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The Splitters and Equalizers of Triangles

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Abstract. The splitters of a triangle are the lines that bisect its perimeter and the equalizers are those lines that bisect both its perimeter and area. In recent studies, it is proved that a triangle can have either one, two or three equalizers that pass through its incenter. The studies, mainly, concentrate on the existence of the equalizers. Our approach, in this article, is more elementary and algebraic in terms of the side-lengths $c \geq a \geq b$ of $\triangle ABC$ and it provides a comprehensive overview on the equalizers of the triangle. It is based on the fact that a cevian from a vertex and through the *Nagel center* is a splitter. So if, say AA' is a *Nagel* splitter, then a line joining two points, M of $A'C$ and N of AC' is an equalizer if and only if $A'M = AN = x$ and $2x^2 + (a + b - 3c)x - c(b - c) = 0$. So, by finding all possible solutions, we proved that every triangle can have either one, two or three equalizers, their distribution and locations on the sides are determined, and their geometric construction by compass and ruler is shown. A summary of these results is given in the conclusions section and to make these results more feasible, a visual diagram that predicts the number of equalizers according with the side-length is drawn. For a scalene $\triangle ABC$, we proved that there are no equalizers that cut the smallest two sides, there is only one equalizer cutting the smallest and largest sides, and a maximum of two equalizers that cut the largest two sides.

Key Words: Nagel center, splitter, Nagel splitter, cleaver, equalizer
MSC 2020: 51M04

1 Preliminaries

The problem of bisecting a triangle by a line into two polygons having equal areas or having equal perimeters has been of interest by mathematicians for some time; see [2, 5–7]. So, a splitter is a line that bisects the perimeter of a triangle, a *Nagel* splitter is a splitter through the *Nagel* center, and a cleaver is a splitter that joins the midpoint of one side and the point that bisects the broken chord of the other two sides. An equalizer is a splitter that bisects its area.

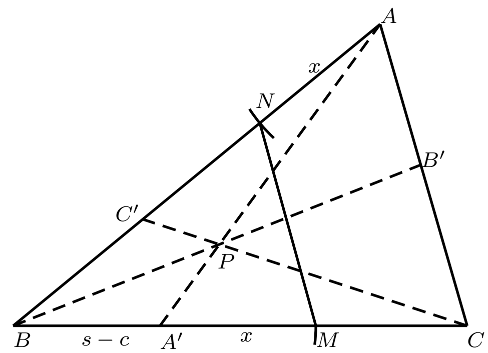


Figure 1: Construction of a general splitter

A well known special splitters are the three cleavers that are attributed to *Archimedes* in his *Broken Chord Theorem* and each joins the midpoint of one side and the point that bisects the broken chord of the other two sides. The three cleavers intersect at the incenter of the medial triangle of $\triangle ABC$, see [7].

Another known special splitters are the three Cevians AA' BB', and CC' that intersect at the *Nagel center* where A', B', and C', are the points of contact of the three excircles of $\triangle ABC$ with the sides BC, AC, and AB, respectively, see [7].

In what follows, let AA', BB', CC' be the *Nagel* splitters and let the side-lengths of $\triangle ABC$ be $c \geq a \geq b$.

A more general type of splitters is obtained by taking a point M of, say $A'C$ and let the length of $A'M = x$. Then we construct by compass the line segment AN of the side AB that has the same length x of $A'M$, as seen in Figure 1. Therefore MN is a splitter. Since $0 \leq x \leq A'C = s - b$, $s = \frac{c+a+b}{2}$, there are infinite number of splitters from $A'C$ to AC'. Note that $MN = AA'$ when $x = 0$ and MN is a cleaver when M is the midpoint of BC.

2 Equalizers

Most recent studies on equalizers have proved that every equalizer of $\triangle ABC$ passes through its incenter and that every triangle can have either one, two or three equalizers by using the concept of an envelope(a curve tangent) to a family of lines that bisect the area of a triangle and that the number of equalizers depends on the location of the incenter with respect to the regions bounded by three hyperbolas and the three medians, see [4, 10], or to rotate a line through the incenter from a normal to an angle bisector and to spot the positions for which the line bisects the area of $\triangle ABC$, see [8]. It is worth mentioning here that there is a kind of similarity in these two studies; in one the three hyperbolas, three medians, and the incenter played an essential role while in the second study the three angle bisectors, three normal lines, and the incenter played a similar role to prove the existence of either one, two, or three equalizers. We will see also in our study that the three *Nagel* spliters and the three medians will play a basic role. Other approaches and generalizations appeared in [1, 3, 9].

Before proceeding with our search for equalizers of $\triangle ABC$, let $c \geq a \geq b$ denote its side-lengths, r the inradius, and $[*]$ the area of any polygon. Let E, F, H be the midpoints of BC, CA, AB, respectively, and AA', BB', CC' be the *Nagel* splitters of $\triangle ABC$. Then it

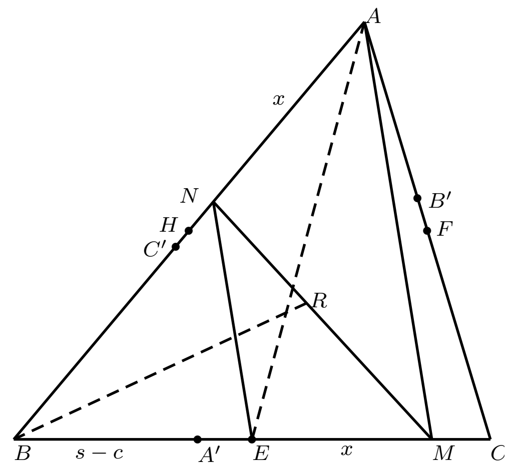


Figure 2: Illustrating the proof of Lemma 1

is clear that

$$BA' = AB' = s - c, \quad CB' = BC' = s - a, \quad \text{and} \quad AC' = CA' = s - b$$
$$\text{where} \quad s = \frac{a+b+c}{2}, \quad A' \text{ lies in } BE,\ B' \text{ in } FA,\ C' \text{ in HB.} \tag{1}$$

