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The Flat Translation Surfaces in the 3-dimensional Lorentz Heisenberg Group $\mathbb{H}_3$

Rafik Medjati, Said Taïfour, Hanifi Zoubir

ENP Oran – Maurice AUDIN (Ex ENSET)
rafik.medjati@enp-oran.dz, said.taifour@enp-oran.dz, zoubirhanifi@yahoo.fr

Abstract. In the Lorentz-Heisenberg space $\mathbb{H}_3$ endowed with flat metric g_3, a translation surface is parametrized by $r(x,y) = \gamma_1(x) * \gamma_2(y)$, where γ_1 and γ_2 are two planar curves lying in planes, which are not orthogonal. In this article, we classify translation surfaces in $\mathbb{H}_3$, with vanishing Gaussian curvature in Lorentz-Heisenberg space $\mathbb{H}_3$.

Key Words: Gaussian curvature, Lorentz Heisenberg space, First Fundamental Form, Second Fundamental form, Translation surface, Flat surface

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1 Introduction

In differential geometry, we have always been interested in the study of the curvatures of surfaces in 3-dimensional spaces. In general the surfaces having a constant Gaussian curvature or a constant mean curvature in 3-dimensional spaces have been studied in [4–7, 10, 12–14, 17, 20, 23, 26, 28].

In particular in [1, 2, 8, 9, 16, 18, 19, 21, 22] the authors classify translation surfaces in 3-dimensional spaces.

It is well known that the surface S is called as flat or minimal surface if the Gaussian curvature or the mean curvature vanishes, respectively. In [3, 11, 27, 29] we can see that the study of flat or minimal surfaces have found many applications in differential geometry and physics.

Recently on the one hand L. Belarbi in [6] classifies $(G_i)_{i=1-2}$-invariant surfaces of the Heisenberg group $\mathbb{H}_3$ with constant extrinsically Gaussian curvature K_{ext}, including extrinsically flat G_1-invariant surfaces. On the other hand in [15] A. Kelleci gets the complete classification of Translation-Factorable (TF) surfaces with vanishing Gaussian curvatures in 3-spaces.

In [25] and [24] N. Rahmani and S. Rahmani have showed that, modulo an automorphism of the Lie algebra, the 3-dimensional Lorentz Heisenberg group $\mathbb{H}_3$ has the following classes

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of left-invariant Lorentz metrics:

$$\begin{aligned} g_1 &= -\,\mathrm{d}x^2 + \mathrm{d}y^2 + (x\,\mathrm{d}y + \mathrm{d}z)^2 \\ g_2 &= \mathrm{d}x^2 + \mathrm{d}y^2 - (x\,\mathrm{d}y + \mathrm{d}z)^2 \\ g_3 &= \mathrm{d}x^2 + (x\,\mathrm{d}y + \mathrm{d}z)^2 - [(1-x)\,\mathrm{d}y - \mathrm{d}z]^2. \end{aligned}$$

They proved that the metrics g_1, g_2, g_3 are non-isometrics, and that g_3 is flat.

In this paper, we classify flat translation surfaces in Lorentz Heisenberg group $\mathbb{H}_3$ endowed with flat metric g_3.

2 Definition of Translation Surfaces in $(\mathbb{H}_3, g_3)$ and Their Types

2.1 The Lorentz-Heisenberg Space $\mathbb{H}_3$

In this paragraph we recall that the Heisenberg group $\mathbb{H}_3$ is a Lie group which is diffeomorphic to 3-dimensional real space $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - x\overline{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -xy - z)$. The left invariant Lorentz metric on $\mathbb{H}_3$ is

$$g_3 = \mathrm{d}x^2 + (x\,\mathrm{d}y + \mathrm{d}z)^2 - [(1-x)\,\mathrm{d}y - \mathrm{d}z]^2.$$

The following set of left-invariant vector fields forms pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ for corresponding Lie-algebra

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + (1-x)\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}.$$

The characterizing properties of this algebra are the following commutation relations:

$$[e_2, e_3] = 0, \quad [e_3, e_1] = e_2 - e_3, \quad [e_2, e_1] = e_2 - e_3.$$

with

$$g_3(e_1, e_1) = 1, \quad g_3(e_2, e_2) = 1, \quad g_3(e_3, e_3) = -1.$$

If ∇ is the Levi-Civita connection and R is the curvature tensor of ∇, we have

$$\begin{aligned} &\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_1} e_3 = 0, \\ &\nabla_{e_2} e_1 = \nabla_{e_3} e_1 = e_2 - e_3, \\ &\nabla_{e_2} e_2 = \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = -e_1. \end{aligned}$$

Let (S, r) be a surface in the 3-dimensional Lorentz Heisenberg group $\mathbb{H}_3$. If (S, r) is parametrized by an immersion

$$r(s, t) = (r_1(s, t), r_2(s, t), r_3(s, t)). \tag{1}$$

Thus, the basis of the tangent space T_pS is

$$r_s = \frac{\partial r}{\partial s}, \quad r_t = \frac{\partial r}{\partial t}. \tag{2}$$

Therefore, the coefficients of the first and second fundamental forms are

$$E = g_3(r_s, r_s), \quad F = g_3(r_s, r_t), \quad G = g_3(r_t, r_t) \tag{3}$$

and

$$L = g_3(\nabla_{r_s} r_s, \mathbb{N}), \quad M = g_3(\nabla_{r_s} r_t, \mathbb{N}), \quad N = g_3(\nabla_{r_t} r_t, \mathbb{N}) \tag{4}$$

where $\mathbb{N}$ is a unit normal vector field on S that satisfies the following system

$$\begin{cases} g_3(r_s, \mathbb{N}) = 0, \\ g_3(r_t, \mathbb{N}) = 0, \\ g_3(\mathbb{N}, \mathbb{N}) = -1. \end{cases} \tag{5}$$

The Gaussian curvature K is defined by

$$K = \frac{LN - M^2}{W^2} \tag{6}$$

where $W = \sqrt{|EG - F^2|}$.

