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Double Contact Conics in Involution

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Abstract. Three coplanar line segments OA, OB, OC are given and three concentric ellipses C_1, C_2, C_3 are defined, so that every two of the segments are conjugate semi-diameters of one ellipse. In previous studies we proved using Analytic Plane Geometry that the problem of finding an ellipse circumscribed to C_1, C_2, C_3 has at most two solutions. The *primary solution* T_1 is always an ellipse. The *secondary solution* T_2 (if it exists) is an ellipse or a hyperbola. We also constructed T_1 using Synthetic Projective Plane Geometry.

This study investigates the existence and the construction of T_2 with Synthetic Projective Geometry, particularly Theory of Involution. We prove that the common diameters of every couple of C_1, C_2, C_3 correspond through an involution f. Criteria of Synthetic Projective Geometry determine whether f is hyperbolic or elliptic. If f is hyperbolic, exactly two double contact conics T_1, T_2 exist circumscribed to C_1, C_2, C_3. T_1 is always an ellipse. T_2 is an ellipse, a hyperbola or a degenerate parabola. The common diameters of T_1, T_2 define the double lines of f. If f is elliptic, still two double contact conics T_1, T_2 exist. Now T_1 is an ellipse circumscribed and T_2 an ellipse inscribed to C_1, C_2, C_3. Regardless of whether f is hyperbolic or elliptic, we construct T_2 using the already constructed ellipse T_1 and the involution f.

Key Words: mutually conjugate ellipses, double contact conic, elliptic/hyperbolic involution, double rays, Frégier point

MSC 2020: 51N15 (primary), 51N20, 68U05

1 Introduction

The present study is a continuation of our study [9]. In that study we considered two concentric conics C_1, C_2 intersecting at four points and we searched all conics having double contact with these two. As a solution we found an one-parameter family of conics, the so-called *double contact conics of* C_1, C_2. We noticed that this family creates a hyperbolic involution f_{AB} on

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the pencil of lines through their common centre O, with double lines the lines of the common diameters AC, BD of C_1, C_2. The lines of the contact diameters of every double contact conic C_3 with C_1, C_2 correspond through f_{AB}.[1]

In the present paper we consider three concentric ellipses, *mutually conjugate*, and we search all conics having double contact with these three. The problem of finding a fourth concentric ellipse circumscribed to all three is solved through the three-dimensional space by G. A. Peschka (1879) in his proof of K. Pohlke's *Fundamental Theorem of Axonometry*. Previous studies of ours (cf. [5, 6]) dealing with the problem as a two-dimensional one, confirmed that there is always the so-called *primary solution* T_1 of the problem, which is an ellipse. That's why the problem is referred as the *Four Ellipses Problem*. T_1 is also constructed in [5, 6] using Synthetic Projective Plane Geometry.

The present study focuses on the investigation of existence and on the construction of the *secondary solution* T_2 of the *Four Ellipses Problem* using methods of Synthetic Projective Plane Geometry, in particular the Theory of Involution.

A projective transformation, which is not the identity, but applied twice yields the identity, is called an *involution* (cf. [2, p. 212] and [4, Vol. I, p. 174]). An involution on a pencil of lines has either two fixed lines *(hyperbolic involution)* or none *(elliptic involution)* (cf. [1, p. 153] and [4, Vol. I, p. 176]). Two pairs of lines (δ_1, δ_1'), (δ_2, δ_2') are needed, in order for an involution f on a pencil of lines to be defined (cf. [1, p. 153] and [4, Vol. I, p. 175]). Then, $f(\delta_1) = \delta_1'$, $f(\delta_2) = \delta_2'$, $f(\delta_1') = \delta_1$ and for any line δ of the pencil, $f(\delta)$ is the line of the pencil defined through the cross ratio equation $(\delta_1, \delta_2, \delta_1', \delta) = (\delta_1', \delta_2', \delta_1, f(\delta))$.

2 Common Diameters of two Double Contact Conics

We consider now two double contact conics T_1, T_2 of C_1, C_2 intersecting at four points A', B', C', D' (Figure 1). We will prove the following:

Proposition 2.1. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Let T_1, T_2 be two of the double contact conics of C_1, C_2 intersecting at four points A', B', C', D'. Then, the common diameters $A'C'$, $B'D'$ of T_1, T_2 and the common diameters AC, BD of C_1, C_2 form a harmonic pencil, i.e. $O(A, B, A', B') = -1$.*[2]

Proof. Let T_1, T_2 be two double contact conics of C_1, C_2 with respect to M_1N_1, M_2N_2, i.e. M_1N_1, M_2N_2 are contact diameters of T_1, T_2 with C_1 respectively (Figure 1). Let t_1, t_2 be the gradients of M_1N_1, M_2N_2. We suppose that

$$t_1 \neq \pm\lambda_1, \qquad t_2 \neq \pm\lambda_1, \tag{1}$$

where λ_1 is the gradient of AC, in order for T_1, T_2 not to degenerate to double lines (cf. [9, Proposition 3]). According to [9, Equation (18)], T_1, T_2 have the following equations:

$$T_1 \colon \alpha_1 x^2 + 2\beta_1 xy + \gamma_1 y^2 + \delta_1 = 0 \tag{2}$$
$$T_2 \colon \alpha_2 x^2 + 2\beta_2 xy + \gamma_2 y^2 + \delta_2 = 0 \tag{3}$$

[1]In what follows, when we refer to corresponding lines of a pencil, we will use the term *common diameter* (resp. *contact diameter*) instead of the term *line of a common diameter* (resp. *line of a contact diameter*) for brevity.

[2]In what follows, the cross ratio of four concurring lines OA, OB, OC, OD (in this order) will be denoted by $O(A, B, C, D)$, instead of (OA, OB, OC, OD), for brevity.

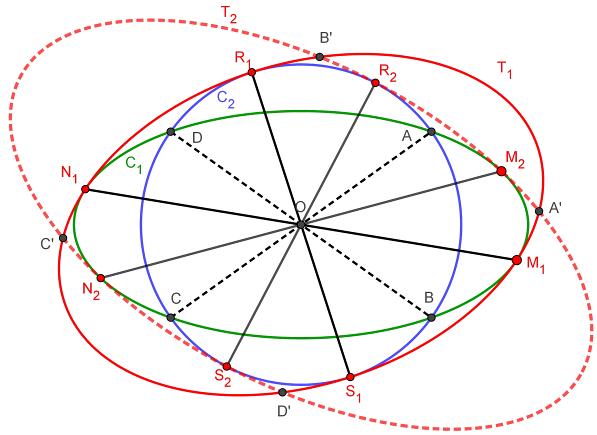

Figure 1: Two intersecting double contact conics.

where α_1, β_1, γ_1, δ_1 and α_2, β_2, γ_2, δ_2 are given by [9, Equations (19)–(22)] considering $\lambda_3 = t_1$ and $\lambda_3 = t_2$ respectively.

