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Geometrical Shapes Allowing the Construction of the Midpoint of a Segment Using a Straightedge Only

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Abstract. Finding the middle point of a segment is one of the basic constructions for the case when there are restrictions requiring the use of an unmarked ruler (a straightedge) only. When a segment is given with its midpoint, this considerably expands the set of possible constructions using a straightedge only. It is well known that the middle point of a segment cannot be found by using a straightedge only; however, when certain geometrical shapes are given in the plane it is possible to carry out this construction. The paper gives 10 examples of shapes allowing elegant constructions of the midpoint of a certain segment using a straightedge only.

Key Words: geometric constructions, constructions using a straightedge only, midpoint of a segment

MSC 2010: 51M15

1. Introduction

Over hundreds of years geometrical constructions were a fascinating subject that occupied the minds of famous geometers and mathematics enthusiasts. Classic geometrical constructions are done using a straightedge and a compass. By restricting the drawing tools to a straightedge only one considerably reduces the set of possible classical constructions. Nonetheless, when additional geometrical shapes are given in the plane, the set of possible constructions increases, and in certain cases the entire set of classical constructions can be carried out (for example, when a circle with its center is given in the plane [1,2]). Some classical constructions can be carried out when a certain segment AB is given in the plane with its middle point [1]. In this case, one can for example:

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- Draw a straight line parallel to the segment AB through any point,
- Construct the midpoint of any segment that is parallel to AB,
- Construct the diameter of any circle (when its center is not given),
- Construct the point C on the line with segment AB, so that BC = AB,
- Divide the segment AB into n equal parts,
- etc.

However, when the midpoint of the segment is not given, one can find (construct) it using a straightedge only when certain shapes in the plane are available. The present paper identifies some of these shapes, which permit elegant constructions for finding the midpoint of the segment.

The constructions are based on known constructions using a straightedge only, as a reminder we list them with the sources in which the construction is described:

$\mathcal{A}$ Constructing the middle point of two parallel segments [1, problem 14];

$\mathcal{B}$ Constructing a line that is parallel to a segment given with its midpoint, through any point [1, problem 13];

$\mathcal{C}$ Constructing a tangent to a conic section at a certain point on the curve, when 4 additional points of the same conic section are given (five points in total are given) [1, problem 9].

$\mathcal{D}$ Constructing a tangent to a given conic section from any point outside the curve of the section [1, problem 5].

$\mathcal{E}$ Finding another point of intersection of a straight line with the curve of a conic section which is defined by five points and the straight line passes through one of them [1, the note on p. 47].

$\mathcal{F}$ Constructing the ray reflected from a straight line based on the incident ray and the perpendicular to the straight line at the point of incidence [3].

$\mathcal{G}$ Constructing the harmonic point of point C relative to the points A and B for any three points A, B, C that lie on the same straight line [1, problem 1].

Any place where these constructions will be used shall be mentioned in the paper.

2. Geometrical shapes permitting the construction of the midpoint of a segment

Shape 1. *Two non-parallel segments with their midpoints*

Given are the segments AB and CD, which are not parallel, and their midpoints M and N, respectively. Also given is some segment EF whose midpoint we want to construct, as shown in Figure 1.

Description of the construction:

If the segment EF is parallel to one of the segments AB or CD, one can construct its midpoint based on construction $\mathcal{A}$. We therefore consider the case where the segment EF is not parallel to the given segments.

Through the point E we construct a straight line parallel to the segment AB based on construction $\mathcal{B}$.

We do the same for the point F, through which we construct in the same manner a straight

line parallel to CD. Both parallel lines that were constructed intersect at the point G.
Based on construction $\mathcal{A}$, and using the segment AB, we construct the midpoint M' of the segment GE. In the same manner, using the segment CD, we construct the midpoint N' of the segment GF. FM' and EN' are medians in the triangle $\triangle$GEF, whose point of intersection is H.
The continuation of GH intersects the segment EF at the point K which is the midpoint of EF.

