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Affine Construction of Conic Sections from Their Five Points

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Abstract. This article deals with an old problem: Constructing a conic section from its five points. Our goal is to provide feasible construction that can be done even without more profound knowledge of projective geometry. For our construction, we use the concept of affinity.

Key Words: affine construction, conic section, ellipse, hyperbola, parabola

MSC 2020: 51M15 (primary), 00A35, 51N05

1 Introduction

There is no topic in classical geometry as well-known and much-studied as constructing conic sections (or briefly conics) from its five points. It's importance is evident in many applications of mechanical engineering, civil engineering, architecture, and other applied sciences. The beauty of the topic is that it raises difficult questions that can be approached with elementary tools. This article provides constructions (and corresponding theories) that can be taught to high school and university students. We recall some crucial facts about conic sections in the rich literature. We use the concepts of affinity in our construction. We assume the reader knows the basic definitions and constructions of conics and the concepts of focus, axis, tangent, and main circle (the circle around the conic center with semi-major axis as radius). We can avoid Pascal's theorem, because we get an elementary construction to the problem: *Construct the second intersection point of a line with a conic through a known point of the curve if we know the other four points of the conic in general position.* To do so we use Apollonius' theorem on "locus concerning three or four lines" which can be found in the third book of Appolonius [1, Prop. III. LV] and was investigated later by Descartes in the first chapter of [3]. (For more information, see Chapter VII in [5].) This problem implies that five points in a general position determine the conic section. In the twentieth century, Budden gave an elementary proof for this in [2], but neither Apollonius nor Budden gave construction.

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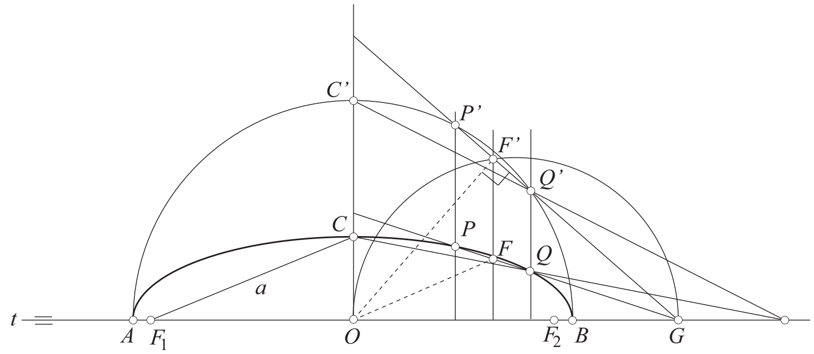

Figure 1: Construction of an ellipse from its centre, a line of its axes and two points not on the axis.

2 Four Constructions Using Affinities

First, we recall the most known definition of conic sections.

Definition 1 (based on foci)**.** The *ellipse* is the locus of those points P of the Euclidean plane E for which the sum of distances from two given points $F_1 \neq F_2$ is a constant $2a$ greater than $|F_1F_2|$. The *hyperbola* is the locus of those points P of the plane E for which the absolute value of the difference of its distances from two given points $F_1 \neq F_2$ is a constant $2a$ less than $|F_1F_2|$. The *parabola* is the locus of those points P of the plane E, which are at equal distances from a given point F and a given line l, where $F \notin l$.

In the following constructions, affine mappings play an essential role. We note that the affine image of an ellipse, parabola and hyperbola is also the same type of conic because points at infinity are sent to points at infinity. The easiest way to solve the following exercises is to state the known properties of affinity, from which follows that the ellipse is the orthogonal affine image of its principal circle, each hyperbola is an affine image of the hyperbola $y = \frac{1}{x}$ of a Cartesian coordinate system, and every two parabolas with a common axis are the orthogonal affine images of each other with a particular axis of affinity perpendicular to the axis of the parabola. For a non-familiar reader, we propose the Chapter 2 of the book [4].

2.1 Ellipse from the Line of one Axis and two Known Points

Let P and Q be the given points and denote by t one of the given axes through the centre O of the ellipse. Let F be the midpoint of the segment PQ and denote by G the intersection point of the line t with the line PQ. The circle with diameter OG meets the vertical line through F in the points F'. The affinity whose axis is t and which sends the points F to F' and P to P' also sends Q to Q'. Now the length of the segment OP' is equal to the length of the segment OQ' implying that the image of the searched ellipse at this affinity is a circle. The radius of this circle is the length of the major semi-axis, and we can get easily also the endpoint of the minor semi-axis using these connections (see Figure 1). The foci are obtained from the axes in the well-known manner.

