "essentially different"? It turns out that there are only *three* independent characteristic CR-values c_j $(j = 1, 2, 3)$:

On line z we add the points $U := z \cap ZA$, $V := z \cap ZB$, $W := z \cap ZC$ and use them for a special coordinate frame in π, such that the 10 points of Δ and those additional points

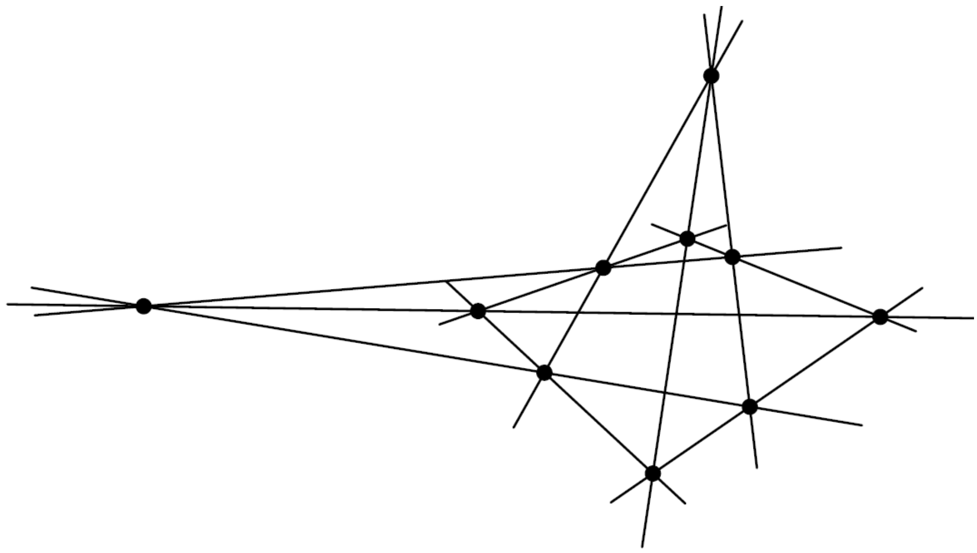

Figure 1: A Desargues configuration

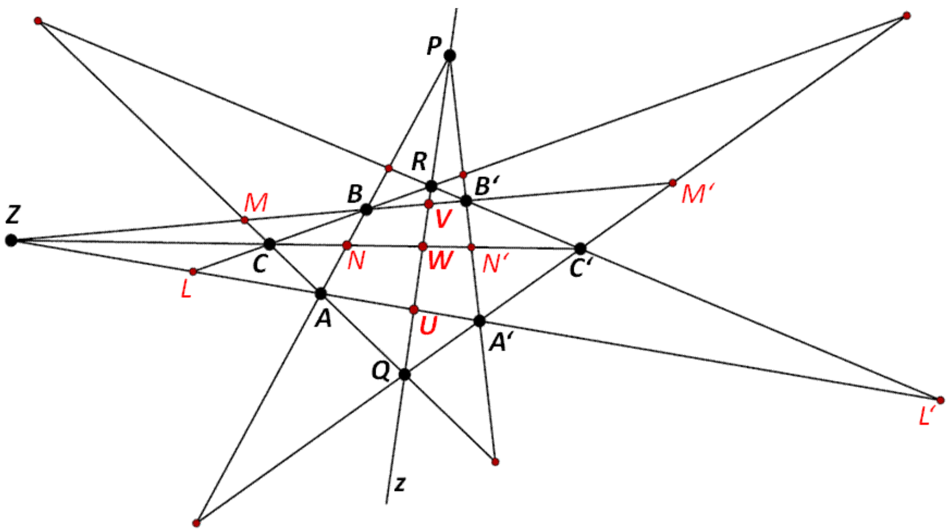

Figure 2: A Desargues figure to the Desargues configuration Fig. 1 labelled according to (2)