So, let MN be a general splitter of $\triangle ABC$ from $A'C$ to AC' such that $A'C = AC' = x$ as shown in Figure 2. Then it follows from (1) that the midpoints E, H lie on the segments $A'C$, AC', respectively. Since AE and CH are medians and they bisect $[\triangle ABC]$, it follows that the splitter MN is an equalizer if and only if M is a point of EC, N is a point of AH, and $[\triangle BMN] = [\triangle BAE]$. Thus MN is an equalizer if and only if $AM \parallel NE$. But $AM \parallel NE$ if and only if $\frac{BN}{AN} = \frac{BE}{ME}$ and since $BN = c - x$, $BE = \frac{a}{2}$, $BM = s - c + x$ and $EM = BM - BE$, we have $EM = \frac{2x+b-c}{2}$. Therefore a splitter

$$MN \text{ is an equalizer} \quad \Longleftrightarrow \quad 2x^2 + (a+b-3c)x - c(b-c) = 0, \quad 0 < x < s - b.$$

Next, we show that a splitter MN is an equalizer if and only if MN passes through the incenter of $\triangle ABC$. So let an angle bisector, say BR meet MN at R, the distances from R to BA, BC be h and from R to AC be k. Then $[\triangle ABC] = 2[\triangle BMN] = (BN + BM)h = \frac{(a+b+c)h}{2}$. But also, $[\triangle ABC] = [\triangle RAB] + [\triangle RBC] + [\triangle RCA] = \frac{(c+a)h}{2} + \frac{bk}{2}$. Therefore $\frac{(a+b+c)h}{2} = \frac{(a+c)h+bk}{2}$ and hence $h = k$ and R is the incenter of $\triangle ABC$.

Also, conversely let MN be a splitter of $\triangle ABC$ that passes through the incenter R. Then

$$[\triangle ABC] = \frac{(a+b+c)r}{2} \quad \text{and} \quad [\triangle BMN] = \frac{(BM+BN)r}{2}. \text{ But } BM + BN = \frac{(a+b+c)}{2}.$$

$$\text{Therefore} \quad [\triangle BMN] = \frac{(a+b+c)r}{4} \quad \text{and} \quad [\triangle ABC] = 2[\triangle BMN]. \text{ Thus } MN \text{ is an equalizer.}$$

Thus we have proved:

Lemma 1. *Let E, F, H be the midpoints of the sides BC, AC, AB of $\triangle ABC$, respectively and let MN be a general splitter of the $\triangle ABC$ that joins, say M of EC, N of AH, and let $c \geq a \geq b$, $A'M = x = AN$. Then*

(i) MN is an equalizer of $\triangle ABC$ if and only if

$$AM \parallel NE \iff CN \parallel MH \iff 2x^2 + (a+b-3c)x - c(b-c) = 0,\ 0 < x \leq s - b \tag{2}$$

(ii) MN *is an equalizer* $\iff$ MN *passes through the incenter* R *of* $\triangle ABC$. (3)

Next, by using this basic Lemma, we proceed searching for the number of equalizers of a scalene $\triangle ABC$ and their distribution and exact locations on its sides.

2.1 Searching for Equalizers of Scalene Triangles

Let $\triangle ABC$ be a scalene triangle such that $c > a > b > 0$ and E, F, H be the midpoints of BC, AC, AB, respectively. Then it follows from (1) that

$$BA' - BE = s - c - \frac{a}{2} = \frac{b-c}{2}, \quad CB' - CF = s - a - \frac{b}{2} = \frac{c-a}{2} \quad \text{and}$$
$$AC' - AH = s - b - \frac{c}{2} = \frac{a-b}{2}. \tag{4}$$

(i) First, we show that there is only one equalizer cutting the largest and smallest sides. So, referring to Figure 3(a), let MN be an equalizer of $\triangle ABC$ from the point M of the segment $C'B$ to the point N of the segment CF such that $C'M = CN = x < CF = \frac{b}{2}$. Then by (2) of Lemma 1 and the permutation (a, c, b), we have

$$2x^2 + (c + a - 3b)x - b(a - b) = 0 \quad \text{and} \quad 0 < x < \frac{b}{2}. \text{ Hence}$$
$$x = \frac{(3b - c - a) \mp \sqrt{(3b-c-a)^2 + 8b(a-b)}}{4}. \quad \text{But } a > b; \text{ so,}$$
$$\sqrt{(3b-c-a)^2 + 8b(a-b)} > |3b - c - a|,$$

and hence

$$x > 0 \iff x = \frac{(3b - c - a) + \sqrt{(3b-c-a)^2 + 8b(a-b)}}{4} \quad \text{and}$$
$$x < \frac{b}{2} \iff (3b - c - a)^2 + 8b(a - b) < (2b - (3b - c - a))^2$$
$$\iff 8b(a - b) < (c + a - b)^2 - (3b - c - a)^2 = 4b(a + c - 2b)$$
$$\iff a + c - 2b - 2a + 2b = c - a > 0.$$

But $c > a$. Thus we conclude that: There is only one equalizer MN cutting the largest and smallest sides AB, AC such that