Let T_1, T_2 be either both inscribed to C_1, C_2, or both circumscribed to C_1, C_2, i.e.

$$(\lambda_1^2 - t_1^2)(\lambda_1^2 - t_2^2) > 0. \tag{4}$$

Let also T_1, T_2 have four intersection points A', B', C', D'. Then, C_1 can be considered as a double contact conic of T_1, T_2 with contact diameters M_1N_1, M_2N_2 respectively. Then, according to [9, Proposition 1] it holds that $O(A', B', M_1, M_2) = -1$, i.e.

$$(m_1 + m_2)(t_1 + t_2) = 2(m_1m_2 + t_1t_2) \tag{5}$$

where m_1, m_2 are respectively the gradients of lines $A'C'$, $B'D'$, which join the points of intersection, that are symmetric with respect to centre O. Similarly, C_2 can be considered as a double contact conic of T_1, T_2 with contact diameters say R_1S_1, R_2S_2 respectively. Then it holds that $O(A', B', R_1, R_2) = -1$, i.e.

$$(m_1 + m_2)(s_1 + s_2) = 2(m_1m_2 + s_1s_2) \tag{6}$$

where s_1, s_2 are the gradients of R_1S_1, R_2S_2 respectively. So, (5) and (6) lead to

$$\begin{vmatrix} t_1 + t_2 & m_1m_2 + t_1t_2 \\ s_1 + s_2 & m_1m_2 + s_1s_2 \end{vmatrix} = 0. \tag{7}$$

Since it holds $O(A, B, M_1, R_1) = -1$ and $O(A, B, M_2, R_2) = -1$, according to [9, Equation (6)] we get

$$s_1 = \frac{\lambda_1^2}{t_1}, \qquad s_2 = \frac{\lambda_1^2}{t_2}. \tag{8}$$

Substituting s_1, s_2 through (8), equation (7) leads to

$$(t_1 + t_2)(\lambda_1^2 - t_1t_2)(m_1m_2 - \lambda_1^2) = 0. \tag{9}$$

But $\lambda_1^2 - t_1t_2 = 0$ states that T_1, T_2 form *a couple of double contact conics* (cf. [9, Proposition 6]). Then, one conic is inscribed and the other one circumscribed to C_1, C_2. That means equation $\lambda_1^2 - t_1t_2 = 0$ contradicts to (4). So, equation (9) turns to

$$(t_1 + t_2)(m_1m_2 - \lambda_1^2) = 0. \tag{10}$$

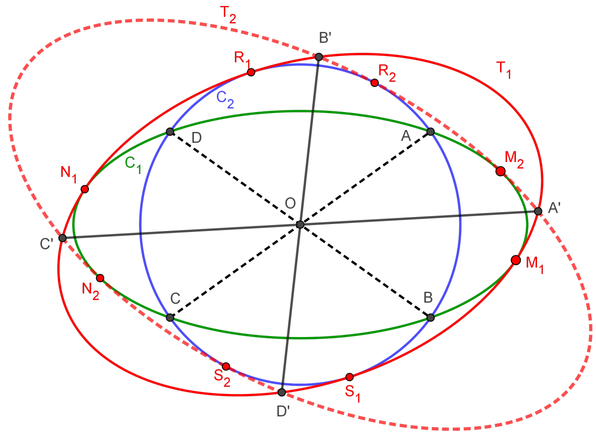

Figure 2: Common diameters $A'C', B'D'$ and AC, BD form a harmonic pencil.

- In case $t_1 + t_2 \neq 0$, equation (10) yields

$$m_1 m_2 = \lambda_1^2 \tag{11}$$

i.e.

$$O(A, B, A', B') = -1. \tag{12}$$

- In case $t_1 + t_2 = 0$, it holds

$$\alpha_2 = \alpha_1, \quad \beta_2 = -\beta_1, \quad \gamma_2 = \gamma_1 \quad \text{and} \quad \delta_2 = \delta_1. \tag{13}$$

So, equation (3) of T_2 turns to $\alpha_1 x^2 - 2\beta_1 xy + \gamma_1 y^2 + \delta_1 = 0$. Obviously, if T_1, T_2 intersect at four points, then the lines of the common diameters $A'C'$, $B'D'$ are the coordinate axes. But the coordinate axes satisfy condition (12) too.

Consequently, line $A'C'$ is always harmonic conjugate to $B'D'$ with respect to AC, BD, i.e. $O(A, B, A', B') = -1$ (Figure 2). □

Shortly, we have proved the following property:

Corollary 2.1. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D and T_1, T_2 be two of the double contact conics of C_1, C_2 intersecting at four points A', B', C', D'. Then OM_1, OM_2 are the rays through the contact points of T_1, T_2 with C_1 and OR_1, OR_2 are the rays through the contact points of T_1, T_2 with C_2 (Figure 2). We proved that the following holds:*

On the pencil of lines with vertex O, (OM_1, OM_2) and (OR_1, OR_2) are two pairs of harmonic conjugate rays with respect to rays OA', OB' and simultaneously (OM_1, OR_1) and (OM_2, OR_2) are two pairs of harmonic conjugate rays with respect to rays OA, OB. This leads to the conclusion that OA, OB are harmonic conjugate rays with respect to OA', OB', under the condition that lines OM_1, OR_2 are not coincident, i.e.:

$$\begin{cases} O(M_1, M_2, A', B') = -1 \\ O(R_1, R_2, A', B') = -1 \\ O(M_1, R_1, A, B) = -1 \\ O(M_2, R_2, A, B) = -1 \end{cases} \Rightarrow O(A, B, A', B') = -1. \tag{14}$$

Remark 2.1. It can be easily verified, that T_1, T_2 have four intersection points, in the following cases:

- T_1, T_2 are both ellipses inscribed to C_1, C_2,
- T_1, T_2 are both ellipses circumscribed to C_1, C_2,
- T_1 is an ellipse and T_2 is a hyperbola or a degenerate parabola, both circumscribed to C_1, C_2.

Remark 2.2. It can be easily proved, that the result of Proposition 2.1. remains true, if the two ellipses C_1, C_2 are replaced by two arbitrary regular conics C_1, C_2 having four intersection points A, B, C, D.

Considering [9, Remark 5], Proposition 2.1 can be formulated as follows (Figure 2):

Lemma 2.1. *Let C_1, C_2 be two arbitrary regular conics with common centre O intersecting at four points A, B, C, D. Let T_1, T_2 be two of the double contact conics of C_1, C_2 intersecting at four points A', B', C', D'. Then, the following hold:*

- *The common diameters AC, BD of C_1, C_2 correspond through the hyperbolic involution $f_{A'B'}$ on the pencil of lines through O, with double lines the common diameters $A'C'$, $B'D'$ of T_1, T_2. The contact diameters M_1N_1, M_2N_2 of C_1 with T_1, T_2 also correspond through $f_{A'B'}$. So do the contact diameters R_1S_1, R_2S_2 of C_2 with T_1, T_2.*
- *The common diameters $A'C'$, $B'D'$ of T_1, T_2 correspond through the hyperbolic involution f_{AB} on the pencil of lines through O, with double lines the common diameters AC, BD of C_1, C_2.*

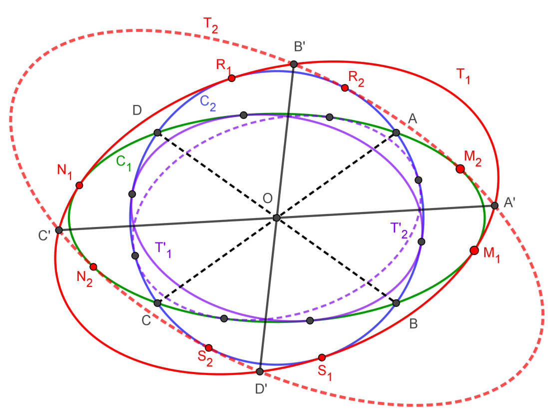

Figure 3: Common diameters of T_1, T_2 and common diameters of T_1', T_2' are collinear.

According to [9, Proposition 6], the double contact conics of C_1, C_2 are in couples, i.e. every diameter of C_1 corresponds to two double contact conics of C_1, C_2, one circumscribed and one inscribed to C_1, C_2. The next proposition relates the common diameters of two circumscribed double contact conics of C_1, C_2 with the common diameters of their corresponding inscribed double contact conics (Figure 3). The result follows directly from Lemma 2.1.