2.2 Translation Surfaces

In this paragraph, we would like to give the definition of the translation surfaces in $(\mathbb{H}_3, g_3)$ defined in [19]. In the Lorentz-Heisenberg space $\mathbb{H}_3$, a translation surface is parametrized by $r(x, y) = \gamma_1(x) * \gamma_2(y)$, where γ_1 and γ_2 are two planar curves lying in planes, which are non orthogonal and $*$ denotes the group operation of $\mathbb{H}_3$.

Definition 1. A translation surface (S, r) in $\mathbb{H}_3$ is surface parametrised by $\gamma_1(x) * \gamma_2(y)$, where $\gamma_1 \colon I \subset \mathbb{R} \to \mathbb{H}_3$, $\gamma_2 \colon J \subset \mathbb{R} \to \mathbb{H}_3$ are curves in two coordinate planes of $\mathbb{R}^3$.

We distinguish six types of translation surfaces in $\mathbb{H}_3$.

2.2.1 Translation Surfaces of Type 1 and Type 2

Let the curves γ_1 and γ_2 be given by

$$\gamma_1(s) = (s, 0, f(s)) \quad \text{and} \quad \gamma_2(t) = (0, t, g(t)).$$

We have two translation surfaces $S(\gamma_1, \gamma_2)$ and $S(\gamma_2, \gamma_1)$ parametrized by, respectively,

$$r(s, t) = \gamma_1(s) * \gamma_2(t) = (s, t, f(s) + g(t) - st) \tag{7}$$

and

$$r(s, t) = \gamma_2(t) * \gamma_1(s) = (s, t, f(s) + g(t)), \tag{8}$$

where f and g are two smooth functions. The surfaces given by (7) and (8) are called the translation surfaces of Type 1 and 2.

2.2.2 Translation Surfaces of Type 3 and Type 4

Let the curves γ_1 and γ_2 be given by

$$\gamma_1(s) = (s, 0, f(s)) \quad \text{and} \quad \gamma_2(t) = (g(t), t, 0).$$

We have two translation surfaces $S(\gamma_1, \gamma_2)$ and $S(\gamma_2, \gamma_1)$ parametrized by, respectively,

$$r(s,t) = \gamma_1(s) * \gamma_2(t) = (s + g(t), t, f(s) - st) \tag{9}$$

and

$$r(s,t) = \gamma_2(t) * \gamma_1(s) = (g(t) + s, t, f(s)), \tag{10}$$

where f and g are two smooth functions. The surfaces given by (9) and (10) are called the translation surfaces of Type 3 and 4.

2.2.3 Translation Surfaces of Type 5 and Type 6

Let the curves γ_1 and γ_2 be given by

$$\gamma_1(s) = (0, s, f(s)) \quad \text{and} \quad \gamma_2(t) = (t, g(t), 0).$$

We have two translation surfaces $S(\gamma_1, \gamma_2)$ and $S(\gamma_2, \gamma_1)$ parametrized by, respectively,

$$r(s,t) = \gamma_1(s) * \gamma_2(t) = (t, g(t) + s, f(s)) \tag{11}$$

and

$$r(s,t) = \gamma_2(t) * \gamma_1(s) = (t, g(t) + s, f(s) - st), \tag{12}$$

where f and g are two smooth functions. The surfaces given by (11) and (12) are called the translation surfaces of Type 5 and 6.

3 Classification of Flat Translation Surfaces in $(\mathbb{H}_3, g_3)$

In this section, we would like to investigate the vanishing Gaussian curvature problem for each type of translation surfaces in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$. As well known, the surfaces with vanishing Gaussian curvature are called flat, and then, we examine when it vanishes. Finally, we give the complete classification of the translation surfaces with vanishing Gaussian curvatures.

3.1 Case of Flat Translation Surfaces of Type 1

Let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 1 which is parametrized as in the formula (7). Thus the basis of the tangent space T_pS is

$$r_s = e_1 + (f_s - t)e_2 - (f_s - t)e_3, \tag{13}$$
$$r_t = g_t e_2 + (1 - g_t)e_3. \tag{14}$$

Therefore, we get

$$\mathbb{N} = -\frac{(f_s - t)}{W} e_1 + \frac{(1 - g_t)}{W} e_2 + \frac{g_t}{W} e_3, \tag{15}$$

where $\mathbb{N}$ is the normal unit vector field on $S(\gamma_1, \gamma_2)$, which satisfies the following system

$$\begin{cases} g_3(r_s, \mathbb{N}) = 0, \\ g_3(r_t, \mathbb{N}) = 0, \\ g_3(\mathbb{N}, \mathbb{N}) = -1 \end{cases} \tag{16}$$

with $W = \sqrt{|EG - F^2|} = \sqrt{|2g_t - (f_s - t)^2 - 1|}$, where E, F and G are the coefficients of the first fundamental form I of $S(\gamma_1, \gamma_2)$ which is defined by

$$I = E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2$$

where

$$E = g_3(r_s, r_s) = 1, \tag{17}$$
$$F = g_3(r_s, r_t) = f_s - t, \tag{18}$$
$$G = g_3(r_t, r_t) = 2g_t - 1. \tag{19}$$