U, V, W have the projective coordinates (see Fig. 2)

$$\begin{gathered} U \mathrel{\widehat{=}} (1,0,0)\mathbb{R}, \quad V \mathrel{\widehat{=}} (1,1,0)\mathbb{R}, \quad W \mathrel{\widehat{=}} (0,1,0)\mathbb{R}, \\ Z \mathrel{\widehat{=}} (0,0,1)\mathbb{R}, \quad A \mathrel{\widehat{=}} (1,0,1)\mathbb{R}, \quad A' \mathrel{\widehat{=}} (1,0,c_1)\mathbb{R}, \\ P \mathrel{\widehat{=}} (1,c_2,0)\mathbb{R}, \quad Q \mathrel{\widehat{=}} (1,c_3,0)\mathbb{R}, \quad R \mathrel{\widehat{=}} (1,c_4,0)\mathbb{R} \implies \\ B := PA \cap ZV \mathrel{\widehat{=}} (c_2, c_2, c_2 - 1)\mathbb{R}, \quad B' \mathrel{\widehat{=}} (c_2, c_2, -c_1(1-c_2))\mathbb{R}, \\ C := QA \cap ZW \cap RB \mathrel{\widehat{=}} (0, -c_3, 1)\mathbb{R} = (0, c_2(c_4 - 1), 1 - c_2)\mathbb{R}, \\ C' :\mathrel{\widehat{=}} (0, -c_3, c_1)\mathbb{R}. \end{gathered} \tag{2}$$

From the description of C in (2) in two ways we have the dependency

$$c_3(1 - c_2) = c_2(1 - c_4). \tag{3}$$

Because we have a $(10_3, 10_3)$-configuration the numbers c_i are restricted by $c_i \notin \{0,1\}$, $(i = 2,3,4)$ and c_2, c_3, c_4 are pairwise different; $c_1 \notin \{0,1\}$ can be chosen freely.

Each of the 10 points in (2) (besides U, V, W) defines a unique pair of (inverse) homologies χ and because their characteristic cross ratios are not essentially different we can restrict ourselves to one representant of each pair. We give a list of these homologies using a description as χ_n (*center, axis, point* $\mapsto$ *image point*):

$$\begin{array}{ccc} & \chi_1(Z, PQ, A \mapsto A'), & \\ \chi_2(P, ZC, A \mapsto B), & \chi_3(Q, ZB, C \mapsto A), & \chi_4(R, ZA, B \mapsto C), \\ \chi_5(A, RC', Z \mapsto A'), & \chi_6(B, QA', Z \mapsto B'), & \chi_7(C, PA', Z \mapsto C'), \\ \chi_8(A', RC, A \mapsto Z), & \chi_9(B', QA, B \mapsto Z), & \chi_{10}(C', PA, C \mapsto Z). \end{array} \tag{4}$$

The homologies $\chi_1, \ldots, \chi_4$ have the following matrix representations:

$$\begin{array}{ll} \chi_1 \ldots \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c_1 \end{pmatrix}, & \chi_2 \ldots \begin{pmatrix} c_2 & 0 & 0 \\ c_2 & c_2 - 1 & 0 \\ 0 & 0 & c_2 - 1 \end{pmatrix}, \\ \chi_3 \ldots \begin{pmatrix} c_3 + 1 & -1 & 0 \\ c_3 & 0 & 0 \\ 0 & 0 & c_3 \end{pmatrix}, & \chi_4 \ldots \begin{pmatrix} c_2 & -c_2 & 0 \\ 0 & -c_3(c_2 - 1) & 0 \\ 0 & 0 & c_2 \end{pmatrix}. \end{array} \tag{5}$$

Now let us consider the product mapping $\kappa := \chi_2 \circ \chi_4 \circ \chi_3$. It maps $Z \mapsto Z, U \mapsto U$ and $A \mapsto A$, i.e., ZU is a pointwise fixed line and therefore κ is a perspective collineation. Furthermore it maps $Q \mapsto P$ and $P \mapsto Q$, therefore its restriction to the line PQ must be an involutoric projectivity, what indicates that κ is a *harmonic homology* meaning that its characteristic cross ratio is -1. Indeed, the product matrix to κ is

$$\kappa \ldots \begin{pmatrix} 1 & -\dfrac{c_2 + c_3}{c_2 c_3} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the point $(c_2 + c_3,\ 2c_2c_3,\ 0)\mathbb{R}$ is the center of κ. Thus its position is described by the *harmonic mean* of the coordinates of P and Q with respect to the projective coordinate frame $\{U, V, W\}$ on z.

Therewith we state

Result 1. *Given a Desargues configuration, the three homologies defined by three collinear points of this configuration as centers* (cf. (4)) *have concurrent axes and their product is a harmonic homology.*

Remark 1. Result 1 is in some sense a generalization of the classical "three reflections theorem" of planar Euclidean (and non-Euclidean) geometry.