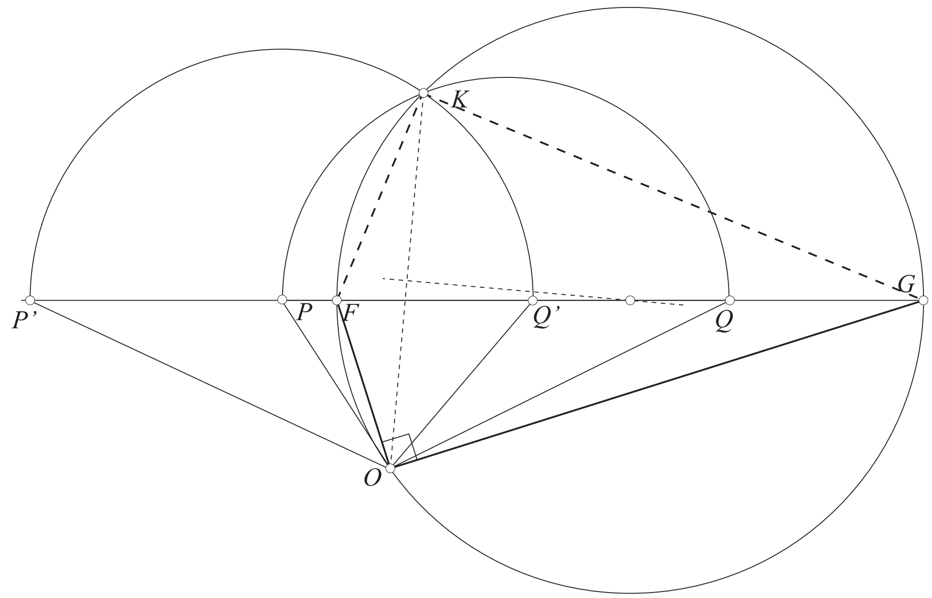

Figure 2: Construction from the direction of the diameters of the ellipse its axes.

2.2 Construction of the Axes of an Ellipse from its Centre and two Pairs of its Conjugate Diameters

A conjugate pair of the diameter of an ellipse is the orthogonal affine image of an orthogonal pair of the diameter of the mean circle concerning the affinity that sends the mean circle into the ellipse. From this, we can see that two pairs of conjugate diameters hold the property that every angle domain of the first pair contains one of the diameters of the second one, moreover the acute domains contain the major axis of the ellipse.

Determine the lines of the axes as follows (see in Figure 2). Consider a skew affinity that sends the two conjugate diameters into pairs with respective orthogonal elements. To do so, intersect the four given diameters with a line at respective points P, Q, P', Q', such that $\angle POQ$ will be obtuse. This line will be the axis of the affinity we search for. (The pairs $\{P,Q\}$ and $\{P',Q'\}$ obviously separates each other.) Consider the intersection point K of the half-circles about the segments PQ and $P'Q'$. A circle passing through the respective centres O and K, centred on the axis of affinity, intersects this line at two points F and G, which are on the legs of the invariant right angle of this affinity. Hence, the points F and G lie on the lines of axes since only one pair of the conjugate diameters exists, whose angle is a right angle, the pair of the two axes. The lines of the axis and the endpoints of the given diameters determine the ellipse by the construction in Subsection 2.1.

2.3 Hyperbola Construction from two Asymptotes and a Point

First, we construct that vertex B of the real axis, which bisects the angle domain of the asymptotes containing the given point P. The product of the affine distances x and y of the point P from the asymptotes is independent of the position of P. (By Theorem 8.1.2 in [4], the area of the corresponding parallelogram with opposite vertices O and P is independent of the choice of the point P. Still, it is equal to $xy \sin\varphi$, where x, y are the affine coordinates of P and φ is the angle of the asymptotes.) For the point B, these distances are equal to each other; we get the expected value as the geometric mean s of x and y. The tangent b at

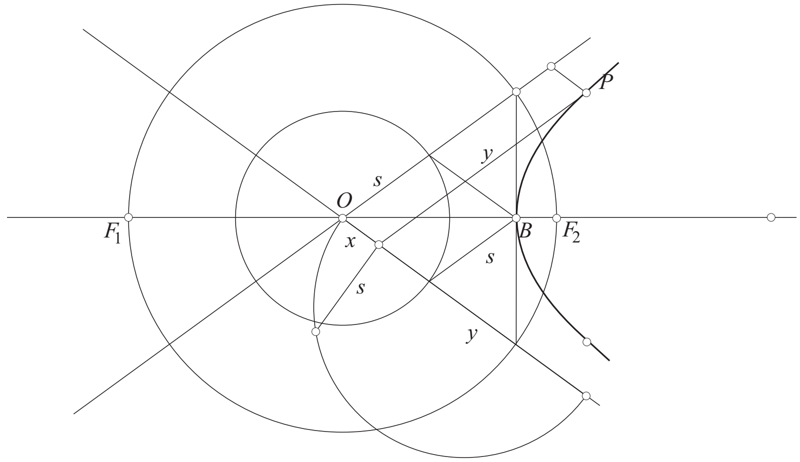

Figure 3: Construction of hyperbola from its asymptotes and one of its points.