To compute the second fundamental form of $S(\gamma_1, \gamma_2)$, we have to calculate the following:

$$\begin{aligned} r_{ss} &= \nabla_{r_s} r_s = f_{ss} e_2 - f_{ss} e_3, \\ r_{st} &= \nabla_{r_s} r_t = 0, \\ r_{tt} &= \nabla_{r_t} r_t = -e_1 + g_{tt} e_2 - g_{tt} e_3. \end{aligned} \tag{20}$$

Which imply the coefficients of the second fundamental form of $S(\gamma_1, \gamma_2)$ are given by

$$L = g_3(\nabla_{r_s} r_s, \mathbb{N}) = \frac{f_{ss}}{W}, \tag{21}$$
$$M = g_3(\nabla_{r_s} r_t, \mathbb{N}) = 0, \tag{22}$$
$$N = g_3(\nabla_{r_t} r_t, \mathbb{N}) = \frac{g_{tt} + f_s - t}{W}. \tag{23}$$

By (17), (18), (19), (21), (22), (19) and (6), the Gaussian curvature K of translation surface S of Type 1 is given by

$$K = \frac{f_{ss}(g_{tt} + f_s - t)}{[2g_t - (f_s - t)^2 - 1]^2}. \tag{24}$$

Now, we would like to investigate the vanishing Gaussian curvature problem, thus we examine translation surfaces of Type 1, whose Gaussian curvature is identically zero.

Let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 1. Thus, from (24), it is clear that it is sufficient that

$$f_{ss}(g_{tt} + f_s - t) = 0. \tag{25}$$

Let us consider on the following possibilities:

Case 1: If $g_{tt} = 0$, so $(g = b_1 t + b_2)$, from (25) we get $f = a_1 s + a_2$, which gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s, t) = (s, t, a_1 s + b_1 t - st + a_2 + b_2) \tag{26}$$

where a_1, a_2, b_1, $b_2 \in \mathbb{R}$.

Case 2: If $f_{ss} = 0$, so ($f = a_1 s + a_2$), then, the Equation (25) is trivially satisfied for all smooth function g and we obtain

$$r(s,t) = (s, t, a_1 s + g(t) - st + a_2) \tag{27}$$

where a_1 $a_2 \in \mathbb{R}$.

Case 3: If $f_{ss} g_{tt} \neq 0$ from (25)) we find

$$f_s = t - g_{tt}. \tag{28}$$

Therefore, both sides have to equal a nonzero constant, namely

$$f_s = \alpha = t - g_{tt} \tag{29}$$

then $f_{ss} = 0$, which is not possible.

As Case 1 is a particular cases of Case 2, we have the following result:

Theorem 1. *Let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 1 in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$. Then, $S(\gamma_1, \gamma_2)$ is a flat surface if and only if it can be parametrized by*

$$r(s,t) = (s, t, a_1 s + g(t) - st + a_2) \tag{30}$$

for all smooth functions g, and $a_1, a_2 \in \mathbb{R}$.

3.2 Case of Flat Translation Surfaces of Type 2

By (8), for a translation surface of Type 2, the basis of the tangent space $T_p S$ is

$$r_s = e_1 + f_s e_2 - f_s e_3, \tag{31}$$
$$r_t = (s + g_t) e_2 + (1 - s - g_t) e_3 \tag{32}$$

and the normal unit vector field $\mathbb{N}$ on $S(\gamma_1, \gamma_2)$

$$\mathbb{N} = -\frac{f_s}{W} e_1 + \frac{(1 - s - g_t)}{W} e_2 + \frac{(s + g_t)}{W} e_3 \tag{33}$$

with

$$W = \sqrt{|EG - F^2|} = \sqrt{|2(g_t + s) - 1 - (f_s)^2|}. \tag{34}$$

Therefore the coefficients of the first fundamental form are

$$E = 1, \quad F = f_s, \quad G = (2g_t + 2s - 1). \tag{35}$$

To compute the second fundamental form of $S(\gamma_1, \gamma_2)$, we have to calculate the following:

$$\begin{aligned} r_{ss} &= \nabla_{r_s} r_s = f_{ss} e_2 - f_{ss} e_3, \\ r_{st} &= \nabla_{r_s} r_t = e_2 - e_3, \\ r_{tt} &= \nabla_{r_t} r_t = -e_1 + g_{tt} e_2 - g_{tt} e_3. \end{aligned} \tag{36}$$

Then the coefficients of the second fundamental form of $S(\gamma_1, \gamma_2)$ are

$$L = \frac{f_{ss}}{W}, \quad M = \frac{1}{W}, \quad N = \frac{f_s + g_{tt}}{W}. \tag{37}$$

By (35), (37) and (6), the Gaussian curvature K of translation surface S of Type 2 is given by

$$K = \frac{f_{ss}(f_s + g_{tt}) - 1}{[2(g_t + s) - 1 - (f_s)^2]^2}. \tag{38}$$

Now let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 2. Thus, from (38), it is clear that it is sufficient that

$$f_{ss}(f_s + g_{tt}) - 1 = 0. \tag{39}$$

It is clear that if $f_{ss} = 0$, Equation (39) is never satisfied, so we consider the following cases:

Case 1: If $g_{tt} = 0$, so $g = b_2 t + b_3$, from (39) we get $f = \pm\frac{1}{3}(2s + a_1)^{\frac{3}{2}} + a_2$, which gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s,t) = (s, t, \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} + b_2 t + a_2 + b_3), \tag{40}$$

where a_1, a_2, b_2, $b_3 \in \mathbb{R}$ and $s \geq -\frac{a_1}{2}$.