Proposition 2.2. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Let T_1, T_2 be two double contact conics circumscribed to C_1, C_2 and T_1', T_2' their corresponding double contact conics of C_1, C_2 inscribed to C_1, C_2. Let T_1, T_2 intersect at four points A', B', C', D'. Then, the common diameters of T_1', T_2' lie on the common diameters of T_1, T_2 respectively.*

Proof. Let $A''C''$, $B''D''$ be the common diameters of T_1', T_2' (Figure 3). According to Lemma 2.1 on the pencil of rays through O two hyperbolic involutions are defined: $f_{A''B''}$ with double lines $A''C''$, $B''D''$ and $f_{A'B'}$ with double lines $A'C'$, $B'D'$. Then, the common diameters AC, BD of C_1, C_2 correspond through both involutions. We will prove that so do the contact diameters M_1N_1, M_2N_2 of C_1 with T_1, T_2. Indeed, M_1N_1, M_2N_2 correspond through $f_{A'B'}$ according to Lemma 2.1 Furthermore they carry the contact diameters of C_2 with T_1', T_2'. Consequently, M_1N_1, M_2N_2 correspond through $f_{A''B''}$, too. So, involutions $f_{A'B'}$, $f_{A''B''}$ coincide, since they have two common pairs: (AC, BD) and (M_1N_1, M_2N_2). Then, their double lines coincide too, i.e. the common diameters of T_1', T_2' lie on the common diameters of T_1, T_2. □

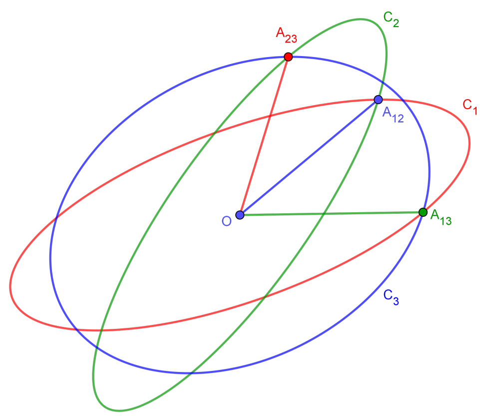

Figure 4: Every two of the three line segments are two conjugate semi–diameters of one of the three ellipses.

3 The Four Ellipses Problem

In [5] the following problem has been studied (Figure 4):

> *Consider three coplanar line segments, having one start point in common, where only two of them are permitted to coincide. Three concentric ellipses can then be defined, say C_i, $i = 1, 2, 3$, such that every two of these three line segments are considered to be two conjugate semi–diameters of each ellipse. Can we determine a concentric to C_i ellipse T, circumscribing all C_i, $i = 1, 2, 3$, using only Synthetic Projective Plane Geometry?*

The above plane–geometric problem (referred by the authors as the *Four Ellipses Problem*) is solved in [5] by presenting one solution T_1. The same problem is also investigated in [10] in order for all existing circumscribing ellipses T of C_i, $i = 1, 2, 3$ to be determined. This time the problem was investigated exclusively with methods of Analytic Geometry. It is proved that, at most, two (concentric to C_i) circumscribing conics of C_i, $i = 1, 2, 3$ exist. One of them, say T_1, is always an ellipse. We shall call it *primary solution* of the problem. The other one, say T_2, if it exists, it is either an ellipse or a hyperbola. We shall call it *secondary solution* of the problem.

In [7] a necessary and sufficient condition for the existence of the two circumscribing ellipses T_1, T_2 is given through the three-dimensional space.

In [6] a new construction of the *primary solution* T_1 is introduced using methods of Synthetic Plane Projective Geometry. In the present study we will go one step further. In case the *secondary solution* T_2 exists (i.e. there exist a second conic circumscribing C_i, $i = 1, 2, 3$), we will use the already constructed T_1 in [6] and a hyperbolic involution to construct T_2, regardless of the type of T_2. So, T_2 will be also constructed using methods of Synthetic Projective Plane Geometry.

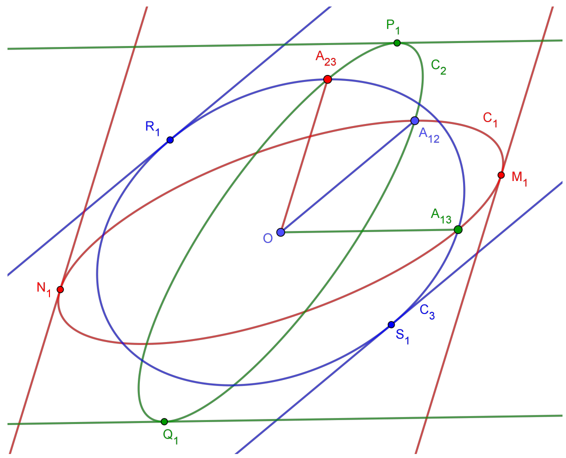

Figure 5: Tangent lines of each ellipse are parallel to the given line segment, which is not a semi-diameter of this ellipse.

3.1 Construction of the Primary Solution T_1

In the real projective plane three line segments are given, having one start point in common, say OA_{13}, OA_{12}, OA_{23} (Figure 4). Following *Rytz's Construction* (cf. [2, p. 357] and [4, Vol. II, Issue B, p. 183]) three concentric ellipses can then be defined, say C_i, $i = 1, 2, 3$, such that every two of these three line segments are two conjugate semi–diameters of each ellipse, i.e.

- OA_{13}, OA_{12} are two conjugate semi–diameters of C_1,
- OA_{12}, OA_{23} are two conjugate semi–diameters of C_2 and
- OA_{13}, OA_{23} are two conjugate semi–diameters of C_3.

In the following C_1, C_2, C_3 are called *mutually conjugate* ellipses (cf. [10, p. 64]).

According to [6] we consider the tangent lines of each ellipse C_i, $i = 1, 2, 3$, that are parallel to the given line segment, which is not a semi-diameter of C_i (Figure 5). The corresponding contact points M_1, N_1, P_1, Q_1, R_1, S_1 determine an ellipse T_1. It is proved that T_1 has double contact with C_1, C_2, C_3 at M_1, N_1, P_1, Q_1, R_1, S_1 respectively. This ellipse is defined as the *primary solution* of the *Four Ellipses Problem* (Figure 6).

Remark 3.1. Obviously, M_1N_1 is the diameter of C_1 whose conjugate diameter lies on OA_{23}, i.e. M_1N_1 corresponds to diameter $A_{23}C_{23}$ through the elliptic involution, through which the conjugate diameters of C_1 correspond. Similarly, P_1Q_1 (resp. R_1S_1) corresponds to $A_{13}C_{13}$ (resp. $A_{12}C_{12}$) through the respective involution of C_2 (resp. C_3).

3.2 Construction of the Secondary Solution T_2

Let T_2 be the *secondary solution* of the problem. Both solutions T_1, T_2 are double contact conics of C_1, C_2, C_3, circumscribed to C_1, C_2, C_3 and T_1 is always an ellipse. T_2 can be an

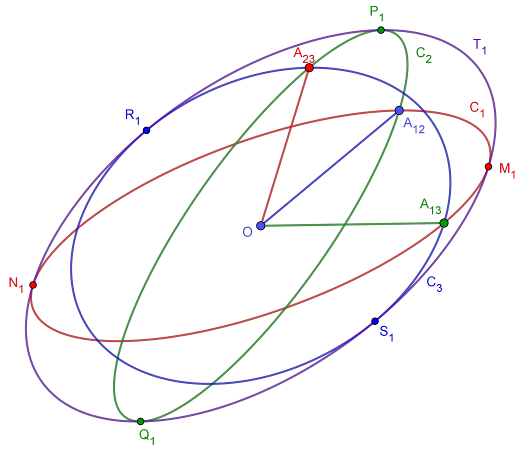

Figure 6: Primary solution T_1 of the Four Ellipses Problem.

ellipse, a hyperbola or a degenerate parabola (i.e. a pair of parallel lines or a double line) (cf. [9, Proposition 3]). So, T_1, T_2 intersect at four points, say A', B', C', D' (cf. Remark 2.1). A', B', C', D' are now unknown and they have to be determined.

