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When is a Triangle Bisected by its Euler Line?

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Abstract. The Euler line divides both right triangles and isosceles triangles into two equivalent parts. We show that these are the only two cases in which this can happen.

Key Words: Euler line, Euclidean Geometry

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1 Introduction

The Euler line is a particular line that can be associated with any triangle. It is a fascinating concept in geometry, named after the swiss mathematician Leonhard Euler who probably discovered it [3]. Its study has so far been the subject of several interesting research (see for example [2, 4] and references therein).

To define Euler line it is essential to recall the definitions of some notable points of the triangles:

Orthocenter: The three altitudes of a triangle concur at the same point. Euler called this point the *orthocenter.* We will denote this point by H.

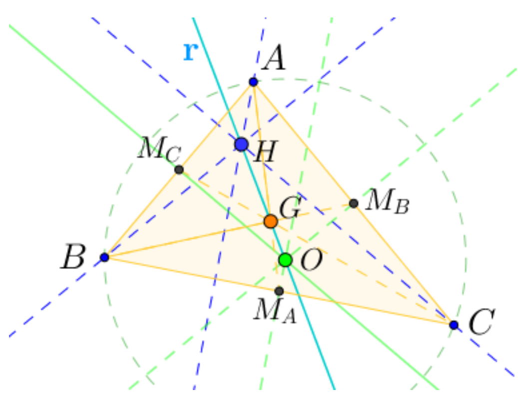

Figure 1: The Euler line in a triangle ABC

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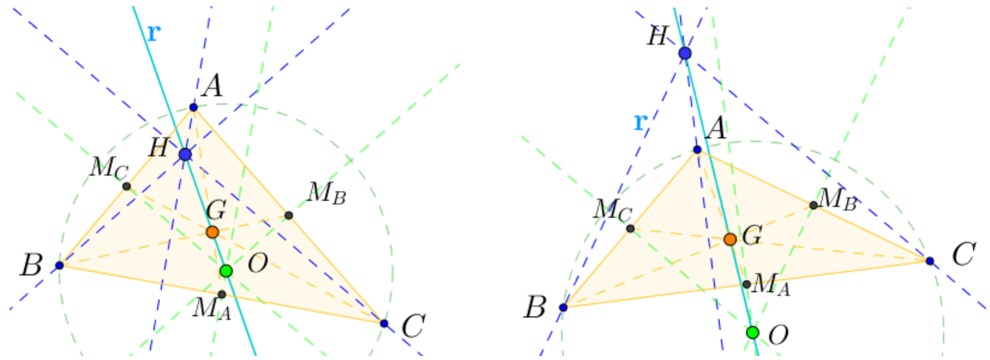


Figure 2: In every triangle the centroid G is always internal. In an acute triangle H and O are also internal; while in an obtuse triangle H and O are external

Centroid: The three medians of a triangle intersect in a point. As we do today, Euler called it *the center of gravity.* Today, this point is usually denoted by G, and it is also called *centroid.*

Circumcenter: The perpendicular bisectors of the sides of a triangle T are concurrent in the same point. Thus, this point is the center of the circumcircle of T. Since about 1890, people have called it the *circumcenter.* We will denote this point by O.

Note that Euler used different symbols, namely E, F and H, to respectively indicate the orthocenter, centroid and circumcenter of a triangle (see [5]).

At first stage of his investigation, Euler also considered the incenter of a triangle (see [5]):

Incenter: The bisectors of the three angles of a triangle are concurrent in the same point. Today, this point is usually denoted by I, and it is also called *incenter.*

The original idea of Euler was to reconstruct a triangle, given the locations of some of its various centers, and in this attempt, referring to the above four centers, he discovered that (see [5]):

If these four points do not coincide, then the triangle is determined. If any two coincide, then all four coincide, and the triangle is equilateral, but it could be any size.