To calculate the characteristic cross ratio values of (χ_i, χ_i^{-1}), $(i = 1, \ldots, 10)$ we need six additional points, where lines of the configuration meet (see Fig. 2):

$$\begin{array}{ll} L := RC \cap UZ \mathrel{\widehat{=}} (c_3, 0, c_4)\mathbb{R}, & L' := RC' \cap UZ \mathrel{\widehat{=}} (c_3, 0, c_1c_4)\mathbb{R}, \\ M := QA \cap VZ \mathrel{\widehat{=}} (c_3, c_3, c_3 - 1)\mathbb{R}, & M' := QA' \cap VZ \mathrel{\widehat{=}} (c_3, c_3, (c_3 - 1)c_1)\mathbb{R}, \\ N := PA \cap WZ \mathrel{\widehat{=}} (0, c_2, -1)\mathbb{R}, & N' := PA' \cap WZ \mathrel{\widehat{=}} (0, c_2, -c_1)\mathbb{R}. \end{array} \tag{6}$$

Using the relation (3) it follows

$$\begin{aligned}
\chi_1 :&\quad d_1 = \mathrm{CR}(A', A, U, Z) = c_1, \\
\chi_2 :&\quad d_2 = \mathrm{CR}(V, U, W, P) = c_2/(c_2 - 1), \\
\chi_3 :&\quad d_3 = \mathrm{CR}(U, W, V, Q) = 1/c_3, \\
\chi_4 :&\quad d_4 = \mathrm{CR}(W, V, U, R) = 1 - c_4 = c_3(1 - c_2)/c_2, \\
\chi_5 :&\quad d_5 = \mathrm{CR}(A', Z, L', A) = c_1(c_4 - c_3)/(c_3(1 - c_1)) = c_1(c_2 - c_3)/(c_2c_3(1 - c_1)), \\
\chi_6 :&\quad d_6 = \mathrm{CR}(B', Z, M', B) = c_1(c_2 - c_3)/(c_3(1 - c_1)(1 - c_2)), \\
\chi_7 :&\quad d_7 = \mathrm{CR}(C', Z, N', C) = -c_1(c_2 - c_3)/((1 - c_1)c_2), \\
\chi_8 :&\quad d_8 = \mathrm{CR}(Z, A, L, A') = c_3(1 - c_1)/(c_3 - c_4) = -(1 - c_1)c_2c_3/(c_2 - c_3), \\
\chi_9 :&\quad d_9 = \mathrm{CR}(Z, B, M, B') = -c_3(1 - c_1)(1 - c_2)/(c_2 - c_3), \\
\chi_{10} :&\quad d_{10} = \mathrm{CR}(Z, C, N, C') = c_2(1 - c_1)/(c_2 - c_3).
\end{aligned} \tag{7}$$

In accordance with Result 1 we receive conditions for products of homologies with collinear centers:

$$\begin{array}{lllll}
d_3d_7/d_5 = -1, & d_5d_8/d_1 = -1, & d_2d_9/d_8 = -1, & d_2d_3d_4 = -1, & d_4d_6/d_7 = -1, \\
d_6d_9/d_1 = -1, & d_3d_8/d_{10} = -1, & d_2d_5/d_6 = -1, & d_7d_{10}/d_1 = -1, & d_4d_{10}/d_9 = -1,
\end{array} \tag{8}$$

and furthermore the conditions

$$d_2d_5d_9 = d_3d_7d_8 = d_4d_6d_{10} = d_1. \tag{9}$$

Putting triple products (8) and (9) together we can add the strange relation

$$\prod_{i=1}^{10} d_i = d_1^4, \tag{10}$$

where we have to take into account that we choose a certain representant in each pair of inverse homologies. By eventually changing the representant in each pair of homologies and choosing the "right" CR-values (1) we can state

Result 2. *A (non degenerate) Desargues configuration gives rise to 20 homologies grouped in 10 pairs* (χ_i, χ_i^{-1}), $(i = 1, \dots, 10)$, *having characteristic cross ratios* d_i, d_i^{-1}. *One can choose a representant* χ_i, $i \in \{1, \dots, 10\}$, *in each pair such that the product of the 10 characteristic cross ratios* d_i *of these homologies equals the* 4^{th} *power of any single* d_j, $j \in \{1, \dots, 10\}$.

Remark 2. For (8), (9) and (10) we refer also to the so-called *Peczar-identity* for the product of cross ratios of quadruplets out of 5 collinear points *(cf. [1, p. 132])*. The terms (8) relate to the well-known Menelaos theorem in affine planes concerning the product of three ratios defined by a triangle and a transversal of it.