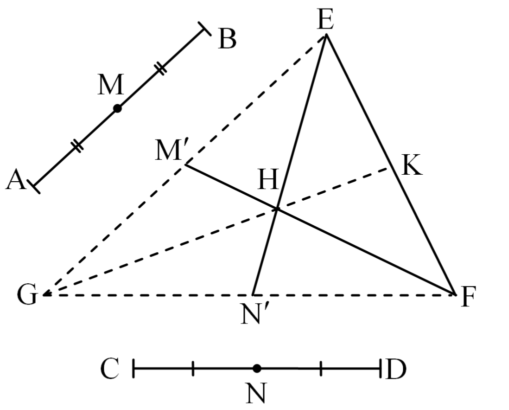

Figure 1: Constructing the midpoint of a segment based on two segments with their midpoints

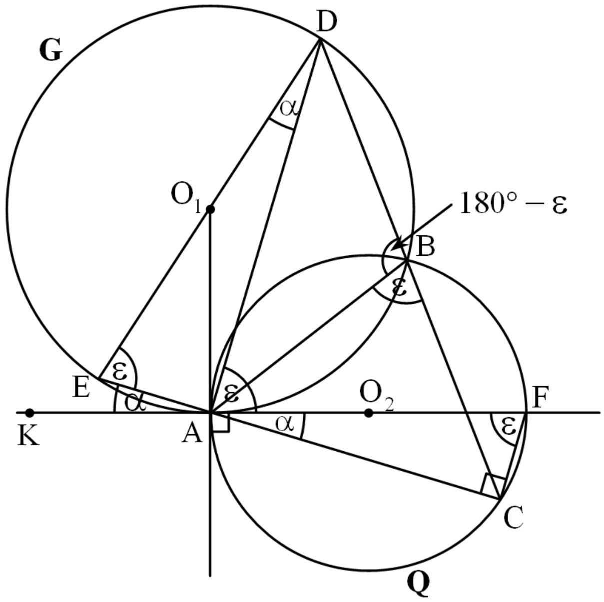

Figure 2: Constructing the midpoint of a segment from two circles intersecting at a right angle

Shape 2. *Two perpendicular circles*

Two circles, G and Q (Figure 2) intersect perpendicularly to each other (in other words, their tangents at the points of intersection are perpendicular to each other). Given are the two points of intersection A and B. Also given are three additional points on G and one additional point C on the circle Q (the circles themselves are not given, and they are shown for illustration purposes only). Using these data one is to construct the midpoint of some segment.

Description of the construction:

Stage a) – data analysis:

Using straight lines, we connect the point C with the points of intersection of the circles A and B, the continuations of the straight lines intersect the circle G at the points D and E.
We first prove that the straight line ED is a diameter in the circle G (Figure 2).

$$\angle \mathrm{KAE} = \angle \mathrm{EDA} = \alpha, \quad \angle \mathrm{FAD} = \angle \mathrm{AED} = \varepsilon, \quad \angle \mathrm{DBA} = 180^\circ - \varepsilon,$$

and therefore $\angle$ABC $= \varepsilon$, and also $\angle$AFC $= \varepsilon$. We have $\angle$ACF $= 90^\circ$ (because AF is the diameter of the circle). Hence $\varepsilon + \alpha = 90^\circ$, and therefore $\angle$EAD $= 90^\circ$, and the chord ED is a diameter.

Stage b) – construction:
Using straight lines, we connect the point C with points A and B. From the construction $\mathcal{E}$ one can find the points of intersection D and E of the straight lines with the circle G.
After obtaining the diameter ED of the circle G, one constructs two tangents to the circle at the points E and D (based on construction $\mathcal{C}$). The tangents are parallel to each other, and based on construction $\mathcal{A}$ one can construct the midpoint of some segment that lies on one of the tangents.

Note 1. As seen, Shape 2 does not permit the finding of the middle of any segment, but rather only the midpoint of a segment perpendicular to the diameter ED. However, if another (fourth) point C' given on the circle Q on the same side of AB as C, then another diameter E'D' of G can be constructed to find the center of the circle G. Thus, we have constructed two segments that are not parallel to each other (ED and E'D') with their midpoint (the center of the circle G), and then one can find the midpoint of any segment in the plane.

Note 2. As an alternative to adding a point C', one can add to the data a tangent at the point C which lies on the circle Q. It is easy to prove that this tangent is parallel to ED, and then to find the midpoint of the segment ED, which is the center of the circle G, and then to construct again the midpoint of any segment in the plane.

Shape 3. *The triangle inscribed in a circle with its orthocenter*

Given is a triangle $\triangle$ABC inscribed in a circle given by five of its points (the points A, B, C and two additional points). The circle itself is not given, and it appears in Figure 3 for illustration purposes.
Also given is the point H — the orthocenter of the triangle.