Probably, Euler noted that the orthocenter, centroid, and circumcenter of any triangle are collinear, but probably he didn't give relevance to this fact. In any case, today, we call the line trough H, G and O the *Euler Line.* The key relationship among H, G and O is expressed as:

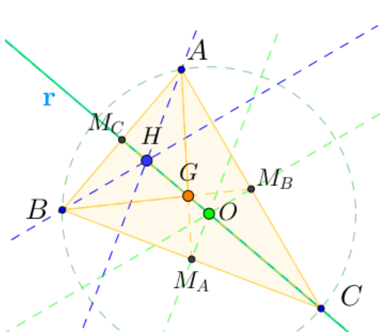
$$HG = 2GO \tag{1}$$

Thus, the distance from the orthocenter to the centroid is twice the distance from the centroid to the circumcenter (see [6] for an analytic proof and [7] for a syntetic proof, for example). Another interesting property of the centroid is the following:

Remark 1. By definition, the centroid G of a triangle T is the intersection of the three medians of T, so that it lies in T. A relevant property is that G divides each median into two segments, with the segment closer to the vertex being twice as long as the segment closer to the midpoint of the opposite side (see Figure 2).

We also highlight that both the orthocenter and the circumcenter of a triangle T may lie in any position, inside or outside of T.

Understanding Euler Line not only deepens one's appreciation for the interconnectedness of geometric properties but also highlights the elegance and universality of certain mathematical relationships.



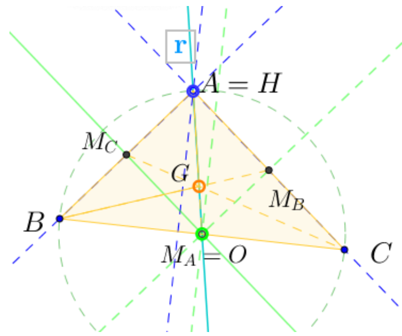


Figure 3: Left: Euler line in an isosceles triangle. Right: Euler's line right triangle

For those concerning geometry problems, the use of dynamic geometry can be very useful (see [1]). In our case it can help to discover the behavior of the Euler line for different configurations of triangles. The reader can view it using the following link: [8]. For these didactic reasons, in some countries, basic properties of Euler line are considered not only in the undergraduate courses, but also in secondary high schools.

Focusing our attention on the mathematical aspects, it is necessary to highlight two elementary properties (see Figure 3):

- *The Euler line of an isosceles triangle T contains the altitude with respect to the base of T.*
- *The Euler line of a right triangle T contains the median from the vertex of the right angle to the hypotenuse of T.*

We observe that in both previous cases the Euler line passes through a vertex and the midpoint of its opposite side; it also divides the triangle into two equivalent parts. Recall that two planar figures are said to be *equivalent* if they have the same area. It is not difficult to see that this property does not hold for any triangle. In this, the use of geogebra can help disprove this general property of the Euler line ([9]). Therefore, naturally, the following opposite question arises (see for example [10]).

> If Euler line divides a triangle T into two equivalent parts, must T necessarily be a right-angled or isosceles triangle?

In this paper we prove that the answer is "yes". Precisely, we prove the following characterization:

Theorem 1. *Let T be a triangle and let r be the Euler line of T. The following properties are equivalent:*

i) The Euler line r passes through one of the midpoints of the sides of T.
ii) One of the vertices of T belongs to r.
iii) The triangle T is a right-angled or isosceles triangle.
iv) The Euler line r divides T into two equivalent parts.

The paper is suitable for a large audience.

2 Proof of the theorem

In the following, given three points A, B and C, we will indicate by ABC the triangle whose vertices are A, B and C as well as the measurement of its area. Furthermore, if there is no