Case 2: If $g_{tt} \neq 0$. From (39) we have

$$f_{ss}(f_s + g_{tt}) - 1 = 0, \tag{41}$$

$$g_{tt} = \frac{1 - f_{ss} f_s}{f_{ss}} \tag{42}$$

therefore, both sides have to equal a nonzero constant, namely

$$g_{tt} = b_1 = \frac{1 - f_{ss} f_s}{f_{ss}}. \tag{43}$$

Then

$$f_{ss}(f_s + b_1) = 1 \tag{44}$$

and

$$g = \frac{b_1}{2} t^2 + b_2 t + b_3 \tag{45}$$

where $b_1 \in \mathbb{R}^*$ and b_2, $b_3 \in \mathbb{R}$.

After solving (44), we find

$$f = \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} - b_1 s + a_2 \tag{46}$$

where a_1, $a_2 \in \mathbb{R}$, $b_1 \in \mathbb{R}^*$ and $s \geq -\frac{a_1}{2}$. This gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s,t) = (s, t, \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} + \tfrac{b_1}{2} t^2 + b_2 t - b_1 s + a_2 + b_3). \tag{47}$$

As Case 1 is particular case of Case 2, then have the following result:

Theorem 2. *Let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 2 in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$. Then $S(\gamma_1, \gamma_2)$ is a flat surface if and only if it can be parametrized by*

$$r(s,t) = (s, t, \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} + \tfrac{b_1}{2} t^2 + b_2 t - b_1 s + a_2 + b_3), \tag{48}$$

where a_1, a_2, b_1, b_2, $b_3 \in \mathbb{R}$ and $s \geq -\frac{a_1}{2}$.

3.3 Case of Flat Translation Surfaces of Type 3

Let $S(\gamma_1,\gamma_2)$ be a translation surface of type 3, by (9), Thus the basis of the tangent space T_pS is

$$r_s = e_1 + (f_s - t)e_2 - (f_s - t)e_3, \quad r_t = g_t e_1 + e_3, \tag{49}$$

and the normal unit vector field $\mathbb{N}$ on $S(\gamma_1,\gamma_2)$

$$\mathbb{N} = \frac{(f_s - t)}{W}e_1 - \frac{(g_t(f_s - t) + 1)}{W}e_2 + \frac{g_t(f_s - t)}{W}e_3 \tag{50}$$

with $W = \sqrt{|(f_s - t)^2 + 2g_t(f_s - t) + 1|}$.

Therefore the coefficients of the first fundamental form are

$$E = 1, \quad F = f_s + g_t - t, \quad G = (g_t)^2 - 1. \tag{51}$$

The covariant derivatives are:

$$\begin{aligned} r_{ss} &= \nabla_{r_s} r_s = f_{ss}e_2 - f_{ss}e_3, \\ r_{st} &= \nabla_{r_s} r_t = 0, \\ r_{tt} &= \nabla_{r_t} r_t = (g_{tt} - 1)e_1 + g_t e_2 - g_t e_3. \end{aligned} \tag{52}$$

Then the coefficients of the second fundamental form of $S(\gamma_1,\gamma_2)$ are

$$L = -\frac{f_{ss}}{W}, \quad M = 0, \quad N = \frac{[(g_{tt} - 1)(f_s - t) - g_t]}{W}. \tag{53}$$

By (51), (53) and (6), the Gaussian curvature K of translation surface S of Type 3 is given by

$$K = \frac{-f_{ss}[(g_{tt} - 1)(f_s - t) - g_t]}{[(f_s - t)^2 + 2g_t(f_s - t) + 1]^2}. \tag{54}$$

Let $S(\gamma_1,\gamma_2)$ be a translation surface of Type 3. Thus, from (54), it is clear that the equation of flat surface is

$$f_{ss}[(g_{tt} - 1)(f_s - t) - g_t] = 0 \tag{55}$$

so we consider the following cases:

Case 1: If $g_t = 0$, so $g = b$, by following the same steps as Case 1 of translation surface of Type 1, from (55)) we obtain the same result, which provides that the surface $S(\gamma_1,\gamma_2)$ is

$$r(s,t) = (s + b, t, a_1 s + a_2 - st) \tag{56}$$

Case 2: If $g_{tt} = 0$, so $g = b_1 t + b_2$, by following the same steps as Case 2 of translation surface of Type 1, from (55) we obtain the same result, which provides that the surface $S(\gamma_1,\gamma_2)$ is

$$r(s,t) = (s + b_1 t + b_2, t, a_1 s + a_2 - st) \tag{57}$$

where a_1, a_2, b_1, $b_2 \in \mathbb{R}$.