3. Interpretation of Δ as "mixed figure" quadrangle plus quadrilateral

We can receive the ten points and lines of a Desargues configuration Δ as the sum of a complete quadrangle $\Omega := (Z, A, B, C)$ (i.e., four points and six lines) and a complete quadrilateral $\Omega^* := (z,\ a := B'C',\ b := C'A',\ c := A'B')$ (i.e., four lines and their six intersection points), which are in an "incident" position. Thus we can define a canonical correlation $\kappa : \pi \to \pi^*$, which maps $Z \mapsto z$, $A \mapsto a$, $B \mapsto b$, $C \mapsto c$.

As we can choose the non degenerate quadrangle Ω in $10 \times 9 \times 7 \times 4$ ways, we could expect 2520 such correlations κ associated with Δ and again we can look for dependencies. From Section 2 we know, Δ depends on three parameters only, so this must be the case for the (homogeneous) (3×3)-matrices describing κ too. Therefore we can expect that three "independent" correlations are sufficient to describe the whole set of correlations associated to Δ.

For each $\kappa : \pi \to \pi^*$ there is a "point conic" of incident points and lines as well as a — in general different — "line conic", the dual of a point conic. What can be said about the pair of those conics and its position to the given Desargues figure? G.K.Ch. Staudt [9, p. 135] seems to have been the first to notice that there is a unique *polarity* connected with every (general) Desargues figure (see also [2, 5]). So the first guess of many correlations is totally wrong! Here comes the calculation:

Assuming a commutative coordinate field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ we use the coordinate frame

$$A \mathrel{\widehat{=}} (1,0,0)\mathbb{F}, \quad B \mathrel{\widehat{=}} (0,1,0)\mathbb{F}, \quad C \mathrel{\widehat{=}} (0,0,1)\mathbb{F}, \quad Z \mathrel{\widehat{=}} (1,1,1)\mathbb{F}. \tag{11}$$

The lines a, b, c, z can be assumed to be represented by vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{z}$

$$a \mathrel{\widehat{=}} \mathbb{F}(a_0, a_1, a_2) = \mathbb{F}\vec{a}, \ \ldots, \ c \mathrel{\widehat{=}} \mathbb{F}(c_0, c_1, c_2) = \mathbb{F}\vec{c}, \ z \mathrel{\widehat{=}} \mathbb{F}(z_0, z_1, z_2) = \mathbb{F}\vec{z}, \tag{12}$$

such that the condition $\vec{z} = \vec{a} + \vec{b} + \vec{c}$ is fulfilled. From the six "Desargues conditions" each of the triplets

$$\{ZC, a, b\}, \ \{ZA, b, c\}, \ \{ZB, c, a\}, \ \{z, BC, a\}, \ \{z, CA, b\}, \ \{z, AB, c\}$$

consists of concurrent lines. This gives the following six homogeneous equations

$$\begin{vmatrix} c_0 & c_1 \\ a_0 & a_1 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} = \begin{vmatrix} b_2 & b_0 \\ c_2 & c_0 \end{vmatrix} = \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}. \tag{13}$$

Without loss of generality we can norm these determinants with value 1. Of all possible determinants occurring in the transformation $(\vec{a}, \vec{b}, \vec{c}) \mapsto (\vec{b} \times \vec{c}, \ \vec{c} \times \vec{a}, \ \vec{a} \times \vec{b})$ of the trilateral (a, b, c) to its triangle (A', B', C') we thus find the six (13) with value 1 and there remain exactly three, which can be considered as independent variables, as expected:

$$\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} =: u, \quad \begin{vmatrix} c_2 & c_0 \\ a_2 & a_0 \end{vmatrix} =: v, \quad \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} =: w, \quad (u, v, w) \neq 0, 1\,. \tag{14}$$

Therewith we receive $A' \mathrel{\widehat{=}} (u, 1, 1)\mathbb{F} \in ZA$, $B' \mathrel{\widehat{=}} (1, v, 1)\mathbb{F} \in ZB$, $C' \mathrel{\widehat{=}} (1, 1, w)\mathbb{F} \in ZC$ and finally

$$\kappa := \pi \ \to \ \pi^* \ldots \begin{pmatrix} vw - 1 & 1 - w & 1 - v \\ 1 - w & wu - 1 & 1 - u \\ 1 - v & 1 - u & uv - 1 \end{pmatrix}. \tag{15}$$

Because of symmetry of the matrix (15) this single correlation κ is a polarity mapping the Desargues configuration onto itself. This means that, starting with a Desargues configuration, any chosen quadrangle of points must lead to the same polarity. Therefore we can state

Result 3. *Given a pair (quadrangle, quadrilateral) in a projective plane π such that it forms a (non degenerate) Desargues configuration, then it defines a correlation $\pi \to \pi^*$, which is a (regular) polarity. Based on the given quadrangle as fundamental coordinate figure, this polarity is described by a symmetric matrix (15).*
Given a Desargues configuration of 10 points and 10 lines, any quadrangle of points corresponds to a unique quadrilateral of lines, and all possible pairs (quadrangle, quadrilateral) determine the same polarity.