Description of the construction:
We connect the vertex A with the orthocenter H, and construct the point of intersection D of the straight line AH with the base BC, and the point of intersection E of the straight line AH with the circle (in accordance with construction $\mathcal{E}$). The point D is the midpoint of the segment HD. The proof of the correctness of this property is hinted at using the angle α that appears in Figure 3.
Similarly, one constructs a segment with its midpoint on the straight line BH.
We have now obtained two segments that are not parallel to each other with their midpoint, and one can find the middle of any segment in the plane.

Note 3. Any triangle permits this construction, aside for a right-angled triangle.

Shape 4. *Two intersecting circles*

Given are five points A, B, U, V and W that are located on a single circle, and five points A, B, U', V' and W' that are located on a different circle, so that A and B are the points of intersection of the two circles.
The circles themselves are not given, and they appear in Figure 4 for illustration purposes.

Description of the construction:
Based on the construction $\mathcal{E}$ we construct two straight lines through the points A and B, and find the points of their intersection with the circles (we denote them by D, C, E, F, as shown in Figure 4). It is easy to prove that CD $\parallel$ EF and therefore using construction $\mathcal{A}$ one constructs the midpoint of the chord CD. Now it is easy to draw another straight line through the point B, to find its points of intersection with the circles D' and F', and to obtain another

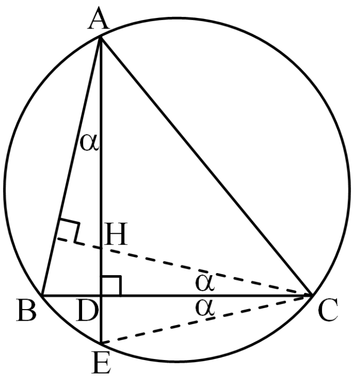

Figure 3: Constructing the midpoint of a segment using a triangle inscribed in a circle with its orthocenter

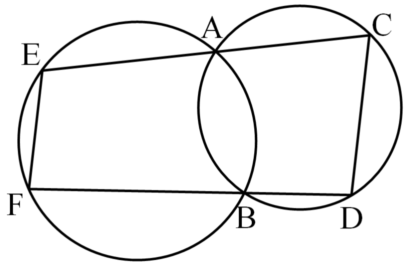

Figure 4: Constructing the midpoint of a segment using two intersecting circles

pair of parallel chords CD' $\|$ EF', to find the midpoint of CD'. Thus we have obtained two non-parallel segments with their middle points. These segments permit construction of the midpoints of any segment.

Shape 5. *Two conic sections with their diameters*

Given are five points A, B, C, D and E that lie on the same conic section G (circle, ellipse or hyperbola), so that the segment AB is the diameter in that section. Also given is another set of five points A', B', C', D', and E', which lie on another conic section Q (circle, ellipse or hyperbola), so that A'B' is the diameter in the second section, AB is not parallel to A'B'.

Description of the construction:

Based on construction $\mathcal{C}$, one draws tangents to the first conic section at the points A and B. Then the tangents are parallel to each other and one can find the midpoint of any segment that lies on one of them. A similar construction can be constructed using the other conic section to obtain the midpoint of another segment that is not parallel to the first segment. So one can construct the midpoint of any segment in the plane.

Shape 6. *Two parabolas with a common vertex and a common axis of symmetry*

Given are two parabolas with a common vertex at the point M, as shown in Figure 5. Also, the parabolas have a common axis of symmetry.

Description of the construction:

Without loss of generality we assume that the vertex of the parabolas is located at the origin (the coordinate system is not shown).
We draw two straight lines through the vertex M and obtain the points A, B, C, D — their intersection with the parabola. We prove that AD $\|$ BC.
Without loss of generality we assume that the equations for the parabolas are: $y = ax^2$, $y = bx^2$ $(a \neq b)$ and the equations of the straight lines are $y = mx$ and $y = nx$ $(m \neq n)$.
We calculate the coordinates of the points A, B, C, D and obtain:

$$\mathrm{A}\left(\frac{m}{a}, \frac{m^2}{a}\right), \quad \mathrm{B}\left(\frac{m}{b}, \frac{m^2}{b}\right), \quad \mathrm{C}\left(\frac{n}{b}, \frac{n^2}{b}\right), \quad \mathrm{D}\left(\frac{n}{a}, \frac{n^2}{a}\right),$$

from here one can show that the slope of the straight line AD is $m + n$ and the slope of the straight line BC is also $m + n$, implying that AD $\|$ BC.