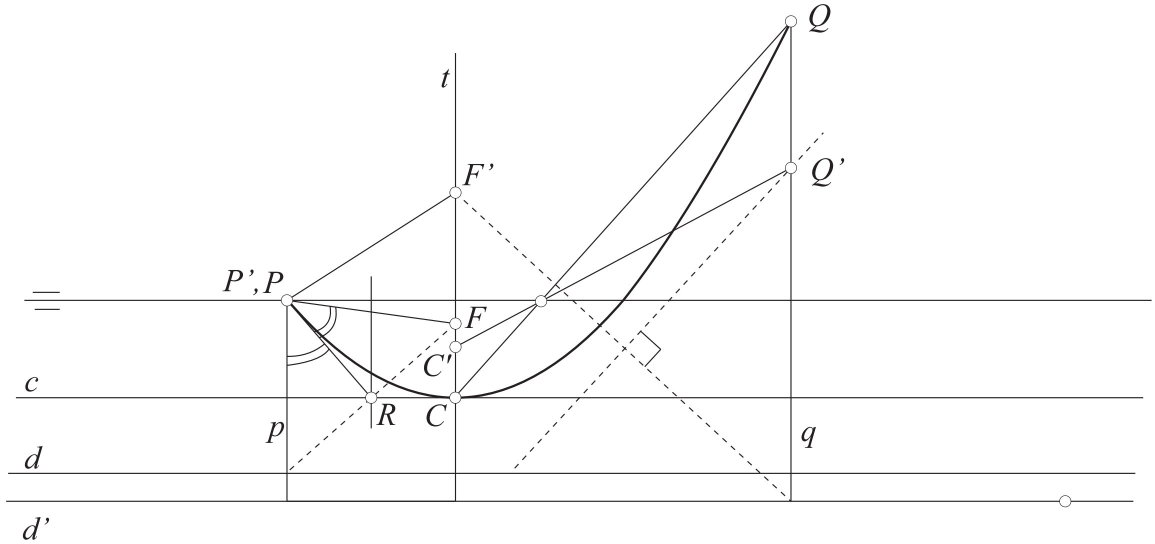

Figure 4: Construction of parabola from its axis and two of its points.

the vertex B intersects the asymptote two points, which lie on the circle with centre O and through the foci (see Figure 3).

2.4 Parabola from its Axis and two of its Points

We use the fact that the affine image of a parabola is also a parabola. If the axis of an orthogonal affinity is parallel to the directrix, then the axis of the image is the same as the axis of the original one. This observation means that the axis of symmetry is unchanged, and the image of the vertex of the parabola is the vertex of the image parabola. On the axis t, choose a point F' and determine the directrix d' such that one of the given points, say P, will be a point of the parabola (F', d'). Denote by C' its vertex. On that diameter q of this parabola, which contains the other given point Q, determine the unique point Q' of the parabola (F', d'). (Of course, if the line q and the line d' intersect at the point E, the orthogonal bisector of the segment $F'E$ intersects the point Q' from the line q.) Consider that orthogonal affinity, whose axis is the line s through the point P and perpendicular to the axis t of (F', d'), sends the point Q' to the point Q. This affinity sends P to $P' = P$ and C' to a point C of t, the vertex of the image parabola. Let p denote the diameter of the two parabolas passing through point P, and let c denote the tangent of the image parabola at the vertex C. Finally, let R be the intersection point of the tangent c with the axis of parallel lines p and t. We know that RP is tangent to the image parabola at point P, and the reflection of the line p to the tangent RP intersects the axis t at the focus F. We also get the directrix d from the points F and C (see Figure 4).

3 Construction of a Conics from its Five Points

We assume the five points in question are in a general position, so three are not collinear. We recall Apollonius' problem of locus, which concerns three or four lines. He investigated and proved the following problem. If given four lines a, b, c, d in general position, then the locus of points P of the plane for which the products $d(P,a)d(P,c)$ and $d(P,b)d(P,d)$ of distances from points to lines has a given ratio forms a conic section. Of course, the intersection points $a \cap b$, $b \cap c$, $c \cap d$ and $d \cap a$ always satisfy this property. The required conic contains these points. It is possible to choose the ratio such that the conic goes through on an arbitrary fifth point P of the plane, which doesn't lie on these lines, respectively. From this, he proved that five points uniquely determine a conic section. Descartes could solve this problem easily by using his coordinate geometry. Shortly recall his argument. He obtained the distance of a point to a line, substituting its coordinates to the normal equation of the line (which is a linear form). Hence, the above products are quadratic expressions of the coordinates of the unknown points. If their ratio is constant, then the coordinates of the points are solutions in a quadratic equation. Hence, in general, the locus is a proper conic section.