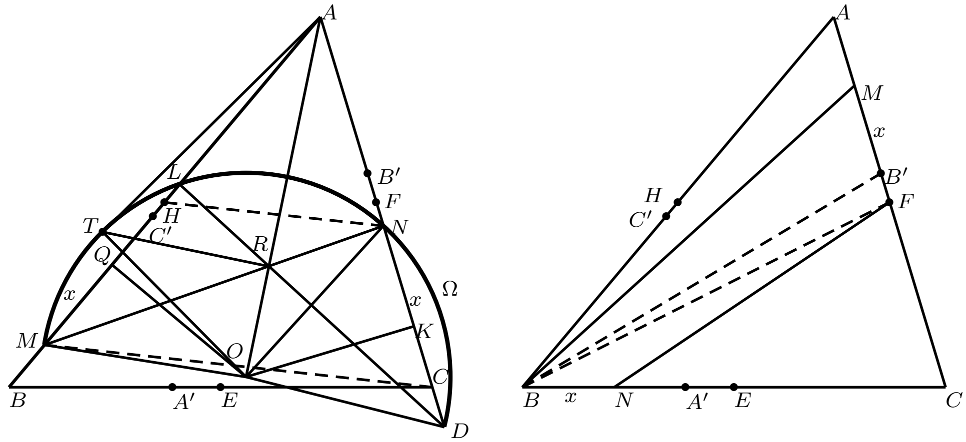
$$x = C'M = CN = \frac{(3b - c - a) + \sqrt{(3b-c-a)^2 + 8b(a-b)}}{4} < s - a, \quad \text{and} \quad NH \parallel CM \tag{5}$$

as required.

2.2 Geometric Construction of *MN* in (i) by Compass and Ruler

Since

$$AM - AN = (s - b + x) - (b - x) = \frac{a + c - b + 2x - 2b + 2x}{2}$$
$$= \frac{a + c - 3b + 4x}{2} = \frac{\sqrt{(3b-c-a)^2 + 8b(a-b)}}{2} > 0,$$



(a) (b)

Figure 3: Construction of MN

we have $AM > AN$. Thus by angle bisector theorem we get that $RM > RN$. So, by reflecting the $\triangle ANM$ about the angle bisector AR, we get $\triangle ALD \cong \triangle ANM$. Therefore $[\triangle ANM] = [\triangle ALD] = \frac{1}{2}[\triangle ABC]$. But we just proved that there is only one equalizer that cuts internally AB and AC. So, LD intersects AC produced at D as shown in Figure 3(a) and it is clear that $RM > RN = RL < RD$, $\angle LDN = \angle NML < 90°$ and hence the quadrilateral $NLMD$ is cyclic,

$$AN = AL, \quad ND = LM = AM - AN = \frac{\sqrt{(3b-c-a)^2 + 8b(a-b)}}{2}$$

and the perpendicular bisectors QO and KO of LM, ND, respectively meet the angle bisector AR produced at the center O of the circle Ω that passes through N, D, M, L. But

$$AK = AQ = AN + \frac{ND}{2} = b - x + \frac{\sqrt{(3b-c-a)^2 + 8b(a-b)}}{4} = \frac{4b - (3b-c-a)}{4} = \frac{s}{2}.$$

Therefore Q is constructed as the midpoint of AB produced by the length of BA' and then QO is constructed as the perpendicular to AB at Q and meets AR produced at center O. Since $RD = RM, OD = OM$, we have $\angle ODR = \angle OMR$. But $ON = OM$. So, $\angle OMR = \angle ONR$ and hence $\angle ODR = \angle ONR$ and the quadrilateral $ODNR$ is cyclic and by symmetry $OMLR$ is cyclic. So, by applying *Euclid's* proposition 36 of Book III, on the cyclic quadrilaterals $LMDN$ and $ROML$ we get $(AT)^2 = (AM)(AL) = (AO)(AR)$. But $(AT)^2 = (AO)^2 - (OT)^2$. Therefore $(OT)^2 = (AO)^2 - (AO)(AR) = (OA)(OR)$ and hence A and R are inverse points with respect to the inversion circle Ω. Since $\angle ATO = 90°$ and $(OT)^2 = (OA)(OR)$, it follows by the inverse of *Euclid's* proposition 36 of Book III that OT is tangent to the circumcircle of $\triangle ART$ and hence $\angle OTR = \angle TAR$ by the alternate segment theorem. Thus $RT \perp AO$. So, the radius of Ω is constructed by drawing a semicircle with diameter AO and then a perpendicular RT to AO that meet the semicircle at T. Thus the circle Ω with center O and radius OT will intersect the sides AB and AC at the endpoints M, N of the equalizer MN.

So, the geometric construction of MN by compass and ruler is complete.

(ii) Next, we show that there are no equalizers cutting the smallest two sides CA, CB. So, referring to Figure 3(b), let M be any point of the segment $B'A$ and N be a point of the segment BA' such that $B'M = BN = x$. We claim that $NF \nparallel BM$. For $CF = \frac{b}{2}, FM =$

$FB' + x, CN = \frac{a}{2} + EN$, and $NB = x$. Therefore $\frac{CF}{FM} = \frac{b/2}{FB'+x}$ and $\frac{CN}{NB} = \frac{a/2+EN}{x}$. But $\frac{b}{2} < \frac{a}{2} < \frac{a}{2} + EN$ and $FB' + x > x$. Thus $\frac{CF}{FM} < \frac{CN}{NB}$ and hence $NF \nparallel BM$ for every $0 < x < s - c$. So, we conclude that:

$$\text{There are no equalizers cutting the smallest two sides } AC \text{ and } BC. \tag{6}$$

