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In and Ex Spheres of a Tetrahedron

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Abstract. We prove that
(1) a tetrahedron is isosceles if and only if the vertices of its twin tetrahedron are the excenters of the tetrahedron,
(2) if a tetrahedron is orthocentric, and if the orthocenter is either the incenter, the centroid, or the circumcenter, then the tetrahedron is regular,
(3) a tetrahedron is regular if and only if the four ex-spheres are tangent to the in-sphere, and
(4) we prove an inequality relating the in-radius, circumradius, and the distances between the in-center and the vertices of a tetrahedron.

Key Words: in-sphere, in-center, in-radius, ex-sphere, ex-center, ex-radius, twin tetrahedron, isosceles tetrahedron, regular tetrahedron, centroid, circumsphere, circumradius, circumcenter, orthocentric tetrahedron, orthocenter, Lagrange multipliers

MSC 2020: 51M04

1 Introduction

Let us start with definitions.

Definition 1. A triangle ABC is denoted by $\triangle ABC$. A tetrahedron $ABCD$ is denoted by $\nabla ABCD$. The sphere inside $\nabla ABCD$ tangent to the four faces $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, and $\triangle BCD$ is called the *in-sphere* of $\nabla ABCD$. Let S denote the in-sphere of $\nabla ABCD$. The center and radius of S are called *in-center* and *in-radius*, and denoted by I and r, respectively. The sphere, outside of $\nabla ABCD$, on the opposite side of the vertex A with respect to the plane BCD, tangent to the face $\triangle BCD$ and tangent to the extended adjacent faces $\triangle ABC$, $\triangle ACD$, $\triangle ABD$ is called an *ex-sphere*, and it is denoted by S_A. The center and the radius of S_A are called the *ex-center* and the *ex-radius,* and denoted by I_A and r_A, respectively. So there are four ex-spheres S_A, S_B, S_C, S_D of $\nabla ABCD$, whose ex-radii are r_A, r_B, r_C, r_D, and ex-centers I_A, I_B, I_C, I_D, respectively.

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We can similarly define the in-circle and ex-circles for a triangle. If r_1, r_2, r_3 are the ex-radii and r the in-radius of a triangle, then it is known that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$ (see [2, Page 13]). For your information, there is an analogous result which states that if $\nabla ABCD$ is a tetrahedron, then $\frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} + \frac{1}{r_D} = \frac{2}{r}$. The source of this equation is unknown. But this can be proven in a similar way for a triangle by letting $T = \frac{1}{3}(T_A + T_B + T_C + T_D)$, where T_A, T_B, T_C, T_D are the areas of the triangular faces $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, $\triangle ABC$, respectively, of the tetrahedron $\nabla ABCD$.

Definition 2. A tetrahedron is *regular* if all edges have the same length. A tetrahedron $\nabla ABCD$ is said to be *isosceles* or *equifacial* if $|AB| = |CD|$, $|AC| = |BD|$ and $|AD| = |BC|$.

Definition 3. Let us inscribe $\nabla ABCD$ into a parallelepiped so that the edges of the tetrahedron are the diagonals of the six faces of the parallelepiped. We label the diagonally opposite vertices of A, B, C, D of the parallelepiped by A^*, B^*, C^*, D^*, respectively. Hence, for example, as in Figure 1, the faces AD^*BC^* and A^*DB^*C of the parallelepiped are determined by the planes parallel to the lines AB and CD. We will call the tetrahedron $\nabla A^*B^*C^*D^*$ the *twin* of the tetrahedron $\nabla ABCD$. We call the parallelepiped $ABCDA^*B^*C^*D^*$ the *inscribing parallelepiped* of $\nabla ABCD$.

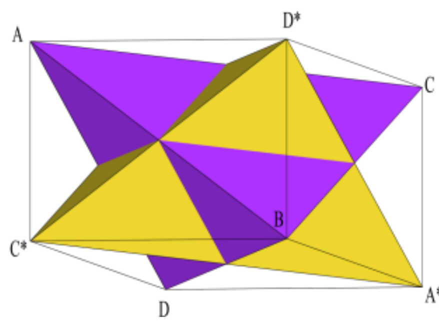

Figure 1: The parallelepiped inscribing a tetrahedron and its twin.

In Theorem 1 of Section 2, we will prove that a tetrahedron $\nabla ABCD$ is isosceles if and only if the vertices of its twin tetrahedron $\nabla A^*B^*C^*D^*$ are the ex-centers of $\nabla ABCD$. Note that the twin tetrahedron of $\nabla A^*B^*C^*D^*$ is $\nabla ABCD$.

The following lemma is well known.