3.1 Construction of Parallel Chords from Five Points of the Conic Section

Our construction is based on the observation that we can consider the conic as the above Apollonius locus. In Figure 5 we denote the given points of the conic by black circles. Four of them successively join with the lines a, b, c and d and the fifth point is P. The perpendicular feet of P to the lines a, b, c, d are P_a, P_b, P_c and P_d, respectively. Consider the line a' through P and parallel to a. We construct its second point Q of intersection concerning the conic section. Thus, we get a pair of parallel chords for the conic.

The similar notation of the point Q gives the feet Q_a, Q_b, Q_C and Q_d, respectively. Denote by B, C, D the intersection points of b, c, d with a' respectively. We get the ratios by the theorem of parallel sections:

$$PP_a = QQ_a, \quad \frac{PP_b}{QQ_b} = \frac{PB}{QB}, \quad \frac{PP_c}{QQ_c} = \frac{PC}{QC}, \quad \text{and} \quad \frac{PP_d}{QQ_d} = \frac{PD}{QD}.$$

Since $PP_a \cdot PP_c = \alpha PP_b \cdot PP_d$ and $QQ_a \cdot QQ_c = \alpha QQ_b \cdot QQ_d$ we get the equality

$$\frac{PB \cdot PD}{PC} = \frac{QB \cdot QD}{QC}.$$

Since P is given, we have to construct point Q on the line a' for which the above equality holds. Consider the top picture in Figure 6. Here M and N are the respective endpoints of the

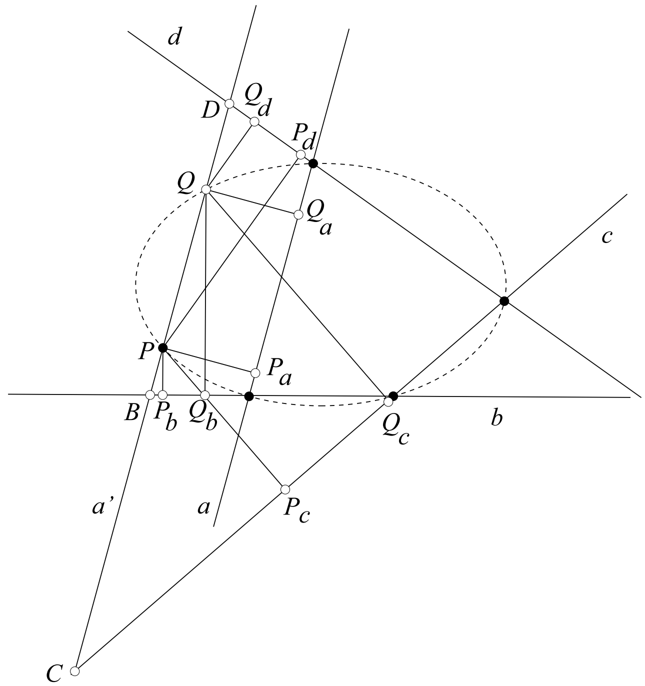

Figure 5: Points on parallel chords.

chords of the Thales circle about B and D perpendicular to a'. Let C' and C'' be the second intersection points of the Thales circles through C and the points M and N, respectively. Since

$$PB \cdot PD = PM^2 = PC \cdot PC',$$
$$QC \cdot QC'' = QN^2 = QB \cdot QD,$$

then

$$\frac{PB \cdot PD}{PC} = \frac{QB \cdot QD}{QC}$$

if and only if $PC' = QC''$. For simpler notation, denote by x and y the lengths of the segment PC' and BP, respectively; moreover, denote by $x' = x$ and y' the lengths of the corresponding segments QC'' and QD at Q. (From the order of the points C, B, P and D in Figure 6 we see that P lies between B and C' and C'' lies between Q and D.) To determine Q, we construct the segment y'. Let r and R be the respective radius of the circle through CMC' and CNC'', and k denotes the radius of the circle BMD. Then we get the equalities:

$$x(2R - x) = QN^2 = y'(2k - y') \quad \text{and} \quad 2R + y' - x = CD = 2r + 2k - y - x.$$

From the second equation, we get

$$2R = 2r + 2k - y - y'$$

and the first one gives the equation

$$x(2r + 2k - y) = y'(2k + x - y'). \tag{1}$$

We can see the construction of $QD = y'$ in the bottom picture in Figure 6.