Case 3: If $f_{ss} = 0$, so $f = a_1 s + a_2$. Then, the Equation (55) is trivially satisfied for all smooth function g and we obtain

$$r(s,t) = (s + g(t), t, a_1 s + a_2 - st) \tag{58}$$

where $a_1, a_2 \in \mathbb{R}$.

Case 4: If $f_{ss}g_{tt} \neq 0$. from (55) we obtain

$$f_{ss}[(g_{tt}-1)(f_s-t)-g_t]=0, \tag{59}$$
$$(g_{tt}-1)(f_s-t)=g_t. \tag{60}$$

Since we cannot have both $g_{tt}=1$ and $g_t=0$, so $g_{tt}\neq 1$, which implies

$$f_s=\frac{g_t}{g_{tt}-1}+t. \tag{61}$$

Therefore, both sides have to equal a nonzero constant, namely

$$f_s=a_1=\frac{g_t}{g_{tt}-1}+t \tag{62}$$

which implies $f_{ss}=0$, which is a contradiction.

As Cases 1 and 2 are particular cases of Case 3, we have the following theorem:

Theorem 3. *Let $S(\gamma_1,\gamma_2)$ be a translation surface of Type 3 in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$. Then, $S(\gamma_1,\gamma_2)$ is a flat surface if and only if it can be parametrized by*

$$r(s,t)=(s+g(t),t,a_1s+a_2-st) \tag{63}$$

for all smooth function g, $a_1\in\mathbb{R}^*$ and $a_2\in\mathbb{R}$.

3.4 Case of Flat Translation Surfaces of Type 4

Let $S(\gamma_1,\gamma_2)$ be a translation surface of Type 4, by (10), we obtain

$$r_s=e_1+f_se_2-f_se_3, \tag{64}$$
$$r_t=g_te_1+(g+s)e_2+(1-g-s)e_3. \tag{65}$$

and normal unit vector field $\mathbb{N}$ on $S(\gamma_1,\gamma_2)$

$$\mathbb{N}=-\frac{f_s}{W}e_1-\frac{(f_sg_t+1-g-s)}{W}e_2+\frac{(g+s-f_sg_t)}{W}e_3 \tag{66}$$

with $W=\sqrt{|2(g+s-f_sg_t)-(f_s)^2-1|}$.

Therefore the coefficients of the first fundamental form are

$$E=1,\quad F=f_s+g_t,\quad G=(g_t)^2+2g+2s-1. \tag{67}$$

The covariant derivatives are:

$$\begin{aligned} r_{ss}&=\nabla_{r_s}r_s=f_{ss}e_2-f_{ss}e_3,\\ r_{st}&=\nabla_{r_s}r_t=e_2-e_3,\\ r_{tt}&=\nabla_{r_t}r_t=(g_{tt}-1)e_1+2g_te_2-2g_te_3.\end{aligned} \tag{68}$$

Then the coefficients of the second fundamental form of $S(\gamma_1,\gamma_2)$ are

$$L=\frac{f_{ss}}{W},\quad M=\frac{1}{W},\quad N=\frac{2g_t-(g_{tt}-1)f_s}{W}. \tag{69}$$

By (67), (69) and (6), the Gaussian curvature K of translation surface S of Type 4 is given by

$$K = \frac{f_{ss}[2g_t - (g_{tt} - 1)f_s] - 1}{[2(g + s - f_s g_t) - (f_s)^2 - 1]^2}. \tag{70}$$

Let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 4. Thus, from (70), it is clear that the equation of flat surface is

$$f_{ss}[2g_t - (g_{tt} - 1)f_s] - 1 = 0. \tag{71}$$

It is clear that if $f_{ss} = 0$, equation (71) is never satisfied, so we us consider the following cases:

Case 1: If $g_t = 0$, so $g = b$, from (71) we get

$$f_{ss} f_s = 1. \tag{72}$$

After solving (72), we find

$$f = \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} + a_2 \tag{73}$$

which gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s,t) = (s + b, t, \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} + a_2), \tag{74}$$

where a_1, a_2, $b \in \mathbb{R}$ and $s \geq -\frac{a_1}{2}$.

Case 2: If $g_{tt} = 0$, so $g = b_1 t + b_2$. Then, the Equation (71) becomes

$$f_{ss}(f_s + 2b_1) = 1. \tag{75}$$

After solving (75), we find

$$f = \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} - 2b_1 s + a_2 \tag{76}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and $s \geq -\frac{a_1}{2}$, which gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s,t) = (b_1 t + s + b_2, t, \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} - 2b_1 s + a_2). \tag{77}$$

Case 3: If $f_{ss} g_{tt} \neq 0$. Taking partial derivative in (71) with respect to t, we find

$$2g_{tt} - g_{ttt} f_s = 0 \tag{78}$$

whence

$$\frac{2}{f_s} = \frac{g_{ttt}}{g_{tt}}. \tag{79}$$

Then both sides have to equal a nonzero constant, namely

$$\frac{2}{f_s} = \alpha = \frac{g_{ttt}}{g_{tt}} \tag{80}$$

which implies $f_{ss} = 0$, which is a contradiction. As Case 1 is particular case of Case 2, we have the following result:

Theorem 4. *Let $S(\gamma_1, \gamma_2)$ be a translation surface of type 4 in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$. Then, $S(\gamma_1, \gamma_2)$ is a flat surface if and only if it can be parametrized by*

$$r(s,t) = (b_1 t + s + b_2, t, \pm\tfrac{1}{3}(2s + a_1)^{\frac{3}{2}} - 2b_1 s + a_2) \tag{81}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$, and $s \geq -\frac{a_1}{2}$.