Therefore using construction $\mathcal{A}$ one can find the midpoint of AD. In the same manner one can draw another straight line through M, which intersects the parabolas are the points C' and D', and then find the midpoint of segment AD'.

Note 4. It can be proven that the graphs of any two functions of the form $y = ax^n$ and $y = bx^n$ $(n \neq 1)$ are suitable for obtaining a pair of parallel segments in a manner similar to that described in the construction above.

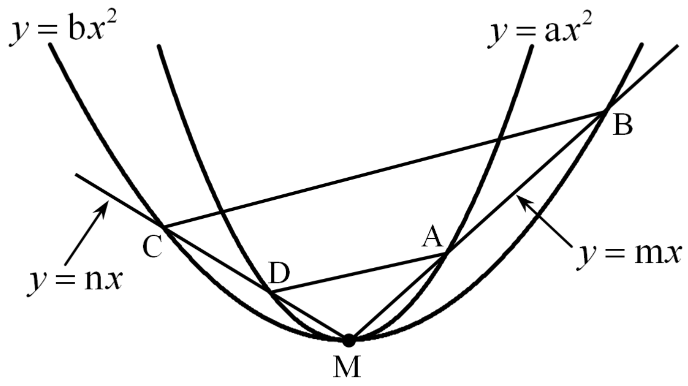

Figure 5: Constructing the midpoint of a segment using two parabolas with a common vertex and a common axis of symmetry

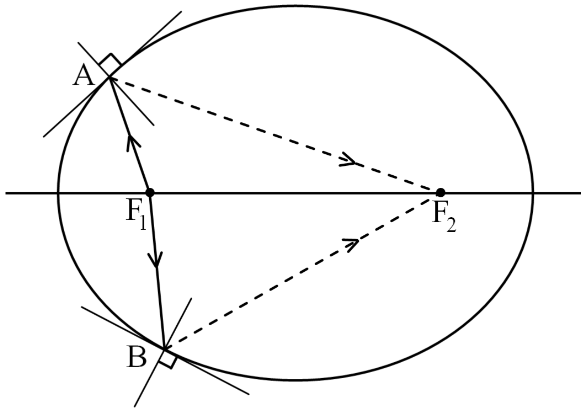

Figure 6: Constructing the other focus of an ellipse using two normals

Shape 7. *Ellipse with two normals to it*

Given is an ellipse with its left focus F_1. In addition, the normals to the ellipse at two points A and B that lie on the ellipse are also given, as shown in Figure 6.

Description of the construction:

Using construction $\mathcal{C}$ we construct the tangents to the ellipse at the points A and B. We connect the point F_1 with the points A and B by straight lines and consider F_1 A and F_1 B as rays impinging on the straight lines.
Using construction $\mathcal{F}$ we construct the reflected rays.
Based on the well-known property that "the normal to the ellipse bisects the angle between the straight lines that connect the point of tangency with its foci", the reflected ray passes through the second (right) focus.
Hence the point of intersection of the two reflected rays is the right focus F_2. We draw the straight line through the two foci to obtain the large axis of the ellipse, as shown in Figure 7.
We select some point C on one side of the continuation of the large axis, and from this point, using construction $\mathcal{D}$, we construct the tangents to the ellipse to obtain the two points of tangency D_1 and D_2. The segment D_1D_2 is bisected by the axis of the ellipse to obtain its midpoint M.
Using the segment D_1D_2 and its midpoints M one can construct using construction $\mathcal{B}$ two parallel straight lines that pass through the foci of the ellipse (Figure 8). The straight lines intersect the ellipse at the points G_1, G_2, G_3, G_4, to form a rectangle. By drawing the diagonals of the rectangles that intersect at the point O, one obtains two additional segments with their midpoints.