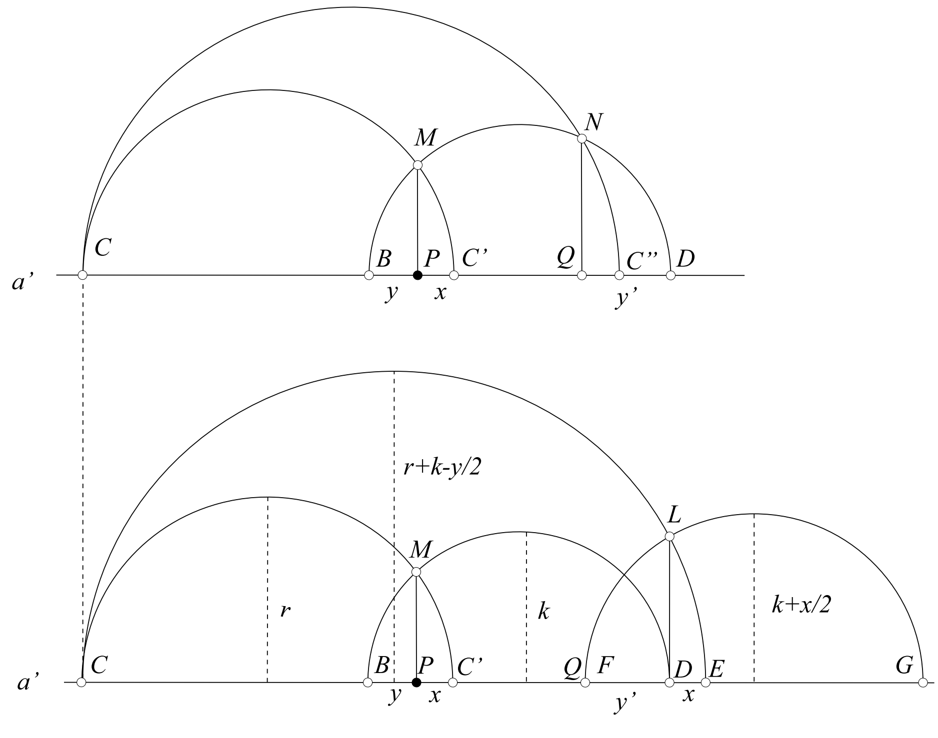

Figure 6: Construction of the second point of intersection on a line parallel to a given chord.

- Let E be the point from which $DE = x$ and the order (CDE) holds on a'. Draw a circle of radius $r + k - \frac{y}{2}$ through C and E. Then the tangent of the circle BMD at D intersects the point L from the circle CLE. (We now know that DL^2 is equal to the left side of Equation (1).
- Draw a circle of radius $k + \frac{x}{2}$ whose centre lies on a' and goes through L. (From the two possibilities, choose the one whose centre doesn't separate the points B and D.) If it intersects the line at the points F and G, then one of the lengths of the segments FE and GE is equal to y' while the other one is equal to $2k + x - y'$. Since $QD = y'$ and $QB = 2k - y'$, the location of Q on the line a' can be determined uniquely from getting metric data. (In Figure 6, $y' = DF$ hence $F \equiv Q$.)

Remark 1. This problem has a solution in all cases which arise in the discussion of the order of the points B, C, D and P in a'. The hardness is that we should use two distinct methods to adopt the required equality of products according to the geometric position of the points in its line. In the above construction, both P and Q separate B, D, and we could use the intersecting secants theorem[1] when both intersection points are inner points of a circle. If this property does not hold, we can also use this theorem for the outer point of a curve. In this situation, e.g. if P doesn't separate B and D, the length of PM is equal to that of the tangent to Thales-circle of BD from P. From Equation (1), we see that the terms' signs may also have plus or minus in both parentheses of the two sides. This fact means there are sixteen different equations on the unknown value of y'. In this paper, we give a construction for three of them and leave the other constructions to the reader. This complicated discussion

[1] The *intersecting secants theorem* says that if we draw lines through a point of the plane which intersects a given circle, then the product of the signed distance of the given point from the points of intersection in the circle is a constant. This constant is positive if the point is outside the circle, negative if it is an inner point and zero if it lies on the circle.