Lemma 1 ([1, Page 97]). *A tetrahedron is isosceles if and only if any of the following three identities holds: the centroid = the in-center, the centroid = the circum-center, or the in-center = the circumcenter.*

In Theorem 2 of Section 3, we will prove that if a tetrahedron $\nabla ABCD$ is orthocentric, and if the orthocenter is either the incenter, the centroid, or the circumcenter of the tetrahedron, then the tetrahedron $\nabla ABCD$ is regular. This is an interesting contrast to the above Lemma 1. We will also prove that $\nabla ABCD$ is regular if and only if S_A, S_B, S_C, S_D are all tangent to S in Theorem 3.

Definition 4. The sphere that contains all four vertices of $\nabla ABCD$ is said to be the circumsphere of $\nabla ABCD$. The center and the radius of the circumsphere are called the *circumcenter* and *circumradius* of $\nabla ABCD$, respectively. The circumradius is denoted by R. Let $L_A = |IA|$, $L_B = |IB|$, $L_C = |IC|$, $L_D = |ID|$, where I is the incenter.

It is known that $R \geq 3r$ (see [5]). In Theorem 4 of Section 4, we will use Lagrange multipliers to extend this inequality to $\frac{r}{R} \leq \frac{1}{4}\left(\frac{r}{L_A} + \frac{r}{L_B} + \frac{r}{L_C} + \frac{r}{L_D}\right) \leq \frac{1}{3}$.

2 Isosceles Tetrahedra

We will prove a characterization of an isosceles tetrahedron in terms of ex-centers.

Definition 5. Let $\nabla ABCD$ be a tetrahedron. Let $\overrightarrow{DA} = \vec{\alpha}$, $\overrightarrow{DB} = \vec{\beta}$, $\overrightarrow{DC} = \vec{\gamma}$. Let Γ be the parallelepiped defined by vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$. Let M and G be points defined by $\overrightarrow{DM} = \frac{1}{2}(\vec{\alpha} + \vec{\beta} + \vec{\gamma})$ and $\overrightarrow{DG} = \frac{1}{4}(\vec{\alpha} + \vec{\beta} + \vec{\gamma})$. Then M is the centroid of the parallelepiped Γ, and G is the centroid of the tetrahedron $\nabla ABCD$.

We use the next lemma to prove Theorem 1 in this section.

Lemma 2 (See [3])**.** *A tetrahedron $\nabla ABCD$ is isosceles if and only if the centroid M of Γ is an ex-center of $\nabla ABCD$.*

Theorem 1. *A tetrahedron $\nabla ABCD$ is isosceles if and only if the vertices of its twin tetrahedron $\nabla A^*B^*C^*D^*$ are the ex-centers of $\nabla ABCD$.*

Proof. Since the parallelepiped $ABCDA^*B^*C^*D^*$ is the inscribing parallelepiped of $\nabla ABCD$, we have

$$\overrightarrow{DB^*} + \overrightarrow{DC^*} = \overrightarrow{DA} = \vec{\alpha}, \quad \overrightarrow{DA^*} + \overrightarrow{DC^*} = \overrightarrow{DB} = \vec{\beta}, \quad \text{and} \quad \overrightarrow{DA^*} + \overrightarrow{DB^*} = \overrightarrow{DC} = \vec{\gamma}.$$

Solving these equations for $\overrightarrow{DA^*}$, $\overrightarrow{DB^*}$ and $\overrightarrow{DC^*}$, we have

$$\overrightarrow{DA^*} = \tfrac{1}{2}(-\vec{\alpha} + \vec{\beta} + \vec{\gamma}), \overrightarrow{DB^*} = \tfrac{1}{2}(\vec{\alpha} - \vec{\beta} + \vec{\gamma}), \quad \text{and} \quad \overrightarrow{DC^*} = \tfrac{1}{2}(\vec{\alpha} + \vec{\beta} - \vec{\gamma}).$$

Hence, we have

$$\overrightarrow{DD^*} = \overrightarrow{DA^*} + \overrightarrow{DB^*} + \overrightarrow{DC^*} = \tfrac{1}{2}(\vec{\alpha} + \vec{\beta} + \vec{\gamma}).$$

This shows that D^* is the centroid of the parallelepiped Γ defined by $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$. Hence, $\nabla ABCD$ is isosceles if and only if $D^* = M$ is one of the ex-centers of the tetrahedron $\nabla ABCD$ by Lemma 2. Similarly, we can show that A^*, B^*, C^* are ex-centers of $\nabla ABCD$. This proves that $\nabla ABCD$ is isosceles if and only if the vertices of $\nabla A^*B^*C^*D^*$ are the ex-centers of $\nabla ABCD$. □

3 Regular Tetrahedra

We will characterize a regular tetrahedron using altitudes. The altitudes of a triangle always concur at a point which is called the *orthocenter*. In comparison, four altitudes of a tetrahedron may not concur.

Definition 6. A tetrahedron is *orthocentric* if its four altitudes are concurrent, and the concurrent point of the altitudes is called its *orthocenter*.