Let $f_{A'B'}$ be the hyperbolic involution defined on the pencil of lines through O with double lines the common chords $A'C'$, $B'D'$ through O of T_1, T_2.

We consider now C_1, C_2 as two double contact ellipses of T_1, T_2, intersecting at four points A_{12}, B_{12}, C_{12}, D_{12}. If $A_{12}C_{12}$, $B_{12}D_{12}$ are the common diameters of C_1, C_2 (Figure 7), according to Lemma 2.1, $A_{12}C_{12}$, $B_{12}D_{12}$ and common diameters $A'C'$, $B'D'$ form a harmonic pencil. So, $A_{12}C_{12}$, $B_{12}D_{12}$ correspond through $f_{A'B'}$. Similarly, considering C_2, C_3 as two double contact conics of T_1, T_2, the common diameters $A_{23}C_{23}$, $B_{23}D_{23}$ of C_2, C_3 (Figure 7) correspond through $f_{A'B'}$. So do the common diameters $A_{13}C_{13}$, $B_{13}D_{13}$ of C_1, C_3.

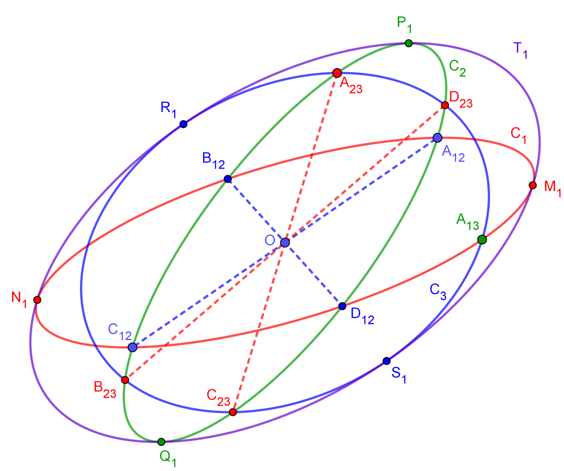

Figure 7: Common diameters $A_{12}C_{12}$, $B_{12}D_{12}$ correspond through $f_{A'B'}$. So do common diameters $A_{23}C_{23}$, $B_{23}D_{23}$.

The two pairs of lines $(A_{12}C_{12}, B_{12}D_{12})$ and $(A_{23}C_{23},\ B_{23}D_{23})$ through O enable us to determine the hyperbolic involution $f_{A'B'}$, through which the members of the pairs correspond. Then, we can construct the double lines of the hyperbolic involution $f_{A'B'}$. For this purpose we use the following (cf. [2, p. 255] and [4, Vol. I, p. 200, 202]):

Theorem (Frégier's Theorem[3]). *Let f be an involution on a pencil of lines with vertex O. If vertex O lies on a conic c, then the lines, that join the intersection points of corresponding lines of the pencil with the conic, pass through one fixed point F. Point F lies on the line of the pencil, which corresponds to the tangent line of the conic c at point O. Conversely, the intersecting points of conic c and a line through point F define a couple of corresponding lines of the pencil.*

Point F is called the *Frégier point* to c and O (cf. [4, Vol. I, p. 199] and [8, p. 201]). According to the above theorem and Lemma 2.1 we construct T_2 following the next steps:

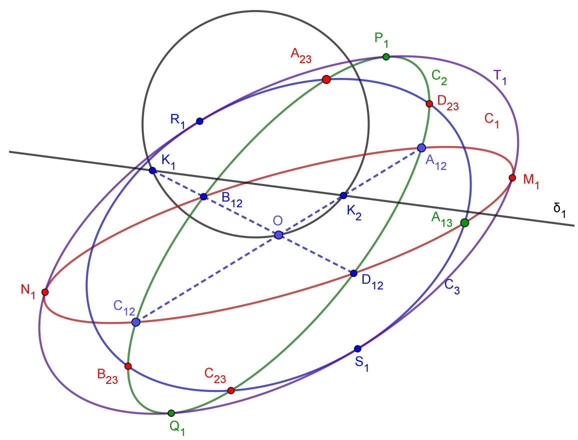

Figure 8: $A_{12}C_{12}, B_{12}D_{12}$ define secant δ_1 of circle c.

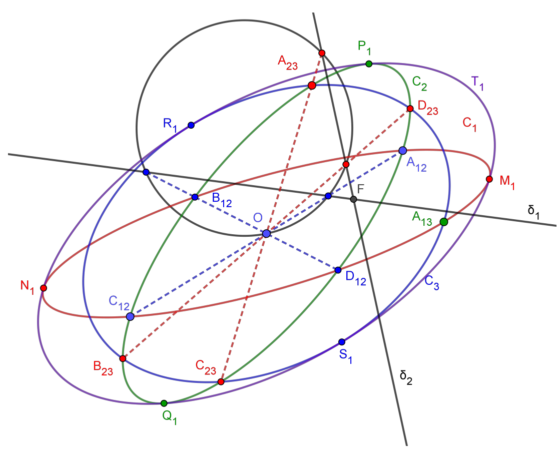

Figure 9: Lines δ_1, δ_2 intersect at Frégier point F.

Step 1: We consider a circle c passing through point O. Let $A_{12}C_{12}, B_{12}D_{12}$ intersect circle c (except of O) at K_1, K_2 respectively. Then, K_1, K_2 define a secant δ_1 of c (Figure 8).
Step 2: Similarly to Step 1, $A_{23}C_{23}$, $B_{23}D_{23}$ define a secant δ_2 of c.
Step 3: Lines δ_1, δ_2 intersect at Frégier point F (Figure 9).

We consider now each of the contact chords M_1N_1, P_1Q_1, R_1S_1 of C_i, T_1, $i = 1, 2, 3$ respectively and we construct its corresponding line through $f_{A'B'}$ in the following way:
Step 4: Line M_1N_1 intersects circle c at point L_1, different than O (Figure 10).

[3]P. F. Frégier, Annales des Math. Pures et Appl., **6** (1815–1816), pp. 321–323.

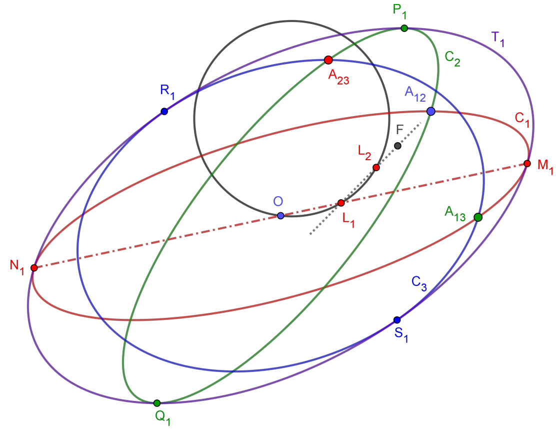

Figure 10: FL_1 intersects c at L_2.

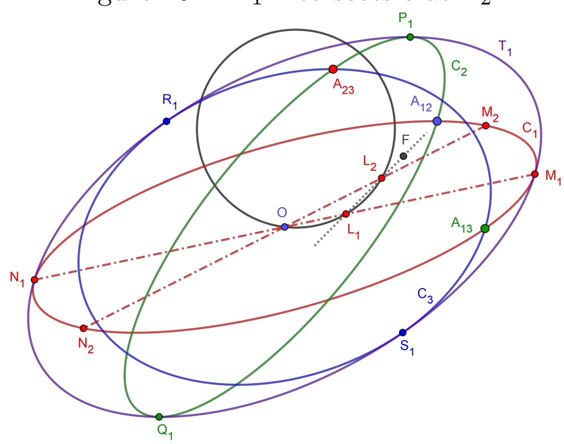

Figure 11: OL_2 corresponds to M_1N_1 through involution $f_{A'B'}$.