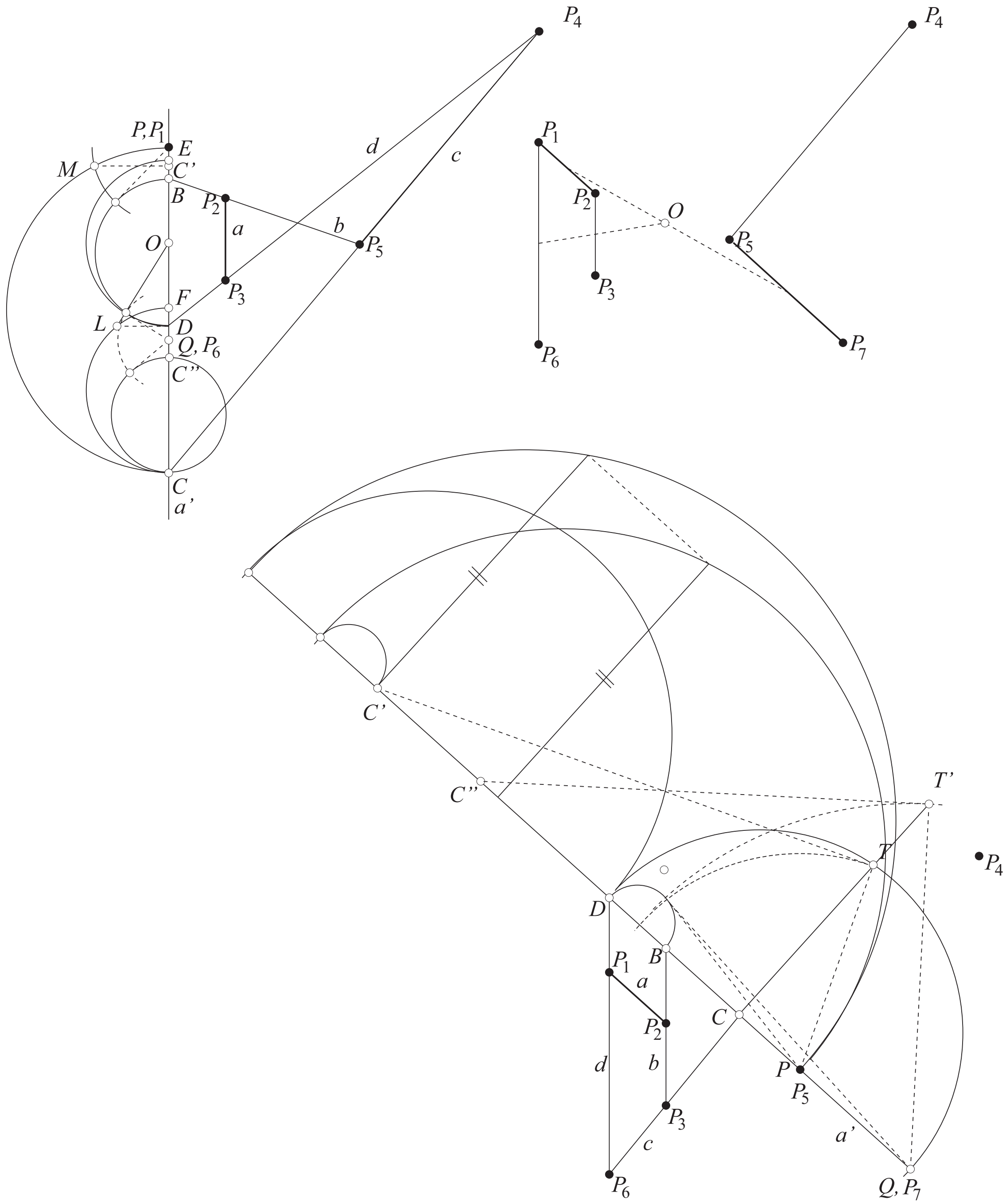

Figure 7: Construction of parallel chords and the centre of the conic.

prevented Apollonius from providing a general explanation of this problem because there is no doubt that he knew all the tools needed for such constructions. The remaining two analogous constructions of this article will prove where they arise.

3.2 Construction of a Conic Section from its Five Points

Let P_1, P_2, $\ldots$, P_5 be five points on the plane. We construct the essential data of that conic section defined by these points. First, we determine two pairs of parallel chords of the conic with the method of Subsection 3.1, see the left top and bottom pictures in Figure 7.

First, we construct point $Q = P_6$ in the top left picture. The given data forces solution different from those we investigated in Subsection 3.1. Thus, we prove it. (In the left top picture of Figure 7 we can follow the proof.) Since the order of the points $P = P_1$, B, D and C holds $(PBDC)$ in a', C' has to lie on the segment PB. From this, we also deduce that Q has to lie between D and C'' where C'' is defined by the equalities $QC'' \cdot QC = QD \cdot QB$, and $QC'' = PC'$. If $PC' = QC'' = x$ and $C'B = y$, $QD = y'$ and as the earlier construction $BD = 2k$, $CC' = 2r$ and $CC'' = 2R$ we get the following equation system:

$$QC'' \cdot QC' = x(2R + x) = y'(2k + y') = QD \cdot QB \quad \text{and} \quad 2R = 2r - 2k - x - y - y'.$$

From which we deduce the equality

$$x(2r - 2k - y) = y'(2k + y' + x).$$

This equality differs from Equation (1) only the signs of the terms in the parenthesizes. From this, we can construct y' and Q by the following steps.

- The construction of C' is standard; around P, we draw a circle with a radius of the tangents from P to the circle with diameter BD and draw a tangent from C to this circle. The point of tangency is M, (which is also the intersection of the circle above and the Thales circle of the segment PC). Its orthogonal projection to a' is C'.
- The circle CLF has diameter $2k+x = CD+DF = CD+PC'$, the chord DL orthogonal to a' has length $x \cdot CD = x(2r - 2k - y)$.
- Draw a circle with diameter $DE = BD+x = 2k+x$ and construct the tangent of it with a length of DL with one endpoint on a'. (This tangent goes through the intersection of the segment LO with the circle if O is the centre of the circle.) We thus get Q as the endpoint of this tangent on the line a' because $QD \times QE = QD \times (QD + 2k - x)$ implies that $QD = y'$.