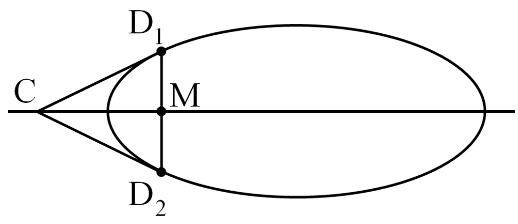

Figure 7: Constructing tangents to an ellipse from a point on its axis

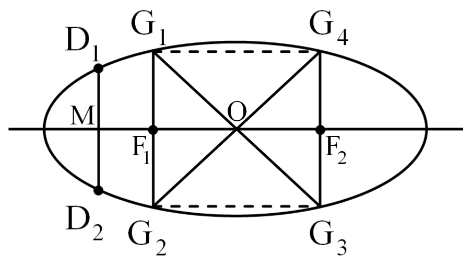

Figure 8: Constructing perpendiculars to the axis of an ellipse through its foci

Shape 8. *Hyperbola with two normals to it*

Given is a hyperbola with its left focus F_1 and two points A and B on its left branch. Also given are the normals to the hyperbola at these points, as shown in Figure 9.

Description of the construction:

Using construction $\mathcal{C}$ we construct the tangents to the hyperbola and the points A and B. We connect the point F_1 with the points A and B using straight lines and consider F_1 A and F_1 B as rays that impinge on the tangents. Using construction $\mathcal{F}$ we construct the reflected rays.

Based on the well-known property that "the tangent to the hyperbola bisects the angle between the straight lines that connect the point of tangency with its foci", the reflected ray passes through the second (right) focus.

Thus we have constructed the second focal point F_2. Now we draw the straight line through the two foci of the hyperbola. On the continuation of the straight line we choose some point C on one side as shown in Figure 10. From this point, by construction $\mathcal{D}$, we construct the tangents to the hyperbola, and obtain two points of tangency: D_1 and D_2. The segment D_1D_2 is bisected by the line F_1F_2 and the midpoint M of D_1D_2 is obtained. Using the segment D_1D_2 and its midpoint M, and through construction $\mathcal{B}$, one can construct two parallel lines that pass through the foci of the hyperbola, as shown in Figure 11. The parallel lines intersect the hyperbola at the points G_1, G_2, G_3, G_4, which form a rectangle. By drawing the diagonals of the rectangle that intersect at the point O, one obtains two additional segments with their midpoints.

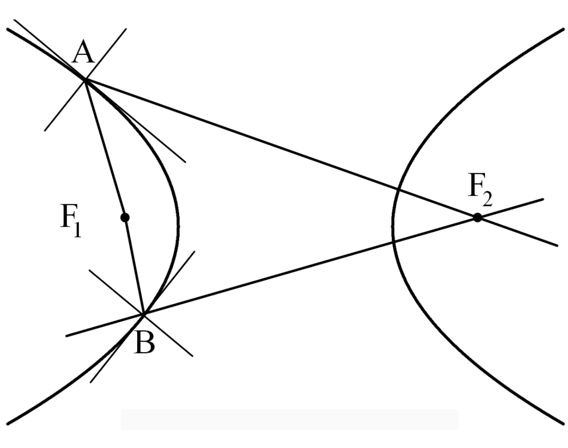

Figure 9: Constructing the second focus of a hyperbola using two normals

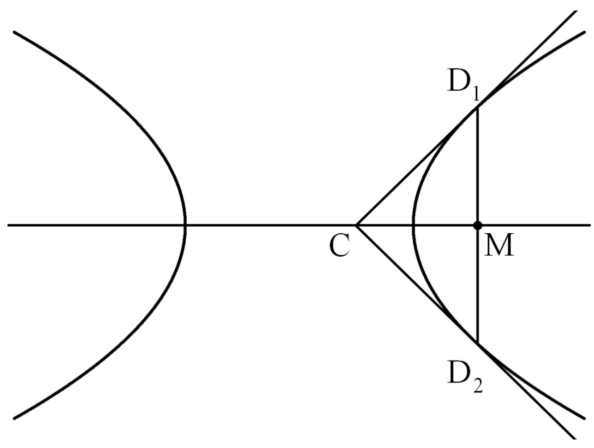

Figure 10: Constructing tangents to a hyperbola from a point on its axis

Shape 9. *Parabola with two normals to it*

Given is a parabola with its focus F and two normals to the parabola at two points A and B on it.
The symmetry axis and the vertex are not given, as shown in Figure 12.