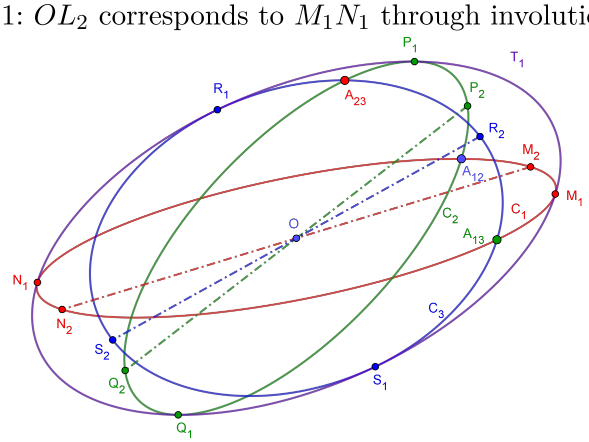

Figure 12: T_2 passes through M_2, N_2, P_2, Q_2, R_2, S_2.

Step 5: We join point L_1 and Frégier point F.
Step 6: Line FL_1 intersects c at L_2.
Step 7: Then, line OL_2 is the corresponding line of M_1N_1 (Figure 11) and its intersection points M_2, N_2 with C_1 are the contact points of T_2, C_1.

We repeat Steps 4–7 to construct the contact points P_2, Q_2 of T_2, C_2 and the contact points R_2, S_2 of T_2, C_3.

Final Step: We construct T_2 passing through M_2, N_2, P_2, Q_2, R_2, S_2 (Figure 12).

Hence, we have constructed T_2 using T_1 and the involution defined by two pairs of common

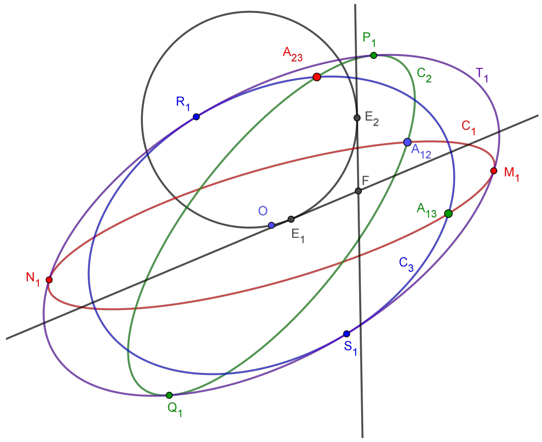

Figure 13: FE_1, FE_2 are the tangent lines of c through F.

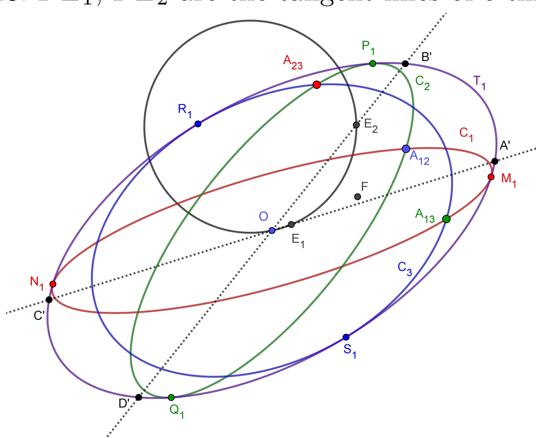

Figure 14: OE_1, OE_2 are the double lines of the involution $f_{A'B'}$.

diameters of the ellipses C_1, C_2, C_3. Since T_1, T_2 are double contact conics of C_1, C_2, C_3 and their contact diameters with C_1, C_2, C_3 correspond through this involution, T_1, T_2 are called *double contact conics in involution*.

In the sequel, in order to determine the double lines of the involution $f_{A'B'}$, we consider the tangent lines of c through point F. Since $f_{A'B'}$ is a hyperbolic involution, Frégier point F lies outside circle c. So, there are two tangent lines of c passing through F. Let E_1, E_2 be their contact points with c (Figure 13). Then, lines OE_1, OE_2 are the double lines of the hyperbolic involution $f_{A'B'}$ (Figure 14). Their intersection points with T_1 are exactly the intersection points A', B', C', D' of T_1, T_2. So, T_2 passes through A', B', C', D' too (Figure 15).

The *secondary solution* T_2 of the *Four Ellipses Problem* can be an ellipse, a hyperbola or a degenerate parabola (i.e. a pair of parallel lines or a double line) (Figures 16, 15, 17 respectively).

Remark 3.2. The secondary solution T_2 of the Four Ellipses Problem degenerates to a pair of parallel lines, in case three endpoints of the common diameters of C_1, C_2, C_3 through O are collinear, i.e. if A_{12}, A_{23}, A_{13} are collinear (Figure 17 left) or A_{12}, A_{23}, C_{13} are collinear (Figure 17 right). In this case, lines of T_2 are parallel to the line that carries the three collinear points.

Remark 3.3. The secondary solution T_2 degenerates to a double line, in case C_1, C_2, C_3 are

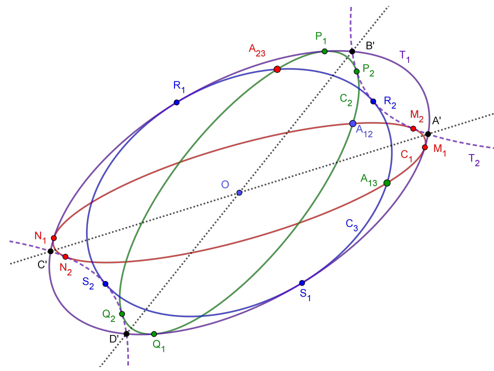

Figure 15: T_2 passes through A', B', C', D' and M_2, N_2, P_2, Q_2, R_2, S_2.

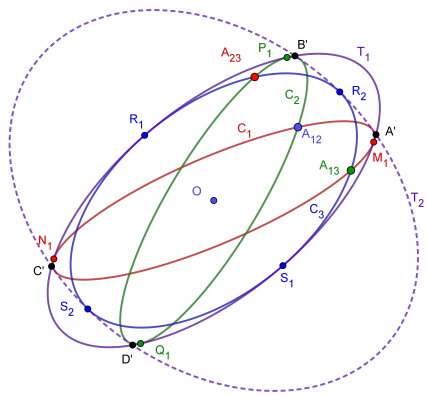

Figure 16: T_2 as an ellipse.

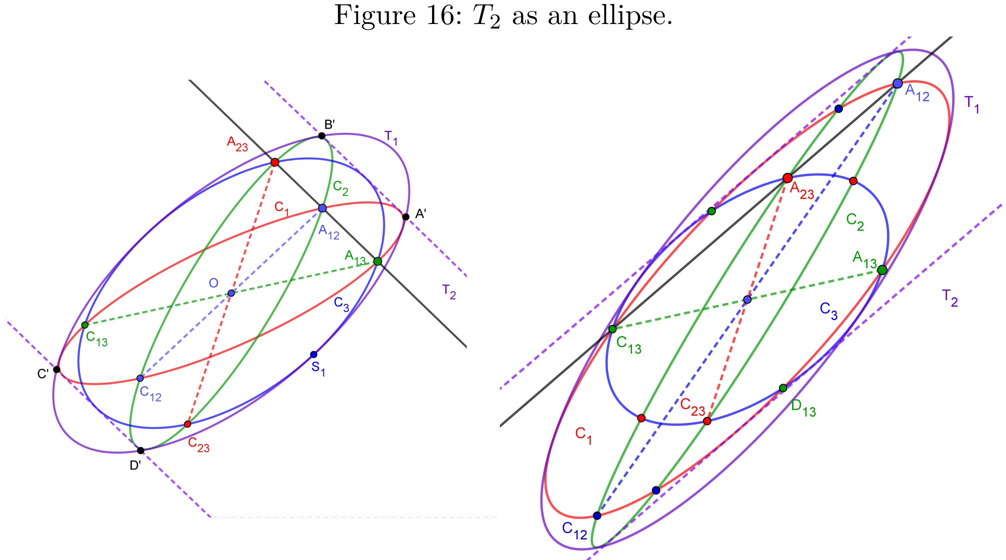

Figure 17: T_2 as a pair of parallel lines, in case points A_{12}, A_{23}, A_{13} are collinear (left) or A_{12}, A_{23}, C_{13} are collinear (right).

concurrent, i.e. three common diameters coincide (Figure 18). In this case, the double line T_2 carries the triple common diameter. Now involution $f_{A'B'}$ can not be defined and Frégier point F lies on circle c.