(iii) Finally, we show that there are either no equalizers, one equalizer, or two equalizers cutting the largest two sides AB, BC and we show how these equalizers can be geometrically constructed by compass and ruler. So, referring to Figure 4, let MN be an equalizer of $\triangle ABC$ joining M from the segment EC to N from the segment AH. Then it follows from (2) of Lemma 1 that

$$2x^2 + (a + b - 3c)x - c(b - c) = 0 \quad \text{and} \quad 0 < x < s - b = \frac{c + a - b}{2}.$$

Hence

$$x = \frac{(3c - a - b) \mp \sqrt{(3c - a - b)^2 - 8c(c - b)}}{4}.$$

Since $c > a > b$, we have $3c - a - b > 0$. So,

$$x > 0 \iff (3c - a - b)^2 - 8c(c - b) \geq 0 \iff \text{ there exists an } a \text{ such that}$$
$$0 < c - b < a \leq \text{ minimum of } (3c - b - \sqrt{8c(c - b)}, c).$$

But

$$c - b < 3c - b - \sqrt{8c(c - b)} \iff 8c(c - b) - 4c^2 = 4c(c - 2b) < 0 \iff 2b > c,$$

and

$$3c - b - \sqrt{8c(c - b)} \geq c \iff (2c - b)^2 - 8c(c - b) = b^2 + 4bc - 4c^2 \geq 0$$
$$\iff b \geq 2(\sqrt{2} - 1)c.$$

Thus

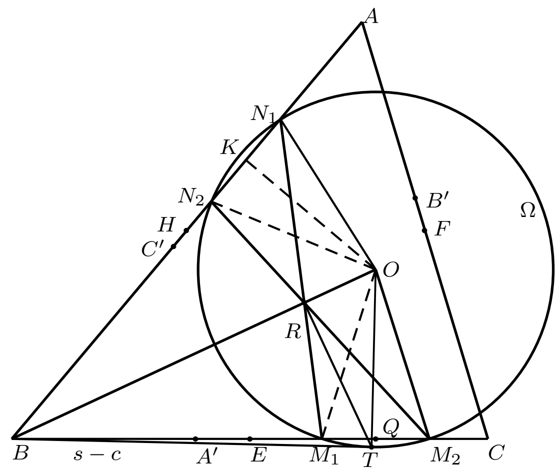
$$x > 0 \quad \text{when} \quad \frac{c}{2} < b < 2(\sqrt{2} - 1)c \quad \text{and} \quad b < a \leq 3c - b - \sqrt{8c(c - b)} < c$$
$$\text{or when} \quad 2(\sqrt{2} - 1)c \leq b < a < c.$$

So, let

$$x_1 = \frac{(3c - a - b) - \sqrt{(3c - a - b)^2 - 8c(c - b)}}{4},$$
$$x_2 = \frac{(3c - a - b) + \sqrt{(3c - a - b)^2 - 8c(c - b)}}{4}.$$

Then $0 < x_1 \leq x_2$ for every $c > a > b$ such that $\frac{c}{2} < b < 2(\sqrt{2} - 1)c$ and $b < a \leq 3c - b - \sqrt{8c(c - b)} < c$ or when $2(\sqrt{2} - 1)c \leq b < a < c$. Next, we show that

$$s - b - x_2 = \frac{2(a + c - b)}{4} - \frac{(3c - a - b) + \sqrt{(3c - a - b)^2 - 8c(c - b)}}{4} > 0.$$



(c)

Figure 4: Illustrating the proof of (iii-2)

Since $a > b > \frac{c}{2}$, it follows that $2(a+c-b)-(3c-a-b) = 3a-b-c > 0$. Therefore,

$$(3a-b-c))^2-(3c-a-b)^2+8c(c-b) = 8(a+c-b)(a-c)+8c(c-b) = 8a(a-b) > 0,$$

and hence $x_1 \le x_2 < s-b$.

Thus we conclude that:

(iii-1) There is one equalizer MN cutting the largest two sides when $x = x_1 = x_2$, $b < a = 3c-b-\sqrt{8c(c-b)} < c$, $\frac{c}{2} < b < (2\sqrt{2}-2)c$, $A'M = AN = x = \sqrt{\frac{c(c-b)}{2}}$,

(iii-2) There are two equalizers M_1N_1, M_2N_2, when $A'M_1 = x_1 < x_2 = A'M_2$, $b < a < 3c-b-\sqrt{8c(c-b)} < c$, $\frac{c}{2} < b < (2\sqrt{2}-2)c$, or when $2(\sqrt{2}-1)c \le b < a < c$ where

$$x_1, x_2 = \frac{(3c-a-b) \mp \sqrt{(3c-a-b)^2-8c(c-b)}}{4},$$

(iii-3) There are no equalizers cutting the largest two sides when $b \le \frac{c}{2}$, or $\frac{c}{2} < b < (2\sqrt{2}-2)c$ and $c > a > 3c-b-\sqrt{8c(c-b)} > b$.

2.3 Geometric Construction of Equalizers by Compass and Ruler

(1) Since the one equalizer MN in (iii-1) passes through the incenter R and

$$BM = BA' + A'M = s-c+x = \frac{a+b-c}{2} + \sqrt{\frac{c(c-b)}{2}}$$

and $a = 3c-b-\sqrt{8c(c-b)}$, it follows that

$$BM = c - \sqrt{\frac{c(c-b)}{2}} = c - x = BN$$

and $MN \perp BR$. Thus MN is constructed by drawing the perpendicular to the angle bisector BR at the incenter R.