Remark 3. If $\mathbb{F} \cong \mathbb{R}$, this polarity is the polar system of a real or imaginary conic ω, which might be interpreted as the absolute conic of a hyperbolic or elliptic geometry.

Thus the Result 3 connects to the well-known

Theorem 1. *Polarising a triangle Δ (identified with its trilateral) with respect to a given polarity κ results in a trilateral $\Delta^\kappa = \bar{\Delta}$ (identified with its triangle) such that $\Delta, \bar{\Delta}$ are in Desargues position with (centre, axis) $= (Z, z)$ a pair (pole, polar line) in κ.*

Result 4. *A non-Euclidean interpretation finds Z as the orthocenter of Δ (see [2, p. 62]). Re-labelling of the Desargues configuration implies that the following is true also in non-Euclidean planes: Of four points $\{A, B, C, Z\}$, whereof Z is the orthocenter of the triangle formed by $\{A, B, C\}$, each point is the orthocenter of the triangle formed by the remaining triplet of points.*

The polarity allows another explanation of Result 1:

Result 5. *Fixing a line of the Desargues configuration (see Fig. 2), we have three points (e.g., Z, A, A') and three additional intersection points of ZA with other lines of the configuration (U, L, L'). There is a canonically defined linear mapping of these three intersection points to the original point triplet (from our assumptions we have six different points): $U \mapsto Z$, $L \mapsto A'$, $L' \mapsto A$. This mapping is induced by the polarity and thus the three pairs of points suit into an involutoric projectivity.*

4. Multi-perspective triangles

We note that for two triangles there are six possibilities to define a correspondence between their vertices. Let us start with two triangles (assuming to have six different points), such that one of their possible correspondences is a perspectivity, i.e., they are in Desargues position. We ask for positions of the two triangles such that a re-labelling of them delivers an additional Desargues position. An example in the Euclidean plane: Reflect an isosceles triangle at a line parallel to its basis. Then, by re-ordering corresponding points, the now two triangles are also centrally symmetric. A more projective geometric formulation of this example connects a harmonic homology to the two triangles (see Fig. 3, left): The axis x of this special homology connects two homologous vertices A, A' and contains the "Desargues centers" Z_1, Z_2. Figure 3 (at left) shows a case with exactly two Desargues positions. Thereby the six vertices of both triangles do not necessarily belong to a conic.

Another case derived from a Euclidean figure is shown in Fig. 3, right: If the six vertices are collinear to a regular hexagon, then there occur even four Desargues centers $Z_1, \ldots, Z_4$, whereof three are collinear with the polar line of the fourth with respect to the circumscribed conic.

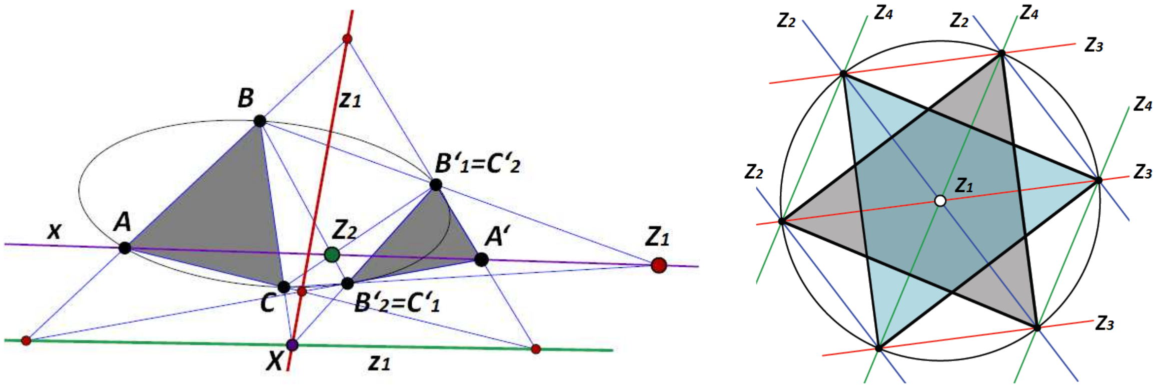

Figure 3: Two triangles admitting exactly two (left) and four (right) Desargues positions

A more general situation is shown in Fig. 4 (at left): Starting with two triangles admitting two Desargues positions (the centers Z_1, Z_2 of which are not collinear with the same pair of corresponding points) it can be shown by using Pappus theorem that there always exists a third Desargues center. There might even occur a fourth one, Z_4 in Fig. 4, right: in this case the lines Z_4Z_i $(i = 1, 2, 3)$ carry corresponding points of the triangles. (Thereby the six vertices of both triangles do not necessarily belong to one conic.)