3.5 Case of Flat Translation Surfaces of Type 5

In this case, the translation surface $S(\gamma_1, \gamma_2)$ is parametrized as in (11), thus the basis of the tangent space T_pS is

$$r_s = (f_s + t)e_2 - (f_s + t - 1)e_3, \tag{82}$$
$$r_t = e_1 + tg_te_2 + (1 - t)g_te_3. \tag{83}$$

Therefore, the normal unit vector field $\mathbb{N}$ on $S(\gamma_1, \gamma_2)$ is

$$\mathbb{N} = \frac{f_sg_t}{W}e_1 - \frac{(f_s + t - 1)}{W}e_2 + \frac{(f_s + t)}{W}e_3 \tag{84}$$

with $W = \sqrt{|2(f_s + t) - (f_s)^2(g_t)^2 - 1|}$.

Therefore the coefficients of the first fundamental form are

$$E = 2(f_s + t) - 1, \quad F = (f_s + 2t - 1)g_t, \quad G = (2t - 1)(g_t)^2 + 1. \tag{85}$$

To compute the second fundamental form of $S(\gamma_1, \gamma_2)$, we have to calculate the covariant derivatives:

$$r_{ss} = -e_1 + f_{ss}e_2 - f_{ss}e_3, \tag{86}$$
$$r_{st} = -g_te_1 + e_2 - e_3, \tag{87}$$
$$r_{tt} = -(g_t)^2e_1 + (2g_t + tg_{tt})e_2 - (2g_t + (t - 1)g_{tt})e_3. \tag{88}$$

These imply the coefficients of the second fundamental form of $S(\gamma_1, \gamma_2)$ are given by

$$L = \frac{f_{ss} - f_sg_t}{W}, \quad M = \frac{1 - f_s(g_t)^2}{W}, \quad N = \frac{2g_t - f_sg_{tt} - f_s(g_t)^3}{W}. \tag{89}$$

By (85), (89), (6), the Gaussian curvature K of translation surface S of Type 5 is given by

$$K = \frac{f_{ss}(2g_t - f_sg_{tt} - f_s(g_t)^3) + (f_s)^2g_tg_{tt} - 1}{[2(f_s + t) - (f_s)^2(g_t)^2 - 1]^2}. \tag{90}$$

Let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 5. Thus, from (90), the equation of flat surface is

$$f_{ss}(2g_t - f_sg_{tt} - f_s(g_t)^3) + (f_s)^2g_tg_{tt} - 1 = 0. \tag{91}$$

It is clear that if $f_sg_t = 0$, equation (91) is never satisfied, so we us consider the following cases:

Case 1: If $f_{ss} = 0$, so $f = a_1s + a_2$ with $a_1 \neq 0$, then, the Equation (91) becomes

$$g_tg_{tt} = \frac{1}{a_1^2}. \tag{92}$$

After solving (92), we obtain

$$g = \pm\frac{a_1^2}{2}\left(\frac{2}{a_1^2}t + c_1\right)^{\frac{3}{2}} + c_2 \tag{93}$$

where a_2, c_1, $c_2 \in \mathbb{R}$, $a_1 \in \mathbb{R}^*$ and $t \geq -\frac{c_1(a_1)^2}{2}$; which gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s,t) = \left(t, \pm\frac{a_1^2}{2}\left(\frac{2}{a_1^2}t + c_1\right)^{\frac{3}{2}} + s + c_2, a_1 s + a_2\right). \tag{94}$$

Case 2: If $g_{tt} = 0$, so $g = b_1 t + b_2$ with $b_1 \neq 0$, then, the Equation (91) becomes

$$f_{ss}(2 - b_1^2 f_s) = \frac{1}{b_1}. \tag{95}$$

After solving (95), we obtain

$$f = \pm\frac{1}{2b_1^3}(d_1 - 2b_1 s)^{\frac{3}{2}} + \frac{2}{b_1^2}s + d_2 \tag{96}$$

where b_2 d_1 $d_2 \in \mathbb{R}$, $b_1 \in \mathbb{R}^*$ and $s \leq \frac{d_1}{2b_1}$; which gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s,t) = (t, b_1 t + s + b_2, y(s)) \tag{97}$$

such that

$$y(s) = \pm\frac{1}{2b_1^3}(d_1 - 2b_1 s)^{\frac{3}{2}} + \frac{2}{b_1^2}s + d_2. \tag{98}$$

Case 3: Here, $f_{ss}g_{tt} \neq 0$. If one takes

$$U = U_1 e_1 + U_2 e_2 + U_3 e_3 \tag{99}$$

such as

$$U_1 = 2\frac{f_{ss}}{f_s}, \tag{100}$$

$$U_2 = -f_{ss}(f_s g_{tt} + f_s g_t^3), \tag{101}$$

$$U_3 = f_{ss}(f_s g_{tt} + f_s(g_t)^3) + \frac{1 - f_s^2 g_t g_{tt}}{f_s + t}. \tag{102}$$