Lemma 3 (See [4, Page 64])**.** *A tetrahedron is orthocentric if and only if the three pairs of opposite edges are mutually perpendicular.*

Lemma 4. *An isosceles tetrahedron is orthocentric if and only if it is regular.*

Proof. A regular tetrahedron is isosceles and orthocentric. So let a tetrahedron $\nabla ABCD$ be isosceles and orthocentric. Let $ABCDA^*B^*C^*D^*$ be a parallelepiped inscribing $\nabla ABCD$. See Figure 1. Since the tetrahedron $\nabla ABCD$ is isosceles, $AB = CD = C^*D^*$ so that the face AC^*BD^*, for example, is a rectangle. Hence, $ABCDA^*B^*C^*D^*$ is a rectangular box. By Lemma 3, the three pairs of opposite edges of $\nabla ABCD$ are mutually perpendicular. This implies that the two diagonals of a rectangular face are perpendicular, i.e., the rectangle is a square. Hence, the rectangular box $ABCDA^*B^*C^*D^*$ must be a cube. Therefore, this shows that all edges of $\nabla ABCD$ have the same length. That is, the tetrahedron $\nabla ABCD$ is regular. □

Lemma 5. *If a tetrahedron is orthocentric, and if the orthocenter is the incenter, then the tetrahedron is regular.*

Proof. Let $\nabla ABCD$ be an orthocentric tetrahedron whose orthocenter is the incenter. Let I be the orthocenter = the incenter. Let A', B', C', D' be the feet of AI, BI, CI, DI on the faces $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, $\triangle ABC$, respectively. Since I is the incenter of $\nabla ABCD$, the plane ABA' bisect the dihedral angle between the faces $\triangle ABC$ and $\triangle ABD$. Since the segment AA' is normal to the plane BCD, the line BA' is perpendicular to the edge CD and bisects $\sphericalangle CBD$. Hence, the line BA' bisects the edge CD. Similarly, CA' and DA' bisect the edges BD and BC, respectively. Hence, A' is the centroid of the face $\triangle BCD$. Similarly, B', C', D' are the centroids of the faces $\triangle ACD$, $\triangle ABD$, $\triangle ABC$, respectively. Therefore, the point I is the centroid of the tetrahedron $\nabla ABCD$. This shows that the tetrahedron is isosceles by Lemma 1. By Lemma 4, the tetrahedron $\nabla ABCD$ is regular. □

Theorem 2. *If a tetrahedron is orthocentric, and if the orthocenter is either the incenter, the centroid, or the circumcenter, then the tetrahedron is regular.*

Proof. If the orthocenter is the incenter, then the tetrahedron is regular by Lemma 5.

Suppose the orthocenter is the centroid $\nabla ABCD$. Let G be the centroid = the orthocenter of a tetrahedron $\nabla ABCD$. Then the plane ABG bisects the edge CD. Since G is also the orthocenter of $\nabla ABCD$, the plane ABG is perpendicular to the edge CD. Hence, the plane ABG bisects the dihedral angle between the faces $\triangle ABC$ and $\triangle ABD$. Similarly, the plane BCG bisects the dihedral angle between the faces $\triangle ABC$ and $\triangle BCD$, and the plane CDG bisects the dihedral angle between the faces $\triangle ACD$ and $\triangle BCD$. Thus, G is the in-center of $\nabla ABCD$. Therefore, $\nabla ABCD$ is regular by Lemma 5.

Next, suppose the circumcenter is the orthocenter of $\nabla ABCD$. Let P be the circumcenter = the orthocenter of $\nabla ABCD$. Since P is the circumcenter of $\nabla ABCD$, P is on the perpendicular bisecting plane Ω of the edge CD. Since P is also the orthocenter of $\nabla ABCD$, the plane ABP is normal to the edge CD. But since $P \in \Omega$ and Ω is normal to the edge CD, the plane Ω must be the plane ABP. Hence, the plane ABP bisects the dihedral angle between the faces $\triangle ABC$ and $\triangle ABD$. Similarly, the plane BCP bisects the dihedral angle between the faces $\triangle ABC$ and $\triangle BCD$, and the plane CDP bisects the dihedral angle between the faces $\triangle ACD$ and $\triangle BCD$. Thus, P is the in-center of $\nabla ABCD$. Therefore, $\nabla ABCD$ is regular by Lemma 5. □

Remark 1. It can be shown that a triangle is equilateral if and only if the orthocenter and the incenter are the same.