Description of the construction:

We construct the tangents to the parabola at the points A and B using construction $\mathcal{C}$.
Then we connect the point F with the points A and B using straight lines.
We consider FA and FB to be rays impinging on the tangents to the parabola, and using the normals at the points A and B we construct the reflected rays using construction $\mathcal{F}$ (as shown in Figure 12).
The rays reflected from the points A and B are parallel to the axis of symmetry of the parabola, from a well-known geometric property. Hence the two rays reflected are parallel to each other. Therefore one can select a segment on one of them and find its midpoint. Now using construction $\mathcal{B}$ one can draw the straight line parallel to the reflected rays through the focus F. This line is the axis of symmetry of the parabola and its point of intersection with the parabola is the vertex of the parabola.
At some location on the axis of symmetry outside the parabola we select the point C and construct from it two tangents to the parabola (based on construction $\mathcal{D}$). We obtain the points of tangency D and E. The segment that connects them is perpendicular to the axis of symmetry of the parabola and is bisected by it. Thus we obtained another segment with its midpoint.

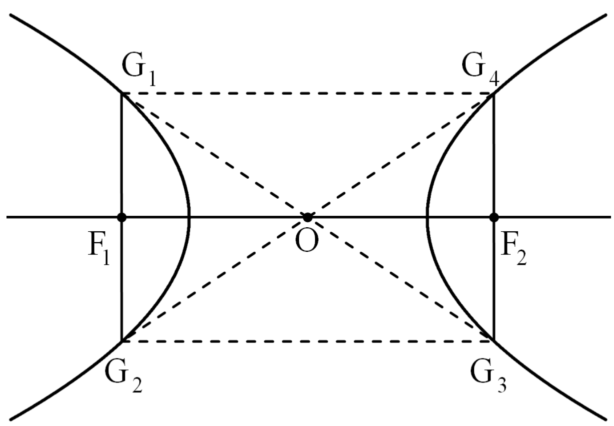

Figure 11: Constructing perpendiculars to the axis of the hyperbola through its foci

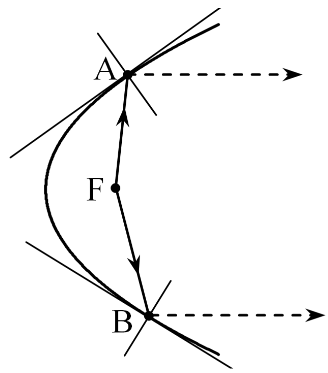

Figure 12: Constructing the midpoint of a segment using a parabola with two normals

Shape 10. *Two segments divided by a given ratio*

Given are two segments, AB and A'B', which are not parallel, and the points C and C' on them, so that $\frac{\mathrm{AC}}{\mathrm{CB}} = m$ and $\frac{\mathrm{A'C'}}{\mathrm{C'B'}} = n$ (m and n are natural numbers larger than 1).

Description of the construction:

We construct the harmonic point C_1 to C relative to A and B (construction $\mathcal{G}$), as shown in Figure 13.
Then $\frac{\mathrm{AC}}{\mathrm{CB}} = \frac{\mathrm{AC}_1}{\mathrm{C}_1\mathrm{B}} = m$, and hence it follows that $\frac{\mathrm{AC}_1 - \mathrm{C}_1\mathrm{B}}{\mathrm{C}_1\mathrm{B}} = m - 1$. Since $\mathrm{AC}_1 - \mathrm{C}_1\mathrm{B} = \mathrm{AB}$, we have $\frac{\mathrm{AB}}{\mathrm{BC}_1} = m - 1$. Now we obtained a segment AC_1 that is divided by the point B in such

a manner that $\frac{\mathrm{AB}}{\mathrm{BC}_1} = m - 1$. Similarly, one can construct the point C_2 that is harmonic to B relative to A and C_1, so that $\frac{\mathrm{AC}_1}{\mathrm{C}_1\mathrm{C}_2} = m - 2$. When we continue the constructions in a similar manner $m - 1$ times, we obtain the point C_{m-1}, such that $\frac{\mathrm{AC}_{m-2}}{\mathrm{C}_{m-2}\mathrm{C}_{m-1}} = m - (m - 1) = 1$, in other words the point C_{m-2} shall be the middle point of the segment AC_{m-1}.
In a similar manner, one can construct a segment with its middle point on the straight line A'B'.

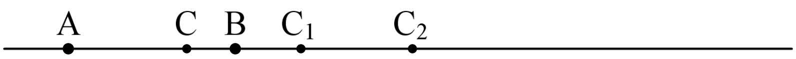

Figure 13: Two segments divided by a given ratio

Note 5. When m and n are equal to 1, we obtain Shape 1 considered above.

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