Remark 3.4. It is worth noting that, although $A_{12}C_{12}$, $B_{12}D_{12}$ correspond through $f_{A'B'}$ and contact diameters M_1N_1, P_1Q_1 form with $A_{12}C_{12}$, $B_{12}D_{12}$ a harmonic pencil, M_1N_1, P_1Q_1 do

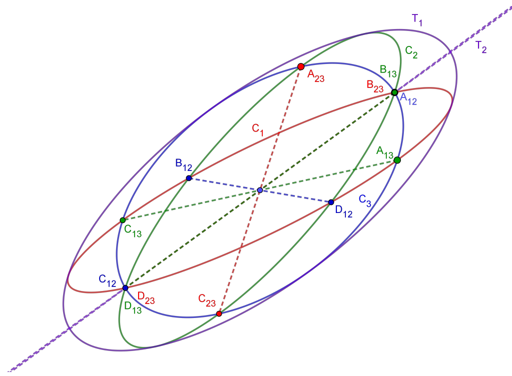

Figure 18: T_2 as a double line, in case C_1, C_2, C_3 are concurrent. Common diameters $A_{12}C_{12}$, $B_{13}D_{13}$, $B_{23}D_{23}$ coincide.

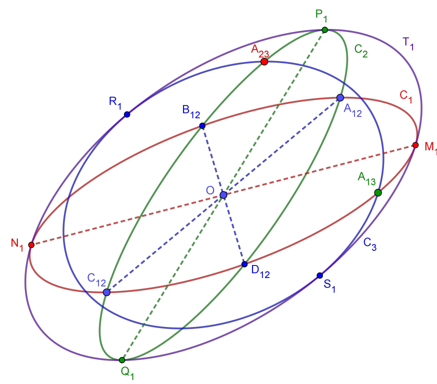

Figure 19: Common diameters $A_{12}C_{12}, B_{12}D_{12}$ and contact diameters M_1N_1, P_1Q_1 form a harmonic pencil, but only $A_{12}C_{12}, B_{12}D_{12}$ correspond through $f_{A'B'}$.

not correspond through $f_{A'B'}$ (Figure 19). Instead, contact diameter M_1N_1 corresponds to contact diameter M_2N_2 through $f_{A'B'}$, where M_2, N_2 are the contact points of C_1, T_2. But M_1N_1, P_1Q_1 do correspond through the hyperbolic involution $f_{A_{12}B_{12}}$ defined on the pencil of lines through O with double lines $A_{12}C_{12}$, $B_{12}D_{12}$. In our study we restricted our interest to the hyperbolic involution $f_{A'B'}$.

4 The Involution Defined by the Pairs of Common Diameters

In the general case, if three line segments are given, having one start point in common, say OA_{13}, OA_{12}, OA_{23}, then three concentric *mutually conjugate* ellipses C_1, C_2, C_3 are defined.

Let C_1, C_2 (resp. C_2, C_3) intersect at four points A_{12}, B_{12}, C_{12}, D_{12} (resp. A_{23}, B_{23}, C_{23}, D_{23}) and $A_{12}C_{12}$, $B_{12}D_{12}$ (resp. $A_{23}C_{23}$, $B_{23}D_{23}$) be their common diameters. Then, using the two pairs of lines ($A_{12}C_{12}$, $B_{12}D_{12}$), ($A_{23}C_{23}$, $B_{23}D_{23}$) we determine an involution f on the pencil of lines through O, through which the members of the pairs correspond. Involution f can be either elliptic or hyperbolic depending on whether the pairs $(A_{12}C_{12}, B_{12}D_{12})$, $(A_{23}C_{23}, B_{23}D_{23})$ are mutually separated or not (cf. [2, p. 211] and [4, Vol. I, p. 177]). In Figure 20 (left) the pairs $(A_{12}C_{12}, B_{12}D_{12})$, $(A_{23}C_{23}, B_{23}D_{23})$ define an elliptic involution f. In Figure 20 (right) they define a hyperbolic involution f.

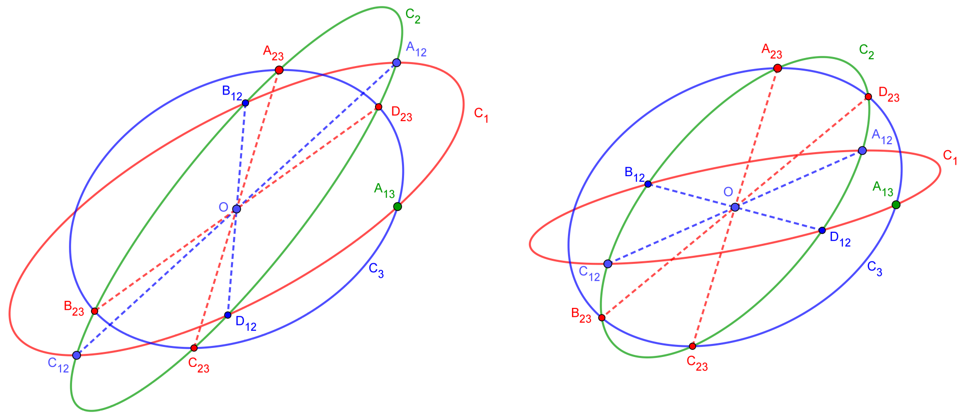

Figure 20: $A_{12}C_{12}$, $B_{12}D_{12}$ and $A_{23}C_{23}$, $B_{23}D_{23}$ are mutually separated on the left, but not on the right.

4.1 The Equation of Involution f

First we determine the equation of involution f. Let λ_{12}, μ_{12}, λ_{23}, μ_{23} be the gradients of lines $A_{12}C_{12}$, $B_{12}D_{12}$, $A_{23}C_{23}$, $B_{23}D_{23}$ respectively. We assume that $\lambda_{12} \neq \mu_{12}$ and $\lambda_{23} \neq \mu_{23}$, so that neither $A_{12}C_{12}$, $B_{12}D_{12}$, nor $A_{23}C_{23}$, $B_{23}D_{23}$ coincide. Let also μ, μ' the gradients of a line OM and its corresponding line OM' through f. Then it holds $O(A_{12}, B_{12}, A_{23}, M) = O(B_{12}, A_{12}, B_{23}, M')$. So,

$$\frac{\lambda_{23}-\lambda_{12}}{\mu_{12}-\lambda_{23}} \cdot \frac{\mu_{12}-\mu}{\mu-\lambda_{12}} = \frac{\mu_{23}-\mu_{12}}{\lambda_{12}-\mu_{23}} \cdot \frac{\lambda_{12}-\mu'}{\mu'-\mu_{12}} \tag{15}$$

or equivalently

$$\begin{vmatrix} \lambda_{12}\mu_{12} & \lambda_{12}+\mu_{12} & 1 \\ \lambda_{23}\mu_{23} & \lambda_{23}+\mu_{23} & 1 \\ \mu\mu' & \mu+\mu' & 1 \end{vmatrix} = 0. \tag{16}$$

Equation (16) is exactly the equation of involution f.

In the sequel we prove that the common diameters of C_1, C_3 also correspond through f.