Theorem 3. *A tetrahedron is regular if and only if its ex-spheres are tangent to the in-sphere.*

Proof. If a tetrahedron is regular, then the ex-spheres are all tangent to the in-sphere. So suppose all ex-spheres S_A, S_B, S_C, S_D of a tetrahedron $\nabla ABCD$ are tangent to its in-sphere S at A', B', C', and D', respectively. Then A', B', C', and D', are on the faces $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively. Recall I, I_A, I_B, I_C, I_D are the centers of S, S_A, S_B, S_C, S_D, respectively. Then $A' \in II_A$, $B' \in II_B$, $C' \in II_C$, $D' \in II_D$. The planes ABI and ABI_A are the same since they are the planes bisecting the dihedral angle between the faces $\triangle ABC$ and $\triangle ABD$. Similarly, the planes ACI and ACI_A are the same. This shows that the lines AI and AI_A are the same. Hence, $A' \in II_A \subset AI_A$. Hence, the segment AA' is normal to the plane BCD. Similarly, we can show that BB', CC', DD' are normal to the planes ACD, ABD, ABC, respectively. Hence, I is the orthocenter of $\nabla ABCD$. By Theorem 2, the tetrahedron $\nabla ABCD$ is regular. □

Remark 2. It can be shown that a triangle is equilateral if and only if the ex-circles are tangent to the in-circle.

Remark 3. Let $\triangle ABC$ be a triangle, and let H_A, H_B, H_C be ex-centers. Then the feet of the altitudes of $\triangle H_AH_BH_C$ are A, B and C (see [2, Page 13]). So, for a tetrahedron $\nabla ABCD$, are A, B, C, D the feet of the altitudes of the tetrahedron $\nabla I_AI_BI_CI_D$? The answer is *NO*. Let $\nabla ABCD$ be any isosceles tetrahedron. Then by Theorem 1, we have $A^* = I_A$, $B^* = I_B$, $C^* = I_C$, $D^* = I_D$, or $\nabla I_AI_BI_CI_D = \nabla A^*B^*C^*D^*$. Since the parallelepiped $ABCDA^*B^*C^*D^*$ that inscribes the tetrahedron $\nabla ABCD$ is the rectangular box, and none of the triangular faces $A^*B^*C^*$, $A^*B^*D^*$, $A^*C^*D^*$, and $B^*C^*D^*$ even do not contain any of the points A, B, C or D. So A, B, C, D are not the feet of the altitudes of the tetrahedron $\nabla I_AI_BI_CI_D$.

4 Inequalities Involving Inradius and Circumradius

We will prove the next theorem in this section.

Theorem 4. *Let $\nabla ABCD$ be a tetrahedron. Recall that R is the circumradius of the tetrahedron $\nabla ABCD$, and $L_A = |IA|$, $L_B = |IB|$, $L_C = |IC|$, $L_D = |ID|$. Then*

$$\frac{r}{R} \le \frac{1}{4}\left(\frac{r}{L_A} + \frac{r}{L_B} + \frac{r}{L_C} + \frac{r}{L_D}\right) \le \frac{1}{3}.$$

The inequality on the left becomes equality if and only if $\nabla ABCD$ is isosceles, and the inequality on the right becomes equality if and only if $\nabla ABCD$ is regular.

Veljan [5] proved the following lemma which is similar to Theorem 4.

Lemma 6 (Veljan)**.** *Let a, a'; b, b'; c, c' be the lengths of opposing pairs of the edges of a tetrahedron. If R and r are the circumradius and inradius of the tetrahedron, respectively, then*

$$\left(\frac{r}{R}\right)^2 \le \frac{\sqrt[3]{(-aa'+bb'+cc')(aa'-bb'+cc')(aa'+bb'-cc')}}{3(aa'+bb'+cc')} \le \frac{1}{9}.$$

The left side inequality becomes equality if and only if the tetrahedron is isosceles, and the right side inequality becomes equality if and only if $aa' = bb' = cc'$.

In this paper [5], Veljan gave a nice proof of the inequality $R \ge 3r$. However, he says "Clearly, the equality ($R = 3r$) is attained if and only if it (the tetrahedron) is regular". Maybe it is "clear". But it is not clear to us why $R = 3r$ implies that the tetrahedron is regular. Since this is important to us, we will prove this result next.

Lemma 7. *If $\nabla ABCD$ is a tetrahedron, then $R \geq 3r$. The inequality becomes equality if and only if the tetrahedron is regular.*

Proof. Let A', B', C', D' be the centroids of the faces $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, $\triangle ABC$, respectively.

The tetrahedron $\nabla A'B'C'D'$ is similar to the tetrahedron $\nabla ABCD$, and its edges are exactly $\frac{1}{3}$ the lengths of edges of the tetrahedron $\nabla ABCD$. Therefore, we must have $R = 3R'$, where R' is the circumradius of the tetrahedron $\nabla A'B'C'D'$. Since the in-sphere is the smallest sphere that touches all four faces of the tetrahedron, we must have $R' \geq r$. Hence, we have $R = 3R' \geq 3r$. (Up to this far, this is exactly Veljan's argument in [5].)

