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The Pythagorean Theorem: One More Proof

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Abstract. A new proof of the Pythagorean theorem is made.
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1 Introduction

Let us remember two proofs of the Pythagorean theorem. The central idea in both is to express the area of the same figure in two different ways, match both expressions and deduce what is sought. In the first case (proof attributed to the Pythagoreans in [1, pp. 27–28] (Figure 1, left) the area of the square, in the second case (Garfield proof, Figure 1, central) the area of the right trapezium. Garfield's proof uses half the figure of the first proof. Looking at the previous proofs, the following question emerge: Is it possible to use a quarter of the figure from the first proof to make a new proof?

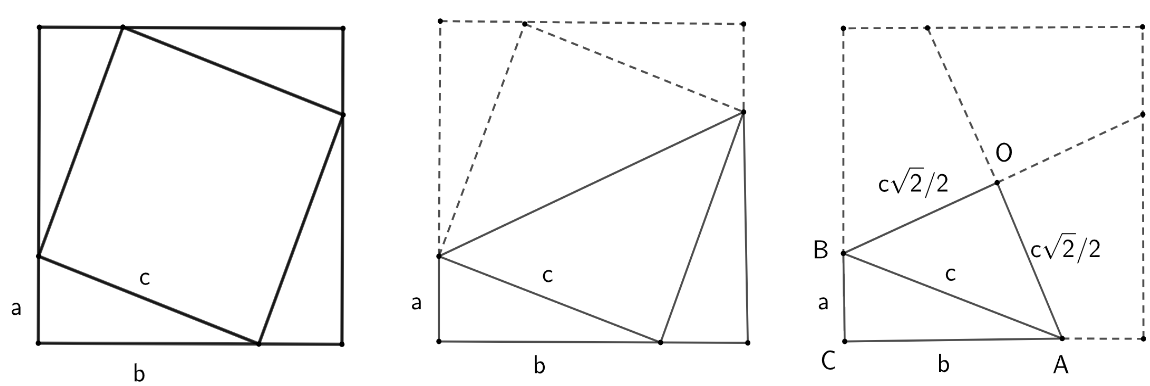

Figure 1: Diagrams to prove the Pythagorean theorem

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2 The Proof

$AOBC$ has right angles AOB and BCA (see Figure 1, right), so $AOBC$ is cyclic (the four vertices belong to the same circle).

$$\text{area}\, AOBC = \text{area}\, ABC + \text{area}\, ABO = \frac{ab}{2} + \frac{c^2}{4},$$

so

$$(\text{area}\, AOBC)^2 = \left(\frac{2ab + c^2}{4}\right)^2 = \frac{c^4 + 4abc^2 + 4a^2b^2}{16}. \tag{1}$$

According to the Brahmagupta formula [2, p. 81] the area of a cyclic quadrilateral with sides a, b, c, d is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ with

$$s = \frac{a+b+c+d}{2}, \quad s = \frac{a+b+c\sqrt{2}}{2}, \quad s-a = \frac{-a+b+c\sqrt{2}}{2} = \frac{c\sqrt{2}-(a-b)}{2},$$
$$s-b = \frac{a-b+c\sqrt{2}}{2} = \frac{c\sqrt{2}+(a-b)}{2}, \quad s-c = s-d = \frac{a+b}{2},$$

$$\begin{aligned}(\text{area}\, AOBC)^2 &= \left(\frac{c\sqrt{2}-(a-b)}{2}\right)\left(\frac{c\sqrt{2}+(a-b)}{2}\right)\left(\frac{a+b}{2}\right)^2\\ &= \left(\frac{2c^2 - a^2 + 2ab - b^2}{4}\right)\left(\frac{a^2 + 2ab + b^2}{4}\right)\\ &= \left(\frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 - 4abc^2)}{16}\right). \end{aligned} \tag{2}$$

From (1) and (2):

$$\begin{gathered}2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 - 4abc^2) = c^4 + 4abc^4 + 4a^2b^2,\\ 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + a^4 + b^4 + c^4 = 0,\\ a^4 + 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4 = 0,\\ (a^2+b^2)^2 - 2c^2(a^2+b^2) + (c^2)^2 = 0,\\ \left((a^2+b^2) - c^2\right)^2 = 0,\\ a^2 + b^2 = c^2.\end{gathered}$$

References

[1] H. Eves: *Great moments in mathematics. Before 1650.* The Mathematical Association of America, 1983.

[2] R. A. Johnson: *Advanced Euclidean Geometry.* Dover, 2007.

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