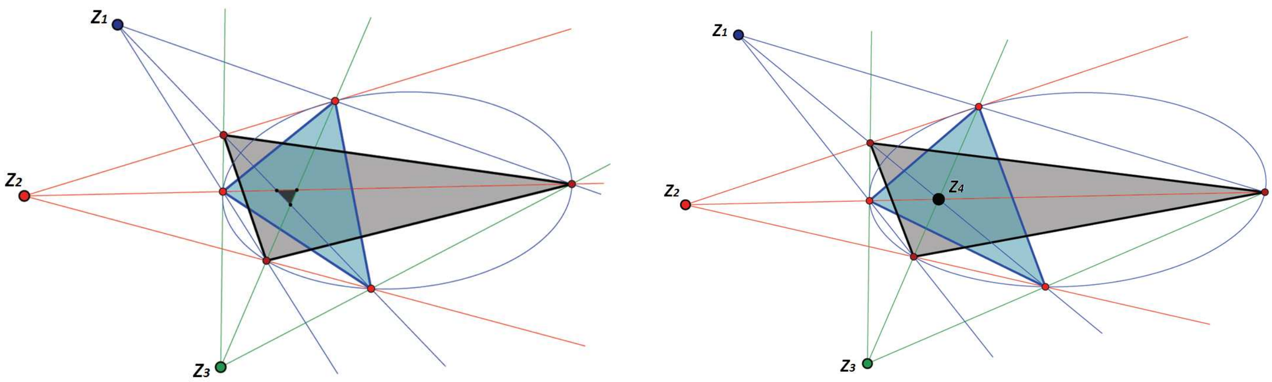

Figure 4: Two triangles admitting exactly three (left) and four (right) Desargues positions

We collect these statements in

Result 6. *Two triangles admit six correspondences between their vertices (and sides). Of these six correspondences there can be at most four perspectivities, i.e., the two triangles are in at most four Desargues positions at the same time. If the triangles admit two perspectivities with centers Z_1, Z_2 collinear with the same pair of corresponding vertices, then there exist either no or two additional perspectivities. If the centers Z_1, Z_2 are not collinear with the same pair of corresponding vertices, then there exists always a third perspectivity and even a fourth perspectivity can occur. For this last special case the six sides of the quadrangle of Desargues centers $\{Z_1, \ldots, Z_4\}$ contain the six vertices of the Desargues axes quadrilateral $\{z_1, \ldots, z_4\}$* (Fig. 5).

In [11] it is proved that for six points forming a "golden Hexagon" *all* partitions into two triplets allow at least one labelling such that the pairs of then labelled triangles are in Desargues position. All golden hexagons are collinear to a Euclidean regular pentagon

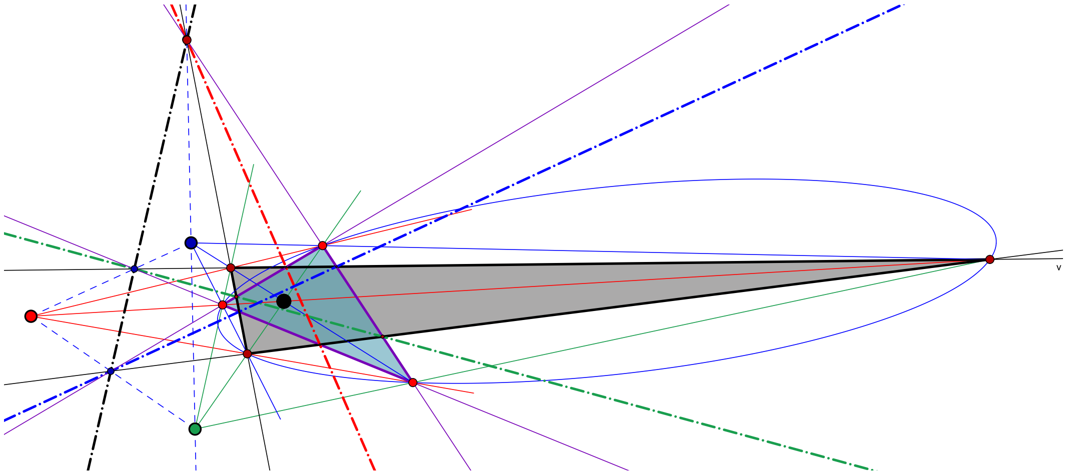

Figure 5: Two triangles being in four Desargues positions together with the quadrangle of Desargues centers and the quadrilateral of Desargues axes