We construct the other pair of parallel chords from the points P_1, P_2, P_3, P_6 and P_5. The seventh point is $Q = P_7$, and the original data differ from the position of C and C' concerning the pair P, D. Our notation again: $PC' = x = QC''$, $PB = y$, $QD = y'$ from which now: $QP = C'C'' = 2r - 2R$ and $C''D = x - y'$. (Again, we have $BD = 2k$, $CC' = 2r$ and $CC'' = 2R$.) The equation system is $x(x - 2R) = y'(y'_2k)$ where $2R = 2k + 2r + y - y'$, and so

$$x(2k + 2r + y - x) = y'(x + 2k - y').$$

The solution requires the use of both versions of the intersecting secants theorem, and hopefully, it can be read from the bottom picture of Figure 7.

Consider a chord AB from the parallel family and connect its midpoint F with the centre O of the conic. (In the case of the parabola, draw a parallel with the axis of the parabola.) Consider the line OF as the axis of a skew affinity sending the point A to the points A' with the property, then $A'F$ is orthogonal to OF. The image of the conic is a conic of the same type, whose diameter is the line OF. Since OF is the perpendicular bisector of the image $A'B'$ of AB, OF is an axis of the image conic. Thus, all chords of the image parallel to $A'B'$ halving by OF, implying that OF halves all chords of the original conic parallel to AB. Hence, a line connecting the midpoints of two parallel chords is always a diameter. Thus, a line connecting the parallel chords' midpoints contains the conic's centre. (In the case of a parabola, it is parallel to the axis; see the right top picture in Figure 7.) From an Euclidean point of view, we have two different options.

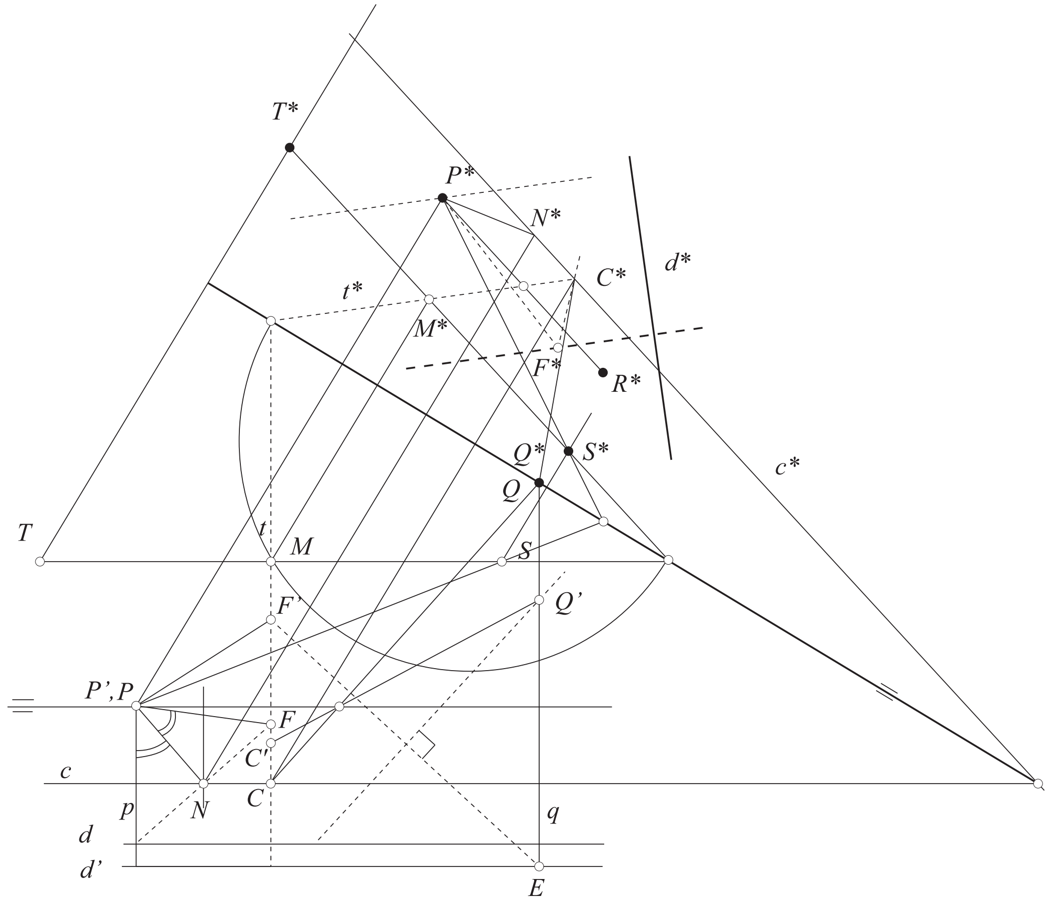

Figure 8: Construction from the direction of the parabola's diameters, focus, and axis.