If the tetrahedron is regular, then $R = 3r$. So suppose $R = 3r$. We will show that the tetrahedron is regular. Again, let R' be the circumradius of $\nabla A'B'C'D'$. Since $R = 3R'$, we must have $R' = r$. Note that the centroids of $\nabla ABCD$ and $\nabla A'B'C'D'$ are identical.[1]Since A', B', C', D' are points on the faces of $\nabla ABCD$, $R' = r$ implies that the centroid and circumcenter of $\nabla A'B'C'D'$ are identical, and it is I. So $\nabla A'B'C'D'$ is isosceles by Lemma 1. Since $\nabla ABCD$ is similar to $\nabla A'B'C'D'$, the tetrahedron $\nabla ABCD$ is also isosceles. Again, by Lemma 1, the point I is also the centroid of $\nabla ABCD$ so that I is the intersection of the segments AA', BB', CC', DD'. Since I is the circumcenter of $\nabla A'B'C'D'$, and since $\{I\} = AA' \cap BB' \cap CC' \cap DD'$, the segments AA', BB', CC', DD' are normal to the faces $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, $\triangle ABC$, respectively. Hence, I is the orthocenter of $\nabla ABCD$. Therefore, the tetrahedron $\nabla ABCD$ is regular by Lemma 4. □

As a corollary of Lemma 7, we have the next lemma.

Lemma 8. *Let $\nabla ABCD$ be an isosceles tetrahedron. Then we have*

$$\cos^{-1}\tfrac{r}{R} \geq \cos^{-1}\tfrac{1}{3} \quad \textit{and} \tag{1}$$

$$\cos^{-1}\tfrac{r}{L_A} + \cos^{-1}\tfrac{r}{L_B} + \cos^{-1}\tfrac{r}{L_C} + \cos^{-1}\tfrac{r}{L_D} \geq 4\cos^{-1}\tfrac{1}{3}. \tag{2}$$

The inequalities in (1) *and* (2) *become equalities if and only if $\nabla ABCD$ is regular.*

Proof. By Lemma 7, we have $R \geq 3r$, or $\frac{r}{R} \leq \frac{1}{3}$. Since the inverse cosine function is decreasing on the interval $[0, 1]$, we have $\cos^{-1}\frac{r}{R} \geq \cos^{-1}\frac{1}{3}$. Since the in-center and the circumcenter of an isosceles tetrahedron are identical, we have $L_A = L_B = L_C = L_D = R$. Therefore, $\cos^{-1}\frac{r}{L_A} + \cos^{-1}\frac{r}{L_B} + \cos^{-1}\frac{r}{L_C} + \cos^{-1}\frac{r}{L_D} = 4\cos^{-1}\frac{r}{R} \geq 4\cos^{-1}\frac{1}{3}$. Again, by Lemma 7, the inequalities in (1) and (2) become equalities if and only if $\nabla ABCD$ is regular. □

Now, we are ready for the next lemma. We use Lagrange Multipliers' method to prove it.

Lemma 9. *Let $\nabla ABCD$ be a tetrahedron. (We are not assuming it to be isosceles.) Then*

$$\tfrac{r}{L_A} + \tfrac{r}{L_B} + \tfrac{r}{L_C} + \tfrac{r}{L_D} \leq \tfrac{4}{3}, \quad \textit{and} \tag{3}$$

$$\cos^{-1}\tfrac{r}{L_A} + \cos^{-1}\tfrac{r}{L_B} + \cos^{-1}\tfrac{r}{L_C} + \cos^{-1}\tfrac{r}{L_D} \geq 4\cos^{-1}\tfrac{1}{3}. \tag{4}$$

The inequalities in both (3) *and* (4) *become equalities if and only if the tetrahedron $\nabla ABCD$ is regular.*

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[1]Let G, G' be the centroids of $\nabla ABCD$ and $\nabla A'B'C'D'$, respectively. We will show that $G = G'$. Using vectors defined in Definition 5, $\overrightarrow{DG} = \frac{1}{4}(\vec{\alpha} + \vec{\beta} + \vec{\gamma})$, $\overrightarrow{DA'} = \frac{1}{3}(\vec{\beta} + \vec{\gamma})$, $\overrightarrow{DB'} = \frac{1}{3}(\vec{\alpha} + \vec{\gamma})$, $\overrightarrow{DC'} = \frac{1}{3}(\vec{\alpha} + \vec{\beta})$, $\overrightarrow{DD'} = \frac{1}{3}(\vec{\alpha} + \vec{\beta} + \vec{\gamma})$. Hence, $\overrightarrow{D'A'} = \overrightarrow{DA'} - \overrightarrow{DD'} = -\frac{\vec{\alpha}}{3}$, $\overrightarrow{D'B'} = -\frac{\vec{\beta}}{3}$, $\overrightarrow{D'C'} = -\frac{\vec{\gamma}}{3}$ so that $\overrightarrow{D'G'} = \frac{1}{4}(\overrightarrow{D'A'} + \overrightarrow{D'B'} + \overrightarrow{D'C'}) = -\frac{1}{12}(\vec{\alpha} + \vec{\beta} + \vec{\gamma})$. Thus, $\overrightarrow{DG'} = \overrightarrow{DD'} + \overrightarrow{D'G'} = \frac{1}{3}(\vec{\alpha} + \vec{\beta} + \vec{\gamma}) - \frac{1}{12}(\vec{\alpha} + \vec{\beta} + \vec{\gamma}) = \frac{1}{4}(\vec{\alpha} + \vec{\beta} + \vec{\gamma}) = \overrightarrow{DG}$. Therefore, $G = G'$.