From (99), (84) and (91), the surface $S(\gamma_1, \gamma_2)$ is flat if and only if $g_3(U, \mathbb{N}) = 0$.
Therefore we have $U = 0$ or $U \in T_p S$.
1) If $U = 0$, from (100) we have $U_1 = 2\frac{f_{ss}}{f_s} = 0$, which is a contradiction.
2) If $U \in T_p S$, there exist α, $\beta \in \mathbb{R}$, such as

$$U = \alpha r_s + \beta r_t. \tag{103}$$

Therefore, from (99), (82) and (83) we obtain the following system of ordinary differential equations

$$2\frac{f_{ss}}{f_s} = \beta, \tag{104}$$

$$U_2 = \alpha(f_s + t) + \beta t g_t, \tag{105}$$

$$-U_2 + \frac{1 - f_s^2 g_t g_{tt}}{f_s + t} = -\alpha(f_s + t - 1) + \beta(1 - t)g_t. \tag{106}$$

Combining Equations (105) and (106) yields

$$1-(f_s)^2 g_t g_{tt} = (\alpha+\beta g_t)(f_s+t). \tag{107}$$

Taking the partial derivative of (107) with respect to s gives

$$-2f_s f_{ss} g_t g_{tt} = (\alpha+\beta g_t) f_{ss}. \tag{108}$$

Since $f_{ss} g_{tt} \neq 0$,

$$f_s = -\frac{\alpha+\beta g_t}{2g_t g_{tt}}. \tag{109}$$

Then both sides have to equal a nonzero constant, namely

$$f_s = \theta = -\frac{\alpha+\beta g_t}{2g_t g_{tt}} \tag{110}$$

We infer $f_{ss}=0$ but this is not possible.

Then we have the following result:

Theorem 5. *Let $S(\gamma_1,\gamma_2)$ be a translation surface of Type 5 in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$. Then, $S(\gamma_1,\gamma_2)$ is a flat surface if and only if it can be parametrized as one of the followings:*

1. *$S(\gamma_1,\gamma_2)$ is a regular surface in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$ parametrized by*

$$r(s,t) = (t, \pm\frac{a_1^2}{2}\left(\frac{2}{a_1^2}t + c_1\right)^{\frac{3}{2}} + s + c_2, a_1 s + a_2) \tag{111}$$

where $a_2,\ c_1,\ c_2 \in \mathbb{R}$, $a_1 \in \mathbb{R}^*$ and $t \geq -\frac{c_1 a_1^2}{2}$.

2. *$S(\gamma_1,\gamma_2)$ is a regular surface in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$ parametrized by*

$$r(s,t) = (t, b_1 t + s + b_2, \pm\frac{1}{2b_1^3}(d_1 - 2b_1 s)^{\frac{3}{2}} + \frac{2}{b_1^2}s + d_2) \tag{112}$$

where $b_2,\ d_1,\ d_2 \in \mathbb{R}$, $b_1 \in \mathbb{R}^*$ and $s \leq \frac{d_1}{2b_1}$.

3.6 Case of Flat Translation Surfaces of Type 6

For the translation surface of Type 6, by (12), the basis of the tangent space T_pS is

$$r_s = f_s e_2 + (1-f_s)e_3, \tag{113}$$
$$r_t = e_1 + (tg_t - s)e_2 + (s - tg_t + g_t)e_3 \tag{114}$$

and the normal unit vector field $\mathbb{N}$ on $S(\gamma_1,\gamma_2)$

$$\mathbb{N} = \frac{(f_s g_t - tg_t + s)}{W}e_1 + \frac{(1-f_s)}{W}e_2 + \frac{f_s}{W}e_3 \tag{115}$$

with $W = \sqrt{|2f_s - (f_s g_t - tg_t + s)^2 - 1|}$.

Therefore the coefficients of the first fundamental form are

$$E = 2f_s - 1, \quad F = (f_s + t - 1)g_t - s, \quad G = 1 - (g_t)^2 + 2g_t(tg_t - s). \tag{116}$$

To compute the second fundamental form of $S(\gamma_1, \gamma_2)$, we have to calculate the covariant derivatives:

$$r_{ss} = -e_1 + f_{ss}e_2 - f_{ss}e_3, \tag{117}$$
$$r_{st} = -g_t e_1, \tag{118}$$
$$r_{tt} = -(g_t)^2 e_1 + (2v_y + yv_{yy})e_2 - (2g_t + (t-1)g_{tt})e_3. \tag{119}$$

which imply the coefficients of the second fundamental form of $S(\gamma_1, \gamma_2)$ are given by

$$\begin{aligned} L &= \frac{f_{ss} - f_s g_t + t g_t - s}{W}, \\ M &= -\frac{(f_s g_t - t g_t + s) g_t}{W}, \\ N &= \frac{-(f_s g_t - t g_t + s)(g_t)^2 + 2g_t + t g_{tt} - f_s g_{tt}}{W}. \end{aligned} \tag{120}$$

By (116), (120), (6), the Gaussian curvature K of translation surface of type 6 is given by

$$K = \frac{1}{W^4}[(f_{ss} - P)(2g_t + t g_{tt} - f_s g_{tt}) - P f_{ss}(g_t)^2] \tag{121}$$

where $P = f_s g_t - t g_t + s$.

Let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 6. Thus, from (121), the equation of flat surface is

$$(f_{ss} - P)(2g_t + t g_{tt} - f_s g_{tt}) - P f_{ss}(g_t)^2 = 0. \tag{122}$$

Remark 1. It is clear that if $g_t = 0$, Equation (122) is satisfied, for all smooth functions f.