(2) Construction of the two equalizers in (iii-2). Since $BM_1 = s - c + x_1$, $BN_2 = c - x_2$, $x_1 + x_2 = \frac{3c-a-b}{2}$, we have $BM_1 - BN_2 = s + x_1 + x_2 - 2c = 0$. Thus $BM_1 = BN_2 < BN_1$ and similarly $BN_1 = BM_2$. But M_1N_1 and M_2N_2 intersect at the incenter R. Thus M_2N_2 is the reflection of N_1M_1 with respect to the angle bisector BR of $\angle B$. Therefore

$$RM_1 = RN_2, \quad RN_1 = RM_2, \quad \triangle RM_2M_1 \cong \triangle RN_1N_2, \quad \angle M_1N_1N_2 = \angle N_2M_2M_1. \tag{7}$$

Since $\angle M_1N_1N_2 = \angle N_2M_2M_1$, it follows that the quadrilateral $M_1M_2N_1N_2$ is cyclic. Let Ω be the circle passing through M_1, M_2, N_1, N_2. Next, we show that the center O of Ω lies on BR produced and the circle can be constructed. Since $BM_1 < BN_1$, it follows by the angle bisector theorem that $RM_1 < RN_1$. But $RM_1 = RN_2, RN_1 = RM_2$ by (7). So, $RM_1 < RM_2$, $RN_2 < RN_1$ and hence the angles RM_2M_1 and RN_1N_2 are equal and acute. Thus the perpendicular bisectors of the segments M_1M_2, N_1N_2 meet BR produced at the center O of Ω, as seen in Figure 4. Let Q, K be the midpoints of M_1M_2, N_1N_2, respectively. Then

$$M_1M_2 = x_2 - x_1, \quad BM_1 = s - c + x_1, \quad BQ = BK = s - c + x_1 + \frac{x_2 - x_1}{2} = s - c + \frac{x_2 + x_1}{2}.$$

Thus $BQ = BK = \frac{c+a+b}{4}$. So, the center O of Ω can be constructed, by compass and ruler, by producing BA by the length of AB' to get a length s whose midpoint is K. Then the perpendicular KO to BK meets BR produced at the center O of Ω. Next, we show also that its radius can be constructed. Note that $OM_1 = ON_2 = OM_2$. Thus $\angle OM_1R = \angle ON_2R = \angle OM_2R$ and hence the quadrilateral OM_2M_1R is cyclic and similarly ON_1N_2R is also cyclic. Let BT be a tangent to the circle Ω. Then

$$(BT)^2 = (BM_1)(BM_2) = (BR)(BO) = (BN_2)(BN_1) \quad \text{and} \quad (BT)^2 = (OB)^2 - (OT)^2.$$
$$\text{So} \quad (OT)^2 = (OB)^2 - (BR)(BO) = (OB)(OB - BR) = (OR)((OB). \tag{8}$$

Therefore B and R are inverse points with respect to the inversion circle Ω. So, as in Section 2.2, the circle Ω with center O and radius OT can be constructed by compass and ruler and the points of intersection of Ω with the sides BC and BA are the endpoints of two equalizers M_1N_1 and M_2N_2.

Thus the geometric construction of the equalizers of scalene triangles by compass and ruler is complete.

2.4 Searching for Equalizers of Isosceles Triangles

Let $\triangle ABC$ be an isosceles triangle such that $BC = a = c = BA$ and E, F, H are the midpoints of BC, AC, AB, respectively. Then

$$BF \text{ is a equalizer}, \quad s = \frac{2a+b}{2}, \quad s - a = s - c = \frac{b}{2}, \quad s - b = \frac{2a-b}{2}, \quad \text{and} \quad b < 2a = 2c. \tag{9}$$

(i) First, we search for equalizers of triangles with side-lengths $a = c > b$ and refer to Figure 5(a) and prove that:

(i-1) The median BF is the only equalizer cutting AC. For if MN is an equalizer, say from FC to BC', then we have by Lemma 1 that $FM = BN$ and $NF \parallel BM$. But $AN > AH > AF$. Thus $\frac{AN}{BN} > \frac{AF}{FM}$ and hence $NF \nparallel BM$, contradicting the assumption. Also, by symmetry, there are no equalizers from $A'B$ to AF.

(ii-1) There are at most two equalizers from from HA to CE. Let MN be an equalizer from HA to CE. Then by Lemma 1 and (9) we have, $C'M = CN = x$, $0 < x < s-b = \frac{2a-b}{2}$, $CM \parallel NH$. Therefore $\frac{BH}{BM} = \frac{BN}{BC}$. But $BM = s-a+x = \frac{b+2x}{2}$, $BN = a-x$ and hence $\frac{a}{b+2x} = \frac{a-x}{a}$. So, we have $2x^2 + (b-2a)x + a^2 - ab = 0$. Thus

$$x_1, x_2 = \frac{2a - b \mp \sqrt{b^2 + 4ab - 4a^2}}{4}.$$

But $a = c > b > 0$, and

$$b^2+4ab-4a^2 < 0 \iff b < (2\sqrt{2}-2)a \quad \text{and} \quad b^2+4ab-4a^2 \geq 0 \iff b \geq (2\sqrt{2}-2)a.$$

Therefore there are no equalizers from HA to CE if and only if $b < (2\sqrt{2}-2)a < a = c$ and hence an isosceles $\triangle ABC$ such that $c = a > b$

$$\text{has one equalizer the median } BF \iff 0 < b < 2(\sqrt{2}-1)c < a = c. \tag{10}$$