Proof of (3). We will

$$\begin{array}{ll} \text{Maximize} & \frac{r}{L_A}+\frac{r}{L_B}+\frac{r}{L_C}+\frac{r}{L_D} \\ \text{Subject to} & \cos^{-1}\frac{r}{L_A}+\cos^{-1}\frac{r}{L_B}+\cos^{-1}\frac{r}{L_C}+\cos^{-1}\frac{r}{L_D}=\theta, \end{array}$$

for some fixed angle $\theta > 0$.

Let $\frac{r}{L_A}=x$, $\frac{r}{L_B}=y$, $\frac{r}{L_C}=z$, $\frac{r}{L_D}=w$ for simplicity. Then, we are to

$$\begin{array}{ll} \text{Maximize} & x+y+z+w \\ \text{Subject to} & \cos^{-1}x+\cos^{-1}y+\cos^{-1}z+\cos^{-1}w=\theta \quad \text{and } 0<x,\ y,\ z,\ w<1. \end{array}$$

Let $f(x,y,z,w)=x+y+z+w$ and $g(x,y,z,w)=\cos^{-1}x+\cos^{-1}y+\cos^{-1}z+\cos^{-1}w$. Then by Lagrange Multipliers' method, a critical point (x,y,z,w) must satisfy $\nabla f(x,y,z,w)=\lambda\cdot\nabla g(x,y,z,w)$ for some λ, where ∇f stands for the gradient of f. Hence,

$$\langle 1,1,1,1\rangle=\lambda\cdot\left\langle -\frac{1}{\sqrt{1-x^2}},-\frac{1}{\sqrt{1-y^2}},-\frac{1}{\sqrt{1-z^2}},-\frac{1}{\sqrt{1-w^2}}\right\rangle.$$

So, $\lambda=-\sqrt{1-x^2}=-\sqrt{1-y^2}=-\sqrt{1-z^2}=-\sqrt{1-w^2}$.

Since x, y, z, $w>0$, we must have $x=y=z=w$. Since $\frac{r}{L_A}=x$, $\frac{r}{L_B}=y$, $\frac{r}{L_C}=z$, $\frac{r}{L_D}=w$, this implies that $L_A=L_B=L_C=L_D=R$. The critical point is when the tetrahedron $\nabla ABCD$ is isosceles.

By Lemma 7, we have $R\geq 3r$ or $\frac{r}{R}\leq\frac{1}{3}$. So when $L_A=L_B=L_C=L_D=R$, we have $\frac{r}{L_A}+\frac{r}{L_B}+\frac{r}{L_C}+\frac{r}{L_D}\leq\frac{4}{3}$. Therefore, for any possible angle $\theta>0$, (that is, for any tetrahedron $\nabla ABCD$), we have $\frac{r}{L_A}+\frac{r}{L_B}+\frac{r}{L_C}+\frac{r}{L_D}\leq\frac{4}{3}$. Since $R=3r$ if and only if the tetrahedron is regular, the inequality becomes equality if and only if $\nabla ABCD$ is regular. □

Proof of (4). As in the above proof, by letting $\frac{r}{L_A}=x$, $\frac{r}{L_B}=y$, $\frac{r}{L_C}=z$, $\frac{r}{L_D}=w$, this problem is simplified to

$$\begin{array}{ll} \text{Minimize} & \cos^{-1}x+\cos^{-1}y+\cos^{-1}z+\cos^{-1}w \\ \text{Subject to} & x+y+z+w=\delta, \quad \text{and} \quad 0<x,y,z,w<1, \end{array}$$

for some fixed $\delta>0$. Let $f(x,y,z,w)=\cos^{-1}x+\cos^{-1}y+\cos^{-1}z+\cos^{-1}w$, and $g(x,y,z,w)=x+y+z+w$. From $\nabla f(x,y,z,w)=\lambda\cdot\nabla g(x,y,z,w)$, the critical point (x,y,z,w) is given by

$$\left\langle -\frac{1}{\sqrt{1-x^2}},-\frac{1}{\sqrt{1-y^2}},-\frac{1}{\sqrt{1-z^2}},-\frac{1}{\sqrt{1-w^2}}\right\rangle=\lambda\cdot\langle 1,1,1,1\rangle \quad \text{for some } \lambda.$$

Hence, $\lambda=-\frac{1}{\sqrt{1-x^2}}=-\frac{1}{\sqrt{1-y^2}}=-\frac{1}{\sqrt{1-z^2}}=-\frac{1}{\sqrt{1-w^2}}$.