So we us consider the following cases:

Case 1: If $f_s = 0$, so $f = a$, from (122) we get

$$(t g_t - s)(2g_t + t g_{tt}) = 0. \tag{123}$$

As $t g_t \neq s$, we have

$$2g_t + t g_{tt} = 0. \tag{124}$$

After solving (124), we obtain

$$g = \pm \tfrac{1}{b_1 t} + b_2 \tag{125}$$

where $b_1 \in \mathbb{R}^*$ and $b_2 \in \mathbb{R}$, $t \neq 0$ which gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s,t) = (t, \pm \tfrac{1}{b_1 t} + s + b_2, a - st). \tag{126}$$

Case 2: If $f_{ss} = 0$, so $f = a_1 s + a_2$ with $a_1 \neq 0$, then, the Equation (122) becomes

$$P(2g_t + t g_{tt} - a_1 g_{tt}) = 0. \tag{127}$$

There exist two cases.

1) If $P = 0$, we obtain

$$(t - a_1) g_t = s. \tag{128}$$

The left hand side in (128) is either a constant or a function of t, while other side is a function of s. That is not possible.

2) If $P \neq 0$, we get

$$2g_t + (t - a_1)g_{tt} = 0. \tag{129}$$

After solving (129), we obtain

$$g = \pm\tfrac{1}{3b_1}(a_1 - t)^3 + b_2 \tag{130}$$

where a_1, $b_1 \in \mathbb{R}^*$ and $b_2 \in \mathbb{R}$, which gives the surface $S(\gamma_1, \gamma_2)$ parametrized as

$$r(s,t) = (t, \pm\tfrac{1}{3b_1}(a_1 - t)^3 + s + b_2, a_1 s - st + a_2). \tag{131}$$

Case 3: If $g_{tt} = 0$, so $g = b_1 t + b_2$ with $b_1 \neq 0$, then, the Equation (122) becomes

$$\frac{(b_1^2 f_s + s - 2)f_{ss} + 2(b_1 f_s + s)}{(b_1 f_{ss} + 2)} = b_1 t. \tag{132}$$

The left hand side in (131) is either a constant or a function of s, while other side is a function of t. That is not possible.

Case 4: Here $f_{ss}g_{tt} \neq 0$. If one takes

$$V = V_1 e_1 + V_2 e_2 + V_3 e_3 \tag{133}$$

such as

$$V_1 = -f_{ss}(g_t)^2 - (2g_t + tg_{tt} - f_s g_{tt}), \tag{134}$$

$$V_2 = f_{ss}(2g_t + tg_{tt} - f_s g_{tt}), \tag{135}$$

$$V_3 = -f_{ss}(2g_t + tg_{tt} - f_s g_{tt}). \tag{136}$$

From (133), (115) and (122), the surface $S(\gamma_1, \gamma_2)$ is flat if and only if $g_3(V, \mathbb{N}) = 0$.

Therefore we have $V = 0$ or $V \in T_p S$.

1) If $V = 0$, from (133) we have

$$f_s = \frac{2g_t + tg_{tt}}{g_{tt}}. \tag{137}$$

Then both sides have to equal a nonzero constant, namely

$$f_s = \alpha = \frac{2g_t + tg_{tt}}{g_{tt}} \tag{138}$$

which implies $f_{ss} = 0$, which is a contradiction.

2) If $V \in T_p S$, there exist $\alpha, \beta \in \mathbb{R}$, such as

$$V = \alpha r_s + \beta r_t. \tag{139}$$

Therefore, from (133) and (113) we obtain the following system of ordinary differential equations

$$f_{ss}(g_t)^2 + (2g_t + tg_{tt} - f_s g_{tt}) = -\beta \tag{140}$$

$$f_{ss}(2g_t + tg_{tt} - f_s g_{tt}) = \alpha f_s + \beta(tg_t - s) \tag{141}$$

$$f_{ss}(2g_t + tg_{tt} - f_s g_{tt}) = \alpha(f_s - 1) - \beta(s - tg_t + g_t). \tag{142}$$

Combining Equations (141) and (142) yields

$$\beta g_t = -\alpha \tag{143}$$

then $g_{tt} = 0$. This is not possible.

Then we have the following theorem:

Theorem 6. *Let $S(\gamma_1, \gamma_2)$ be a translation surface of Type 6 in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$. Then, $S(\gamma_1, \gamma_2)$ is a flat surface if and only if it can be parametrized as one of the followings:*

1. *$S(\gamma_1, \gamma_2)$ is a regular surface in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$ parametrized by*
$$r(s,t) = (t, b + s, f(s) - st), \tag{144}$$
where $b \in \mathbb{R}$ and f any smooth function.
2. *$S(\gamma_1, \gamma_2)$ is a regular surface in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$ parametrized by*
$$r(s,t) = (t, \pm\tfrac{1}{b_1 t} + s + b_2, a - st) \tag{145}$$
where $b_1 \in \mathbb{R}^*$, a, $b_2 \in \mathbb{R}$ and $t \neq 0$.
3. *$S(\gamma_1, \gamma_2)$ is a regular surface in the Lorentz-Heisenberg space $(\mathbb{H}_3, g_3)$ parametrized by*
$$r(s,t) = (t, \pm\tfrac{1}{3b_1}(a_1 - t)^3 + s + b_2, a_1 s - st + a_2) \tag{146}$$
where $a_1, b_1 \in \mathbb{R}^*$ and a_2, $b_2 \in \mathbb{R}$.

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