Thus

$$b^2 + 4ab - 4a^2 = (2a-b)^2 - 8a(a-b) > 0 \iff b > 2(\sqrt{2}-1)a, \quad s - b = \frac{2a-b}{2},$$
$$\text{and} \quad (4(s-b) - 4x_2) = (2a-b) - \sqrt{(2a-b)^2 - 8a(a-b)}.$$

So, if $b > 2(\sqrt{2}-1)a$, then

$$x_1 < x_2 < s-b, \quad x_1 + x_2 = \frac{2a-b}{2} = s - b = C'A = CA',$$
$$AM_1 = x_2 = CN_2, \quad AM_2 = x_1 = CN_1.$$

Therefore a $\triangle ABC$ such that $a = c > b$ has two equalizers from HA to CE if and only if $b > 2(\sqrt{2}-1)c$ and hence such a $\triangle ABC$ has

$$\begin{aligned}&\text{three equalizers the median } BF \text{ and } M_1N_1,\ M_2N_2 \iff 2(\sqrt{2}-1)c < b < a = c,\\ &\text{where} \quad x_1 = C'M_1 = \tfrac{2a-b-\sqrt{b^2+4ab-4a^2}}{4}, \quad x_2 = C'M_2 = \tfrac{2a-b+\sqrt{b^2+4ab-4a^2}}{4},\\ &\text{and } M_2N_2 \text{ is the reflection of } M_1N_1 \text{ with respect to } BF.\end{aligned} \tag{11}$$

Note also that if $c = a > b = 2(\sqrt{2}-1)c$, then $x_1 = x_2 = \frac{2a-b}{4} = \frac{C'A}{2} = \frac{CA'}{2}$ and hence there is only one equalizer MN from HA to CE where M, N are the midpoints of $C'A$, CA', respectively. Thus a $\triangle ABC$ such that $a = c > b$ has

$$\begin{aligned}&\text{two equalizers the median } BF \text{ and } MN \iff 0 < b = 2(\sqrt{2}-1)c < a = c,\\ &\text{where } M,\ N \text{ are the midpoints of } C'A,\ CA', \text{ respectively.}\end{aligned} \tag{12}$$

Note that the two equalizers BF and MN can be constructed by compass and ruler, MN passes through the incenter R, and $BM = BN$. So, MN is normal to the angle bisector BR and in this case

$$\sin \tfrac{B}{2} = \sqrt{2} - 1 \quad \text{and} \quad B = B_0 \approx 48.94^\circ. \tag{13}$$

2.5 Geometric construction of the two equalizers M_1N_1, and M_2N_2

Referring to Figure 5(a), we have

$$BM_1 = s - a + x_1 = \frac{2a + b - \sqrt{b^2 + 4ab - 4a^2}}{4} = a - x_2 = BN_2,$$

$BN_1 = BM_2 = BM_1 + x_2 - x_1 > BM_1$, and BR bisects $\angle B$, we get $RM_1 = RN_2$, $RM_2 = RN_1$, $M_1M_2 = N_2N_1$, and by angle bisector theorem $RN_1 > RM_1 = RN_2$, $RM_2 > RN_2 = RM_2$ and hence $\triangle RM_1M_2 \cong \triangle RN_2N_1$, $\angle M_2N_2N_1 = \angle N_1M_1M_2$, $\angle RM_1M_2 > \angle RM_2M_1$, $\angle RN_2N_1 > \angle RN_1N_2$. Thus $\angle RM_2M_1 = \angle RN_1N_2 < 90°$ and the perpendicular bisectors QO, KO of M_1M_2, N_1N_2, respectively, meet BF at O and since $\angle M_2N_2N_1 = \angle N_1M_1M_2$, it follows that the quadrilateral $M_1M_2N_1N_2$ is cyclic and O is the center of the circle Ω that passes through M_1, M_2, N_1, N_2.

But

$$BK = BQ = BM_1 + x_2 - x_1 = s - a + x_1 + \frac{x_2 - x_1}{2} = s - a + \frac{x_2 + x_1}{2} = \frac{2a + b}{4} = \frac{s}{2}.$$

So, K, Q are the midpoints of BC, BA produced by the lengths CF, AF, respectively. Thus O can be constructed by compass and ruler.

Finally, we show also that if BT is tangent to Ω, then the radius OT of Ω can be constructed. Since $OM_1 = ON_2$ and $RM_1 = RN_2$, we have $\angle ON_2R = \angle OM_1R$. But $ON_1 = OM_1$. So, $\angle ON_1R = \angle OM_1R$. Thus $\angle ON_2R = \angle ON_1R$ and hence $\triangle ORN_2N_1$ is cyclic and by symmetry about BF we have also $\triangle ORM_1M_2$ is cyclic. So, by applying *Euclid's* proposition 36 of Book III on the cyclic quadrilaterals $M_1M_2N_1N_2$ and N_1N_2RO, we get

$$(BT)^2 = (BN_1)(BN_2) = (BO)(BR).$$

But $(BT)^2 = (BO)^2 - (OT)^2$. Therefore $(OT)^2 = (BO)^2 - (BO)(BR) = (OB)(OR)$ and hence B and R are inverse points with respect to the inversion circle Ω. So, as in Section 2.2, the circle Ω with center O and radius OT can be constructed by compass and ruler and the points of intersection of Ω with the sides BA and BC are the endpoints of two equalizers M_1N_1 and M_2N_2.