Since $x,y,z,w>0$, this implies that $x=y=z=w$, which in turn implies that $L_A=L_B=L_C=L_D=R$. Thus, the critical value of $\cos^{-1}\frac{r}{L_A}+\cos^{-1}\frac{r}{L_B}+\cos^{-1}\frac{r}{L_C}+\cos^{-1}\frac{r}{L_D}$ is attained when the tetrahedron is an isosceles tetrahedron. However, among all isosceles tetrahedron $\nabla ABCD$, we have $\cos^{-1}\frac{r}{L_A}+\cos^{-1}\frac{r}{L_B}+\cos^{-1}\frac{r}{L_C}+\cos^{-1}\frac{r}{L_D}\geq 4\cos^{-1}\frac{1}{3}$ by Equation (2) in Lemma 8. This shows that the minimum value of $\cos^{-1}\frac{r}{L_A}+\cos^{-1}\frac{r}{L_B}+\cos^{-1}\frac{r}{L_C}+\cos^{-1}\frac{r}{L_D}$ is attained when the tetrahedron $\nabla ABCD$ is isosceles when $\frac{r}{L_A}+\frac{r}{L_B}+\frac{r}{L_C}+\frac{r}{L_D}=\delta$ for some $\delta>0$.

Therefore, this proves that $\cos^{-1}\frac{r}{L_A}+\cos^{-1}\frac{r}{L_B}+\cos^{-1}\frac{r}{L_C}+\cos^{-1}\frac{r}{L_D}\geq 4\cos^{-1}\frac{1}{3}$ for any tetrahedron $\nabla ABCD$, and the inequality becomes equality if and only if $\nabla ABCD$ is regular by Equation (2) in Lemma 8. □

Remark 4. The inequalities $\frac{r}{L_A}+\frac{r}{L_B}+\frac{r}{L_C}+\frac{r}{L_D}\le\frac{4}{3}$ and $3r\le R$ can be rewritten as

$$\tfrac{1}{L_A}+\tfrac{1}{L_B}+\tfrac{1}{L_C}+\tfrac{1}{L_D}\le\tfrac{4}{3r}\quad\text{and}\quad\tfrac{4}{R}\le\tfrac{4}{3r}.$$

Which is larger, $\frac{1}{L_A}+\frac{1}{L_B}+\frac{1}{L_C}+\frac{1}{L_D}$ or $\frac{4}{R}$? Let us look at the following example:

Let $A=(1,0,0)$, $B=(0,1,0)$, $C=(0,0,1)$, $D=(0,0,0)$. Then the incenter of the tetrahedron $\nabla ABCD$ is $\left(\frac{1}{3+\sqrt{3}},\frac{1}{3+\sqrt{3}},\frac{1}{3+\sqrt{3}}\right)$. Hence,

$$L_A^2=L_B^2=L_C^2=\left(1-\tfrac{1}{3+\sqrt{3}}\right)^2+2\left(\tfrac{1}{3+\sqrt{3}}\right)^2=\tfrac{9+4\sqrt{3}}{(3+\sqrt{3})^2},$$

and $L_D=\frac{\sqrt{3}}{3+\sqrt{3}}$. So,

$$\tfrac{1}{L_A}+\tfrac{1}{L_B}+\tfrac{1}{L_C}+\tfrac{1}{L_D}=3\cdot\tfrac{3+\sqrt{3}}{\sqrt{9+4\sqrt{3}}}+\tfrac{2+\sqrt{3}}{\sqrt{3}}\approx 6.289.$$

On the other hand, its circumcenter of of $\nabla ABCD$ is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, so that $R=\frac{\sqrt{3}}{2}$. Hence, $\frac{4}{R}=\frac{8}{\sqrt{3}}\approx 4.61$. This is the motivation for the next lemma.