Proposition 4.1. *Let C_1, C_2, C_3 be three mutually conjugate ellipses with common centre O. Let f be the involution on the pencil of lines through O determined by the pairs of the common diameters of C_1, C_2 and C_2, C_3. Then, the common diameters of C_1, C_3 also correspond through involution f.*

Proof. Let C_1, C_3 intersect at four points A_{13}, B_{13}, C_{13}, D_{13} and $A_{13}C_{13}$, $B_{13}D_{13}$ be their common diameters with gradients λ_{13}, μ_{13} respectively. Let also

$$C_1\colon \alpha_1 x^2 + 2\beta_1 xy + \gamma_1 y^2 + \delta_1 = 0, \tag{17}$$

$$C_2\colon \alpha_2 x^2 + 2\beta_2 xy + \gamma_2 y^2 + \delta_2 = 0, \tag{18}$$

$$C_3\colon \alpha_3 x^2 + 2\beta_3 xy + \gamma_3 y^2 + \delta_3 = 0 \tag{19}$$

be the equations of C_1, C_2, C_3. So, if line $\varepsilon\colon y = \ell x$ is a secant of C_1, C_2 through O, then it holds

$$\begin{cases} (\alpha_1 + 2\beta_1\ell + \gamma_1\ell^2)x^2 + \delta_1 = 0, \\ (\alpha_2 + 2\beta_2\ell + \gamma_2\ell^2)x^2 + \delta_2 = 0. \end{cases} \tag{20}$$

Therefore, it holds

$$\begin{vmatrix} \alpha_1 + 2\beta_1\ell + \gamma_1\ell^2 & \delta_1 \\ \alpha_2 + 2\beta_2\ell + \gamma_2\ell^2 & \delta_2 \end{vmatrix} = 0. \tag{21}$$

Consequently, gradients λ_{12}, μ_{12} are exactly the roots of the equation

$$\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{vmatrix} \ell^2 + 2 \begin{vmatrix} \beta_1 & \delta_1 \\ \beta_2 & \delta_2 \end{vmatrix} \ell + \begin{vmatrix} \alpha_1 & \delta_1 \\ \alpha_2 & \delta_2 \end{vmatrix} = 0. \tag{22}$$

So, it holds

$$\lambda_{12} + \mu_{12} = -2\frac{\begin{vmatrix} \beta_1 & \delta_1 \\ \beta_2 & \delta_2 \end{vmatrix}}{\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{vmatrix}}, \qquad \lambda_{12} \cdot \mu_{12} = \frac{\begin{vmatrix} \alpha_1 & \delta_1 \\ \alpha_2 & \delta_2 \end{vmatrix}}{\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{vmatrix}}. \tag{23}$$

Similarly, it holds

$$\lambda_{23} + \mu_{23} = -2\frac{\begin{vmatrix} \beta_2 & \delta_2 \\ \beta_3 & \delta_3 \end{vmatrix}}{\begin{vmatrix} \gamma_2 & \delta_2 \\ \gamma_3 & \delta_3 \end{vmatrix}}, \qquad \lambda_{23} \cdot \mu_{23} = \frac{\begin{vmatrix} \alpha_2 & \delta_2 \\ \alpha_3 & \delta_3 \end{vmatrix}}{\begin{vmatrix} \gamma_2 & \delta_2 \\ \gamma_3 & \delta_3 \end{vmatrix}} \tag{24}$$

and also

$$\lambda_{13} + \mu_{13} = -2\frac{\begin{vmatrix} \beta_1 & \delta_1 \\ \beta_3 & \delta_3 \end{vmatrix}}{\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_3 & \delta_3 \end{vmatrix}}, \qquad \lambda_{13} \cdot \mu_{13} = \frac{\begin{vmatrix} \alpha_1 & \delta_1 \\ \alpha_3 & \delta_3 \end{vmatrix}}{\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_3 & \delta_3 \end{vmatrix}}. \tag{25}$$

Using (23), (24) and (25) it can be easily verified that

$$\begin{vmatrix} \lambda_{12}\mu_{12} & \lambda_{12} + \mu_{12} & 1 \\ \lambda_{23}\mu_{23} & \lambda_{23} + \mu_{23} & 1 \\ \lambda_{13}\mu_{13} & \lambda_{13} + \mu_{13} & 1 \end{vmatrix} = 0. \tag{26}$$

So, according to (16), $A_{13}C_{13}$, $B_{13}D_{13}$ correspond through f. □

4.2 The Construction of T_2 Through Involution f

Let now T_1 be the primary solution of the *Four Ellipses Problem* and M_1N_1, P_1Q_1, R_1S_1 the contact diameters of C_i, T_1, $i = 1, 2, 3$ respectively.

We shall prove the contact diameters of C_i, T_1 and C_i, T_2 correspond through f for all $i = 1, 2, 3$, regardless whether f is elliptic or hyperbolic. So, the *secondary solution* T_2 of the problem can be constructed through involution f in any case.

Theorem 4.1. *Let C_1, C_2, C_3 be three mutually conjugate ellipses with common centre O. Let T_1 be the primary solution of the Four Ellipses Problem. Let f be the involution on the pencil of lines through O determined by any two of the three pairs of common diameters of C_1, C_2, C_3. The corresponding lines through f of the contact diameters of C_i, T_1, $i = 1, 2, 3$ determine the secondary solution T_2 of the Four Ellipses Problem.*

Proof. Let T_1 be the *primary solution* of the *Four Ellipses Problem* and M_1N_1, P_1Q_1, R_1S_1 the contact diameters of C_i, T_1, $i = 1, 2, 3$. Let the corresponding line of M_1N_1 through involution f intersect C_1 at M_2, N_2, the corresponding line of P_1Q_1 through f intersect C_2 at P_2, Q_2 and the corresponding line of R_1S_1 through f intersect C_3 at R_2, S_2.

We shall prove that the *secondary solution* of the *Four Ellipses Problem* is exactly the conic T_2 passing through M_2, N_2, P_2, Q_2, R_2, S_2, i.e. the conic passing through M_2, N_2, P_2, Q_2, R_2, S_2 has double contact with C_1 at M_2, N_2, double contact with C_2 at P_2, Q_2 and double contact with C_3 at R_2, S_2.

Since involution f preserves the cross ratio, it holds

$$O(A_{12}, B_{12}, M_2, P_2) = O(B_{12}, A_{12}, M_1, P_1). \tag{27}$$

But contact diameters M_1N_1, P_1Q_1 form with $A_{12}C_{12}$, $B_{12}D_{12}$ a harmonic pencil (cf. [3, p. 287, Case (b)]) i.e. $O(B_{12}, A_{12}, M_1, P_1) = -1$. So,

$$O(A_{12}, B_{12}, M_2, P_2) = -1 \tag{28}$$

i.e. M_2N_2 and P_2Q_2 form with $A_{12}C_{12}$, $B_{12}D_{12}$ a harmonic pencil. Then, according to [9, Proposition 1]

- there is a unique conic K_1 passing through M_2, N_2, P_2, Q_2 and having double contact with C_1 and C_2 at M_2, N_2 and P_2, Q_2 respectively.

Similarly, there is

- a unique conic K_2 passing through M_2, N_2, R_2, S_2 and having double contact with C_1 and C_3 at M_2, N_2 and R_2, S_2 respectively, and
- a unique conic K_3 passing through P_2, Q_2, R_2, S_2 and having double contact with C_2 and C_3 at P_2, Q_2 and R_2, S_2 respectively.