together with its center. For the characteristic CR-values d_j of the homologies χ_j to each pair of triangles occur only the golden mean value and its not essentially different values according to (1).

5. Higher-dimensional interpretations of two Z-perspective polygons

H. Ebisui discovered a nice incidence theorem connected with two Z-perspective *quadrangles* in a projective plane $\pi(\mathbb{R})$ (see [10]). The *"Ebisui-Theorem"* reads as follows:

Theorem 2. *Let* $(A_i, B_i, C_i, D_i), i = 1, 2,$ *be two Z-perspective quadrangles of* π*. Then the intersection points*

$$\begin{aligned} A_1B_1 \cap A_2B_2 &=: E, & C_1D_1 \cap C_2D_2 &=: \bar{E}, \\ B_1C_1 \cap B_2C_2 &=: F, & D_1A_1 \cap D_2A_2 &=: \bar{F}, \\ A_1C_2 \cap A_2C_1 &=: G, & B_1D_2 \cap B_2D_1 &=: \bar{G}, \end{aligned} \tag{16}$$

define three lines $E\bar{E}$, $F\bar{F}$, $G\bar{G}$ *which are incident with a common point* $\bar{Z}$*.*

Note that the last pair $(G, \bar{G})$ stems from "overcrossings" i.e., points of the first quadrangle are connected with points of the second one! There are three possibilities to choose one pair of overcrossing lines such that we expect (at least) three such *"Ebisui points"* $\bar{Z}$ associated with perspective quadrangles. It turns out that there occurs an additional Ebisui point, when choosing two overcrossings, and this additional point completes the configuration:

There are 6 intersection points of corresponding sides of the given quadrangles and 6 such points stemming from overcrossings. They all give rise to 16 lines which connect intersections of opposite pairs of sides according to (16). On each line there are 3 such points and through each point there pass 4 lines, such that we can speak of a $(12_4, 16_3)$-configuration. This configuration possesses 6 diagonals forming a complete quadrangle with the *four* Ebisui points as vertices.

We recall some of the statements from [10] in

Theorem 3. *Let (A_i, B_i, C_i, D_i), $i = 1, 2$, be two Z-perspective quadrangles of π embedded in a projectively enclosed Euclidean 4-space Π^4_E. Every such (not degenerate) "Ebisui figure" allows an interpretation as the perspective projection of a regular cross polytope of Π^4_E into π. Thereby the center of the cross polytope is the preimage of Z.*
Each pair of partial triangles defines a Desargues axis and all such axes form a complete quadrangle. Extending corresponding partial triangles to "overcrossings" leads to a combination of five such complete quadrangles forming the configuration of Desargues in π.

The analysis of the "Ebisui theorem" shows that also in the classical case each planar Desargues figure can be interpreted as the perspective projection of a Euclidean regular octahedron in Π^3_E. Re-partitioning the six points A, B, C, A', B', C' into two triangles leads to exactly four Desargues axis forming a complete quadrangle. In spite of its triviality we formulate this as a

Result 7. *Three Z-perspective non ordered pairs of points $(A, A'), (B, B'), (C, C')$ allow four orderings to Z-perspective triangles, which give rise to four Desargues axes. Each of the four pairs of corresponding triangles results from a linear mapping (central projection) of a pair of opposite (parallel) faces of a regular octahedron. Thereby the center of the octahedron is mapped into Z.*

In [10] also perspective pentagons and n-gons are treated in general and they, too, can be seen as linear images of cross polytopes of a suitably high-dimensional (Euclidean) space. As cross polytopes can be connected with a projective coordinate frame such that there only occur coordinates out of the set $\{0, 1, -1\}$, the Ebisui theorem and its generalizations are valid in projective planes/spaces over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$.