(ii) Finally, in searching for equalizers of triangles with side-lengths $b \geq a = c$ and referring to Figure 5(b), we prove that every such triangle has three equalizers. To see this let M_1N_1 be an equalizer from, say EB to AF such that $A'M_1 = AN_1 = x$. Then $AM_1 \parallel N_1E$ by Lemma 1. Since $CN_1 = b - x$, $CE = \frac{a}{2}$, $CM_1 = s - b + x = \frac{2a-b+2x}{2}$, we have $\frac{CN_1}{CA} = \frac{CE}{CM_1} = \frac{b-x}{b} = \frac{a}{2a-b+2x}$. Therefore

$$2x^2 + (2a - 3b)x + b^2 - ab = (2x - b)(x - b + a) = 0 \quad \text{and hence} \quad x = \frac{b}{2} \quad \text{or} \quad x = b - a.$$

But $s = \frac{2a+b}{2}$, $A'B = a - (s - b) = \frac{b}{2} = AF$. So, if $x = \frac{b}{2}$, then $M_1N_1 = BF$ and if $x = b - a$, then the second equalizer is M_1N_1. Since $BM_1 = s - b + b - a = \frac{b}{2} = CF$ and $CN_1 = b - x = a = CB$, it follows that $\triangle BFC \cong \triangle N_1M_1C$ and hence $N_1M_1 \perp BC, N_1M_1 = BF$. So by symmetry we have the equalizer M_2N_2 from FC to BH where M_2N_2 is the reflection of N_1M_1 with respect to BF, $M_2N_2 \perp BA, M_2N_2 = M_1N_1 = BF$, and M_1, N_2, F are the points of contact of the incircle with the sides of $\triangle ABC$.

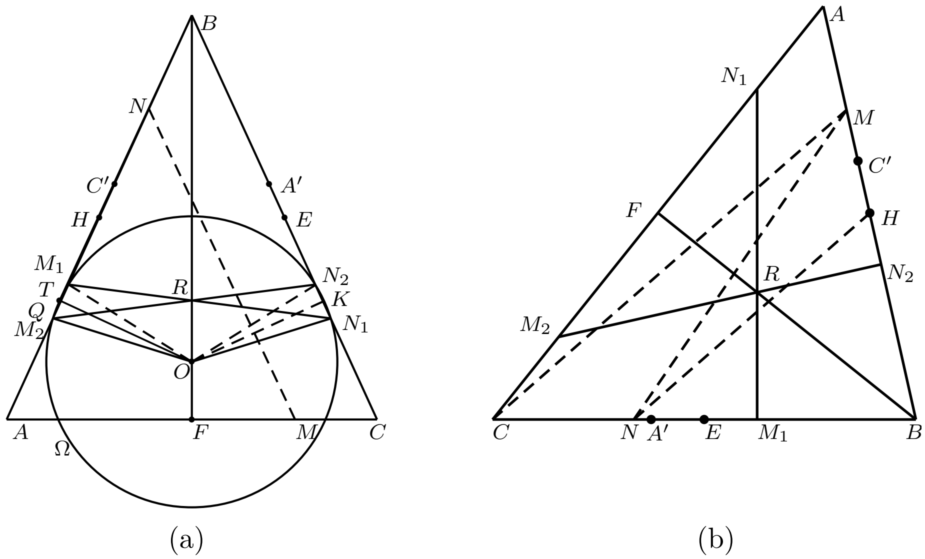


Figure 5: Construction of equalizers of isosceles triangles

Note also that if $b = a = c$, then $A' = E$ and for $x = b - a = 0$ we get $M_1N_1 = AE$ and $M_2N_2 = HC$.

Thus we conclude that every triangle with side-lengths $b \geq a = c$ has three equal equalizers and can be constructed by compass and ruler. (14)

Next, in the conclusions section, a summary for the conditions on the side-length for any $\triangle ABC$ to have either one, two, or three equalizers and their distribution and exact location on the sides is given. To make this summary more feasible, a visual diagram (Figure 6) is constructed that predicts the number of equalizers according with the side-length of $\triangle ABC$.

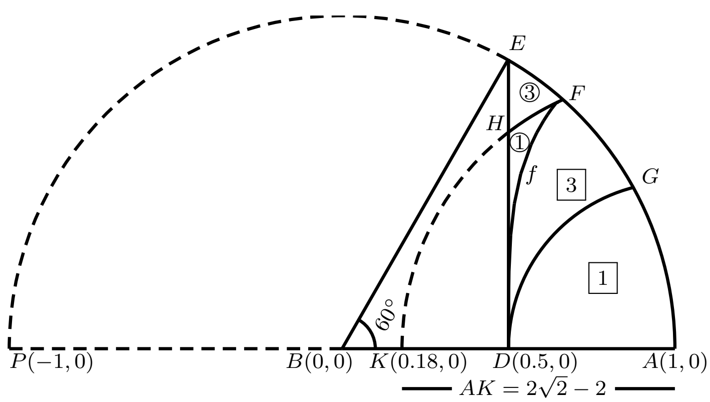


Figure 6: Illustrating the conclusions

3 Conclusions

First, the construction of the visual diagram (Figure 6).

Since every triangle $A^*B^*C^*$ with side-lengths c^*, a^*, b^* is similar to the $\triangle ABC$ with side-lengths $c = 1$, $a = \frac{a^*}{c^*}$, $b = \frac{b^*}{c^*}$ and they have the same number of equalizers, we place the $\triangle ABC$ in the coordinate plane so that $B = (0, 0)$, $A = (1, 0)$, $C = (x, y)$, $CA = b$, $CB = a$.