We will prove that K_1, K_2, K_3 coincide. We have the following cases:

Among K_1, K_2, K_3 we have two ellipses, say K_1, K_2. They are concentric ellipses having double contact at two antipodal points M_2, N_2. Then, all points of the one ellipse – say K_2 – (except M_2, N_2) lie inside the other ellipse-say K_1. So, points R_2, S_2 lie inside K_1. K_1, K_3 are also two concentric conics having double contact at two antipodal points P_2, Q_2, and K_3 passes through points R_2, S_2. So, K_3 is also an ellipse and all points of K_3 (except P_2, Q_2) lie inside K_1. Then, K_2, K_3 are two concentric ellipses inscribed K_1. So, K_2, K_3 intersect at four points, which is absurd, because K_2, K_3 have double contact with C_3 at R_2, S_2 and therefore they have double contact with each other at these two antipodal points. So, K_2, K_3 coincide i.e. all three ellipses K_1, K_2, K_3 coincide. That means there is a unique ellipse passing through M_2, N_2, P_2, Q_2, R_2, S_2 having double contact with C_1, C_2, C_3 at M_2, N_2, P_2, Q_2, R_2, S_2 respectively.

Among K_1, K_2, K_3 we have two hyperbolas, say K_1, K_2. They are concentric hyperbolas having double contact at two antipodal points M_2, N_2. Since they both have double contact with ellipse C_1 at M_2, N_2, all points of the one hyperbola-say K_2-(except M_2, N_2) lie inside the other hyperbola-say K_1. So, similarly to the case of K_1, K_2 being two ellipses, it can be proved that K_2, K_3 coincide i.e. all three hyperbolas K_1, K_2, K_3 coincide. That means there is a unique hyperbola passing through M_2, N_2, P_2, Q_2, R_2, S_2 having double contact with C_1, C_2, C_3 at M_2, N_2, P_2, Q_2, R_2, S_2 respectively.

Among K_1, K_2, K_3 there are neither two ellipses, nor two hyperbolas, i.e. all K_1, K_2, K_3 are degenerate parabolas (couples of parallel lines). But K_1, K_2 have double contact at M_2, N_2 and K_2, K_3 have double contact at R_2, S_2. So, K_1, K_2, K_3 coincide. That means there is a unique degenerative parabola passing through M_2, N_2, P_2, Q_2, R_2, S_2 having double contact with C_1, C_2, C_3 at M_2, N_2, P_2, Q_2, R_2, S_2 respectively.

The *secondary solution* T_2 of the *Four Ellipses Problem* is exactly the unique conic passing through M_2, N_2, P_2, Q_2, R_2, S_2. □

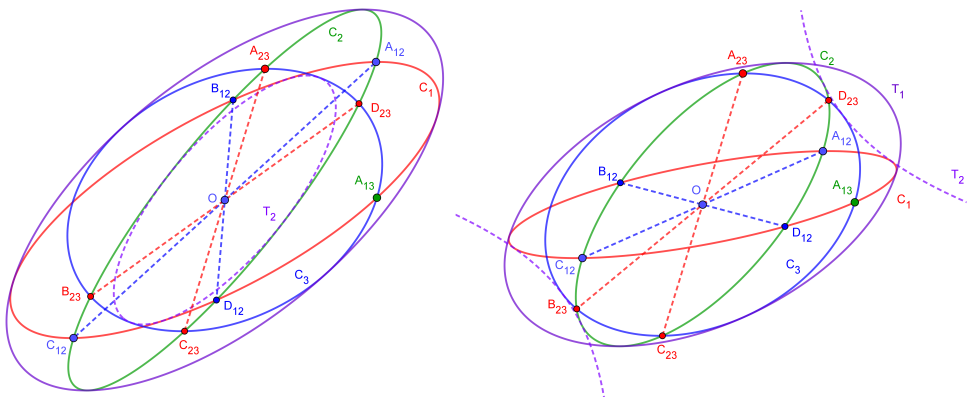

Figure 21: Left: Elliptic involution yields that T_2 is inscribed to C_1, C_2, C_3. Right: Hyperbolic involution yields that T_2 is circumscribed to C_1, C_2, C_3.

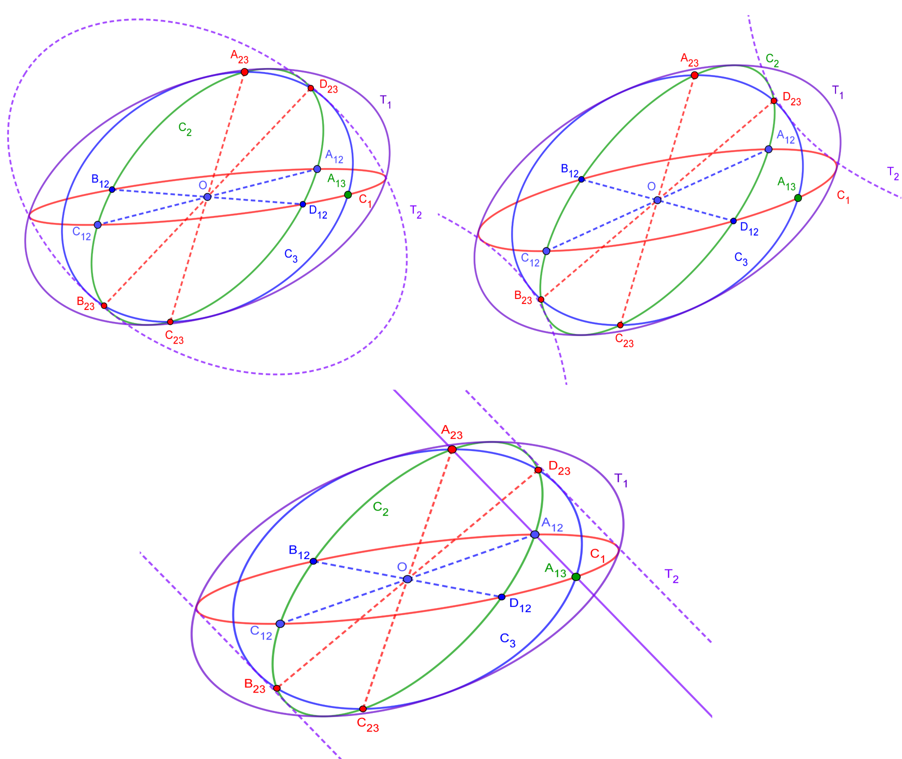

Figure 22: If involution f is hyperbolic, T_2 can be an ellipse (left), a hyperbola (right) or a degenerate parabola (down) circumscribed to C_1, C_2, C_3.

Remark 4.1. In case involution f is hyperbolic the secondary solution T_2 of the Four Ellipses Problem is an ellipse, a hyperbola or a degenerate parabola circumscribed to C_1, C_2, C_3 having four intersection points with the primary solution T_1. In case involution f is elliptic the Four Ellipses Problem has still a secondary solution T_2, but this time T_2 is an ellipse inscribed to C_1, C_2, C_3 (Figure 21).

Conclusion

In the real projective plane three line segments OA, OB, OC are given and three *mutually conjugate* ellipses C_1, C_2, C_3 with common centre O are defined. We proved that the common diameters of every couple of C_1, C_2, C_3 correspond through an involution f. Criteria of Synthetic Projective Plane Geometry determine whether f is hyperbolic or elliptic.

If f is hyperbolic, then there exist exactly two conics T_1, T_2 concentric to C_1, C_2, C_3, that circumscribe C_1, C_2, C_3. The *primary solution* T_1, is always an ellipse, while the *secondary solution* T_2 is an ellipse, a hyperbola or a degenerate parabola, i.e. a pair of parallel lines (Figure 22). In any case, the common diameters of T_1, T_2 define the double lines of f.

If f is elliptic, then there still exist two conics T_1, T_2 concentric to C_1, C_2, C_3, that have double contact with C_1, C_2, C_3. But this time only the *primary solution* T_1 is an ellipse circumscribed to C_1, C_2, C_3, while T_2 is an ellipse inscribed to C_1, C_2, C_3.

Regardless of whether f is hyperbolic or elliptic, T_2 can be constructed using the already constructed T_1 (cf. [5, 6]) and involution f, since the contact diameters of T_1, C_i and T_2, C_i, $i = 1, 2, 3$ correspond through f.

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