Here we just recall the two-dimensional case of two Z-perspective segments: This figure can be interpreted as central projection or, more general, the linear image of a square. Again its center can be taken as the preimage of Z, the homology χ and its image is now harmonic, i.e., the characteristic CR-value equals -1. The collineation from the square figure to the completed quadrangle in π keeps this harmonic homology χ fixed such that we end up with the complete quadrangle together with its diagonal trilateral. A spatial Desargues figure shows this two-dimensional case in each of its 5 planes, while Desargues figures occur in the 3-subspaces of the cross polytope in Π^4 and so on.

6. Iterative processes connected with pairs of triangles

As an extension of Section 3 we here give an outlook to further questions one could pose in connection with Desargues figures or configurations:

Let be given an ordered triangle $\Delta = \{A, B, C\}$ with sides a', b', c' and an ordered trilateral $\nabla = \{a, b, c\}$ with vertices A', B', C' not in Desargues position. Then the lines connecting corresponding vertices form a new trilateral $\nabla_1 = \{a_1, b_1, c_1\}$ (with vertices A'_1, B'_1, C'_1) as well as the intersections of their corresponding sides form a new triangle $\Delta_1 = \{A_1, B_1, C_1\}$ (see Fig. 6). Considering (Δ_1, ∇_1) as a new pair of ordered triangle and ordered trilateral we repeat the construction of connecting corresponding vertices and intersecting corresponding sides.

When starting with an unlabelled pair triangle Δ and trilateral ∇ there will be 6 possibilities to define corresponding vertices and sides. One such starting pair (Δ, ∇) therefore defines six pairs $(\Delta_{1,1}, \nabla_{1,1}), \ldots, (\Delta_{1,6}, \nabla_{1,6})$ of new triangles and trilaterals, respectively, and iteration delivers a fractal based on a projective geometric construction.

A natural question would be, whether the iteration process defined by a labelled pair (Δ, ∇) is attractive or not. As the result of an attractive process we expect that the "limit pair" $(\Delta_{lim}, \nabla_{lim})$ is degenerate. The described iteration process might be even finite for some special start figures (Δ, ∇). One could also ask for the conditions such that the set $\{(\Delta_i, \nabla_i) \mid 0 \leq i \leq n\}$ $((\Delta_0, \nabla_0) := (\Delta, \nabla))$ is even closed for an arbitrarily given $n \in \mathbb{N}$. For such special results we expect that also the start figure is special with respect to the group of projective transformations.

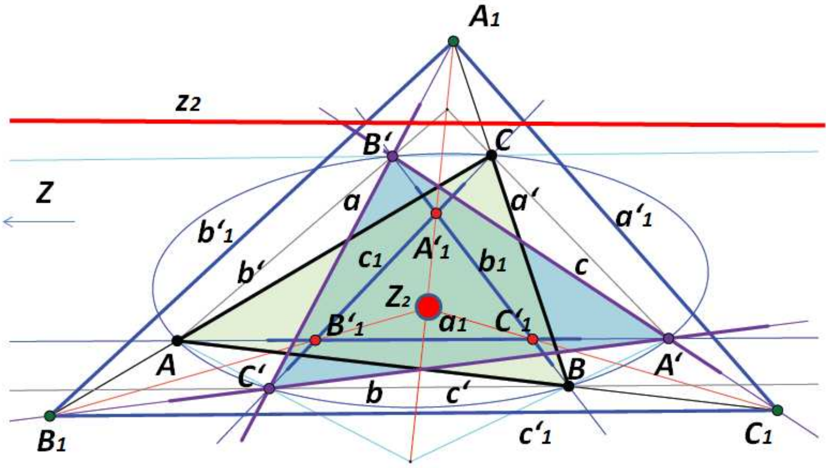

Figure 6: A special pair of labelled triangles $(\Delta = ABC, \nabla = abc)$, which are not in Desargues position, defining a finite iteration process of step length 2

Figure 6 shows an example of such a special case: The labelled start figure $(\Delta = ABC, \nabla = abc)$ is not in Desargues position, but already the first step delivers a pair (Δ_1, ∇_1), which is in Desargues position. The 2^{nd} step delivers the center Z_2 and axis z_2 as the degenerate pair (Δ_2, ∇_2). We collect the conditions for this in

Result 8. *If two labelled triangles $\Delta = \{A, B, C\}$, $\Delta' = \{A', B', C'\}$ which are not in Desargues position allow a re-labelling such that they are in Desargues position and the homology $\chi : \Delta \mapsto \Delta'$ then in addition is harmonic, then they have a common circumscribed conic and the iteration process described above and applied to the originally labelled pair ends already with step 2.*

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