Lemma 10. *For any tetrahedron $\nabla ABCD$, we have $\frac{1}{L_A}+\frac{1}{L_B}+\frac{1}{L_C}+\frac{1}{L_D}\ge\frac{4}{R}$. The inequality becomes equality if and only if the tetrahedron $\nabla ABCD$ is isosceles.*

Proof. This is a problem to

$$\begin{array}{ll}\text{Minimize} & \frac{1}{L_A}+\frac{1}{L_B}+\frac{1}{L_C}+\frac{1}{L_D}\\ \text{Subject to} & \cos^{-1}\frac{r}{L_A}+\cos^{-1}\frac{r}{L_B}+\cos^{-1}\frac{r}{L_C}+\cos^{-1}\frac{r}{L_D}=\theta,\end{array}$$

for some fixed $\theta>4\cos^{-1}\frac{1}{3}$ by Equation 4 in Lemma 9. For the simplicity, let $x=\frac{1}{L_A}$, $y=\frac{1}{L_B}$, $z=\frac{1}{L_C}$, $w=\frac{1}{L_D}$. Then we are to

$$\begin{array}{ll}\text{Minimize} & x+y+z+w\\ \text{Subject to} & \cos^{-1}rx+\cos^{-1}ry+\cos^{-1}rz+\cos^{-1}rw=\theta,\end{array}$$

where $\theta>4\cos^{-1}\frac{1}{3}$, and x, y, z, $w>0$. Let $f(x,y,z,w)=x+y+z+w$, and $g(x,y,z,w)=\cos^{-1}rx+\cos^{-1}ry+\cos^{-1}rz+\cos^{-1}rw$. Then $\nabla f(x,y,z,w)=\lambda\cdot\nabla g(x,y,z,w)$ gives us the critical point (x,y,z,w) for some λ. Hence,

$$\begin{aligned}\langle 1,1,1,1\rangle&=\lambda\left\langle\tfrac{-r}{\sqrt{1-(rx)^2}},\tfrac{-r}{\sqrt{1-(ry)^2}},\tfrac{-r}{\sqrt{1-(rz)^2}},\tfrac{-r}{\sqrt{1-(rw)^2}}\right\rangle,\quad\text{or}\\ -r\lambda&=\sqrt{1-(rx)^2}=\sqrt{1-(ry)^2}=\sqrt{1-(rz)^2}=\sqrt{1-(rw)^2}.\end{aligned}$$

This implies that the critical point (x,y,z,w) is given by $x=y=z=w$ since $x,y,z,w>0$. From Remark 1, this implies that the minimal value of $\frac{1}{L_A}+\frac{1}{L_B}+\frac{1}{L_C}+\frac{1}{L_D}$ is attained only when $L_A=L_B=L_C=L_D=R$ (i.e. when the in-radius is the circumradius), and the minimum value is equal to $\frac{4}{R}$. Hence, $\frac{1}{L_A}+\frac{1}{L_B}+\frac{1}{L_C}+\frac{1}{L_D}$ is minimized if and only if the tetrahedron $\nabla ABCD$ is isosceles by Lemma 1. □

Proof. Proof of Theorem 4 Theorem 4 is a consequence of Equation (3) in Lemma 9 and Lemma 10. □

Remark 5. Let $\triangle ABC$ be a triangle with the in-center H. Let r and R be the in-radius and circumradius of the triangle $\triangle ABC$. Then $R \geq 2r$, called Euler's inequality (see [5]), and the inequality becomes equality if and only if the triangle $\triangle ABC$ is equilateral. Let A', B', C' be the perpendicular feet from H to the edges BC, AC, AB, respectively. Then

$$\sphericalangle AHB' + \sphericalangle BHC' + \sphericalangle CHA' = \pi.$$

Let $L_A = |HA|$, $L_B = |HB|$, $L_C = |HC|$. Then

$$\cos^{-1}\tfrac{r}{L_A} + \cos^{-1}\tfrac{r}{L_B} + \cos^{-1}\tfrac{r}{L_C} = \sphericalangle AHB' + \sphericalangle BHC' + \sphericalangle CHA' = \pi = 3\cos^{-1}\tfrac{1}{2}.$$

This may be an interesting contrast to Equation (2) in Lemma 9.

Now, as in Equation 3 in Lemma 9, we can prove that $\frac{r}{L_A} + \frac{r}{L_B} + \frac{r}{L_C} \leq \frac{3}{2}$, where the equality holds only when the triangle $\triangle ABC$ is equilateral. By rewriting it to $\frac{1}{L_A} + \frac{1}{L_B} + \frac{1}{L_C} \leq \frac{3}{2r}$, and the inequality $2r \leq R$ can be rewritten as $\frac{3}{R} \leq \frac{3}{2r}$. Hence, as in Lemma 10, we can show that $\frac{1}{L_A} + \frac{1}{L_B} + \frac{1}{L_C} \geq \frac{3}{R}$. Thus, we have the next triangle version of Theorem 4.

Corollary 1. *Let r be the in-radius, and R the circumradius of the triangle. Let L_A, L_B, L_C be the lengths between the in-center and the vertices. Then we have*

$$\tfrac{r}{R} \leq \tfrac{1}{3}\left(\tfrac{r}{L_A} + \tfrac{r}{L_B} + \tfrac{r}{L_C}\right) \leq \tfrac{1}{2}.$$

The inequalities become equalities if and only if the triangle is equilateral.

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