Let $P = (-1, 0)$, $K(3-2\sqrt{2}, 0) \approx (0.18, 0)$, $D = (0.5, 0)$ and AP be the semicircle with center B and radius 1 and let E be the point of the semicircle AP so that the central angle $\angle EBA = 60°$. Then every scalene $\triangle ABC$ with $c = 1 > a > b$ has vertex C interior to the region bounded by the line segments AD, ED, and the circular arc AE. Also every isosceles $\triangle ABC$ with $c = a = 1$ has vertex C at the open semicircle AP. Next, we draw the circular arcs GD with center A and radius 0.5 and FK with center A and radius $2\sqrt{2}-2 \approx 0.82$ that meets ED at H, arc AE at F, and AB at K. Since $AF = 2\sqrt{2}-2$ and $BF = AB = 1$, we get $\sin(\frac{\angle FBA}{2}) = \sqrt{2}-1$. So, by (13) $\angle FBA = \angle B_0 \approx 48.94°$. Also we draw the segment FD of the curve f defined by the parametric equations

$$x = 9 - 7t + 2(t-3)\sqrt{2-2t},$$
$$y = \sqrt{(3-t-\sqrt{8-8t})^2 - (9-7t+2(t-3)\sqrt{2-2t})^2}; \quad 0.5 \le t \le (2\sqrt{2}-2),$$

that represents all positions of the vertex $C(x, y)$ such that

$$CB = a = 3 - b - \sqrt{8-8b}, \quad 0.5 \le b = CA \le (2\sqrt{2}-2), \quad AB = 1, \quad t = CA = b.$$

Next, the side-lengths of triangles that have either one, two, or three equalizers are stated and they can be easily predicted by referring to the visual diagram (Figure 5) where the label of each region stands for its interior and for the number of equalizers.

So, let $AB = c = 1$ and the Cevians from the vertices of $\triangle ABC$, through the Nagel center, meet the opposite sides at A', B', C'. Then we conclude from, (5), (iii-2), (10), (11), (12), and (14) that:

(1) $\triangle ABC$ has only one equalizer MN in the following cases:

(i) $c > a > b$ and $c - a < b \le \frac{c}{2}$ (i.e. C is a point of region $\boxed{1}$ or the open $\widehat{GD}$; the hat accent stands for the circular arc connecting G and D),

(ii) $c > a > b$, $\frac{c}{2} < b < (2\sqrt{2}-2)c$ and $3c - b - \sqrt{8c(c-b)} < a < c$, (i.e. C is a point of region ①). Note that in both cases M, N lie on the largest and smallest sides and

$$C'M = CN = \frac{3b - c - a + \sqrt{(3b-c-a)^2 + 8b(a-b)}}{4},$$

(iii) $b < a = c$ and $0 < b < 2(\sqrt{2}-1)c$. (i.e. MN is the median from vertex B and C is a point of the open $\widehat{AF}$).

(2) $\triangle ABC$ has two equalizers M_1N_1, MN in the following cases:

(i) $c > a > b$, $\frac{c}{2} < b < (2\sqrt{2}-2)c$, $a = 3c - b - \sqrt{8c(c-b)} < c$, and so $BM_1 = BN_1 = c - \sqrt{\frac{c(c-b)}{2}}$, $M_1N_1 \perp BR$ and M_1N_1 can be constructed by compass and ruler and MN with endpoints on the largest and smallest sides AB and AC such that

$$C'M = CN = \frac{3b - c - a + \sqrt{(3b-c-a)^2 + 8b(a-b)}}{4}$$

(i.e. C is a point of the open segment FD of the curve f),

(ii) $b = (2\sqrt{2}-2)c < a = c$. (i.e. C is the point F and $\angle B \approx 48.98°$). The equalizers are the median from B and MN where M, N are the midpoints of $C'A$, CA'. So $BM = BN = \frac{2a+b}{4}$ and $MN \perp BR$ can be constructed by compass and ruler.

(3) $\triangle ABC$ has three equalizers in the following cases:

(i) $(2\sqrt{2}-2)c \le b < a < c$ (i.e. C is a point of region ③ or the open $\widehat{FH}$)

(ii) $\frac{c}{2} < b < (2\sqrt{2}-2)c$ and $b < a < 3c-b-\sqrt{8c(c-b)} < c$. (i.e. C is a point of region $\boxed{3}$). M_1N_1, M_2N_2 are equal in either (i) or (ii), have endpoints on the largest two sides and M_2N_2 is the reflection of M_1N_1 with respect to the angle bisector BR, see Figure 4,

$$A'M_1 = AN_1 = \frac{3c-a-b-\sqrt{(3c-a-b)^2-8c(c-b)}}{4},$$
$$A'M_2 = AN_2 = \frac{3c-a-b+\sqrt{(3c-a-b)^2-8c(c-b)}}{4},$$

and MN with endpoints on the largest and smallest sides AB and AC such that

$$C'M = CN = \frac{3b-c-a+\sqrt{(3b-c-a)^2+8b(a-b)}}{4}.$$

(iii) $2(\sqrt{2}-1)c < b < a = c$ (i.e. C is a point of the open $\widehat{FE}$). The equalizers in this case are the median from B, M_1N_1 such that

$$C'M_1 = CN_1 = x_1 = \frac{2a-b-\sqrt{b^2+4ab-4a^2}}{4}$$

and, M_2N_2 such that

$$C'M_2 = CN_2 = x_2 = \frac{2a-b+\sqrt{b^2+4ab-4a^2}}{4},$$

where M_2N_2 is the reflection of M_1N_1 with respect to angle bisector of $\angle B$, see Figure 5(a).

(iv) $c = a = b$ (i.e. C is the point E). The equalizers are the three medians,

(v) $b > c = a$ (i.e. C is a point of the open $\widehat{EP}$).

Note, as we have shown, that all equalizers that exist for a given $\triangle ABC$ can be constructed by compass and ruler.

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