- In the first case, the diameters are parallel, which means that the centre lies at infinity, i.e. the conic section is a parabola. In this case, we know the direction of the axis, and by affinity, we can determine two tangents from the known points. Take an affine image of the data in which a chord and its diameter go to an orthogonal pair of lines. The axis of the image parabola is the image of the diameter. We can construct the focus and directrix with the result of Subsection 2.4; hence, we can also get the tangents at the known points. Using the inverse of two from these tangents, we quickly construct the parabola's further data (see this construction, e.g. in [4].) This construction can be seen in Figure 8, the original data denoted by letters with stars, their affine images by letters, and we denote the helping correspondence by letters with a prime. The diameter is the line that connects the midpoints of the parallel chords $T^\star S^\star$ and $P^\star R^\star$. The axis of the affinity goes through the point $Q^\star$. In the last step, we constructed the focus and the directrix from the tangents at the points $C^\star$ and $P^\star$, respectively.
- In the second case, we must distinguish between the case of a hyperbola and an ellipse. Two pairs of conjugate diameters are available at the centre O, so if we take a line that intersects these lines in four points, we have two possibilities.
  - In the first, the corresponding pairs of points cross each other. As we saw in Subsection 2.2, the searched conic is an ellipse, and we can get its axes with the method of Subsection 2.2. After determining the line of axes, we can use the method of Subsection 2.1 to get the required data of the ellipse.
  - If the two pairs of points do not cross, we get a hyperbola case. First, we define

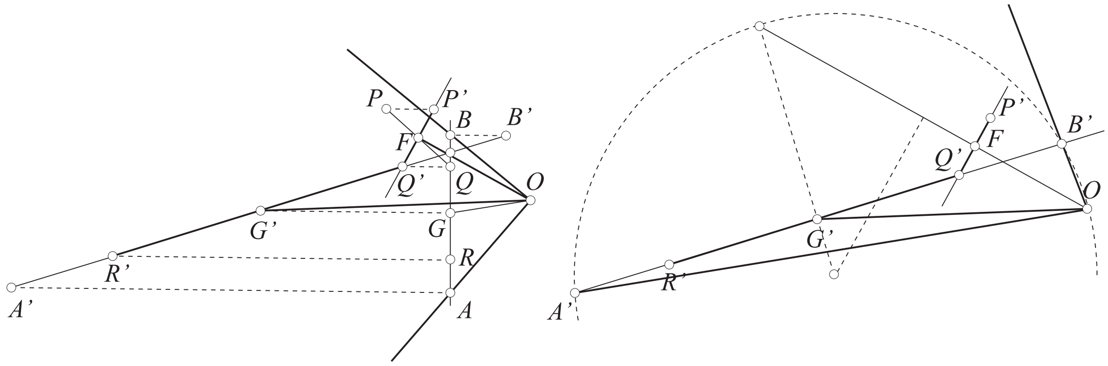

Figure 9: Construction from the direction of the diameters of the hyperbola its asymptotes.

the asymptotes and then apply the construction of subsection 2.3. Note that we can determine the asymptotes of the hyperbola from the points P, Q and R and the centre O (see Figure 9).
In fact, by applying skew-affinity, we can map the segment PQ to the segment $P'Q'$, which is perpendicular to the line OF, where F is the midpoint of PQ (see the left image in Figure 9). This affinity maps R to R' and the midpoint G of the segment QR to the midpoint G' of the segment $Q'R'$. The asymptotes of the new hyperbola defined by the points P', Q', R' and the centre O can be obtained using the method of the right-hand image of Figure 9. In this picture, we use the midpoint G' of the segment $Q'R'$ as the midpoint of the segment $A'B'$, where the sought asymptotes are OA' and OB'. Therefore, we know the vertex O of the triangle $OA'B'$, the angle bisector at this vertex, OF, and the median OG' of the triangle. We know that the segment bisector perpendicular to the side $A'B'$ intersects the angle bisector OF at a point of the circumscribed circle of the triangle so that we can construct the vertices A' and B' as the intersection of the circumscribed circle with the line $A'B'$. Finally, using the previous affinity again, we get the points A and B of the asymptotes from the points A' and B' (see the left image of Figure 9).

Remark 2. Using Pascal's theorem on a hexagon inscribed in a conic section, we can essentially simplify the first part of this construction and the parabola case. Such constructions in a course on projective geometry are practice exercises corresponding to the theorem. The construction of this article is not the simplest but uses the most elementary tools.

4 Conclusion

Given its five points, we show a path of constructions that leads to the solution of a conic section. We used only the elementary properties of conics and affine solutions for engaging exercises from the world of such problems. Our method does not have a universal character in that one construction should solve all metric situations. However, we can analogously reproduce the elements of our path of constructions in all generic cases.

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