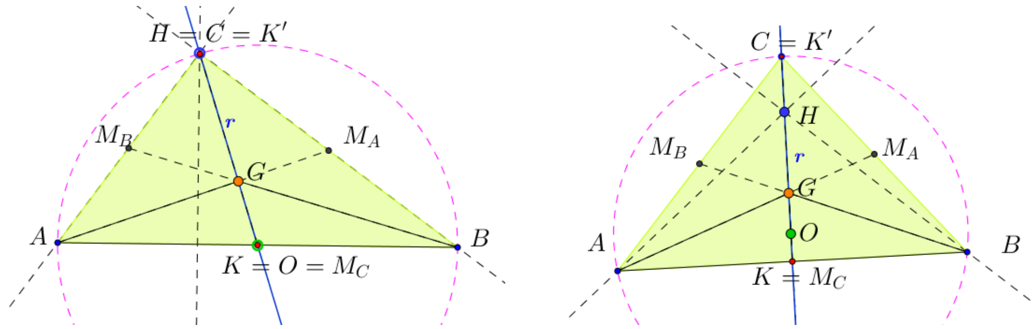


Figure 4: If $M_C \in r$, then T is either a right triangle (see left: $M_C = O$) or an isosceles triangle (see right: $M_C \neq O$)

ambiguity, we will indicate by AB both the straight line passing through A and B and the segment AB. Finally, we will denote by K e K' the intersection points of r with the sides of T.

First, we prove some preliminary results that may have independent interest.

Lemma 1. *If Euler line of a triangle T passes through the midpoints of a side of a T, then it also passes through one of the vertices of T. In this case the triangle is either right angled or isosceles.*

Proof. As we have already observed in the introduction, the centroid G lies inside T (Remark 1).

By hypothesis, r passes through one of the midpoints of the triangle T, say M_C. So that $M_C \neq G$ lie on the Euler line r as well as on the median CM_C. Therefore $r = CM_C$. Now we divide the analysis into two cases:

1. If $MC = O$, then the triangle T is inscribed in a semicircle of diameter AB. In particular, T is right-angled and its orthocenter H coincides with the vertex of the right angle (see Figure 4, left).
2. Suppose now that $M_C \neq O$. By assumption, M_C and O belong to $r = CM_C$, on the other hand they also belong to the perpendicular bisector of AB (see Figure 4, right). We have proved that the median CM_C is also the perpendicular bisector of AB. This is equivalent to say that ABC is an isosceles triangle. □

Lemma 2. *Let $T = ABC$ be a triangle. If the Euler line r passes through the vertex B, then r also passes through the midpoint of the side AC. In this case T is either right-angled or isosceles.*

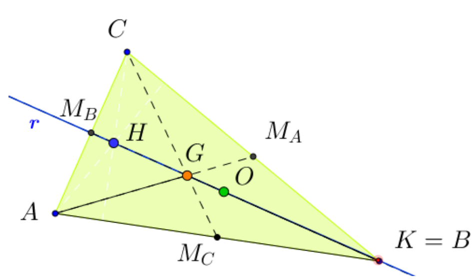


Figure 5: If $B \in r$, then $r = BM_B$

Proof. By hypothesis B and G belong to r (cf. Figure 5). On the other hand B and G also belong to the median BM_B. Thus $r = BM_B$. In particular $M_B \in r$, and by Lemma 1 the triangle T is either right-angled or isosceles. □

We are now in a position to prove the Theorem 1.

Proof. By Lemmas 1 and 2 conditions i and ii are equivalent; moreover, in both the cases T is either an isosceles or a right-angled triangle. In particular, in both the cases the condition iii is satisfied.

The implication $iii \implies iv$ is trivial.

Therefore to complete the proof it is enough to prove that iv implies either i or ii.

Suppose that r bisects T into two equivalent parts and doesn't pass through a vertex or a midpoint of a side of T. Without loss of the generality we assume that r intersects AC at K' between C and M_B and intersects AB at K between B and M_C (see Figure 6).

Thus the area of triangle $AK'K$ is half of area of T and is equal to area of triangle ACM_C. So, triangles CM_CK', KM_CK' have the same area and the same base and hence $CK \| K'M_C$. Therefore triangles CGK and M_CGK' are similar. But $GC = 2GM_C$ by Remark 1. Then,

$$\frac{K'M_C}{CK} = \frac{GM_C}{GC} = \frac{1}{2}. \tag{2}$$

But also $K'M_C \| CK$ in triangle AKC. Therefore

$$\frac{K'M_C}{CK} = \frac{AK'}{AC} > \frac{AM_B}{AC} = \frac{1}{2},$$

contradicting (2). So, r passes through a vertex or midpoint of a side of T as required. □

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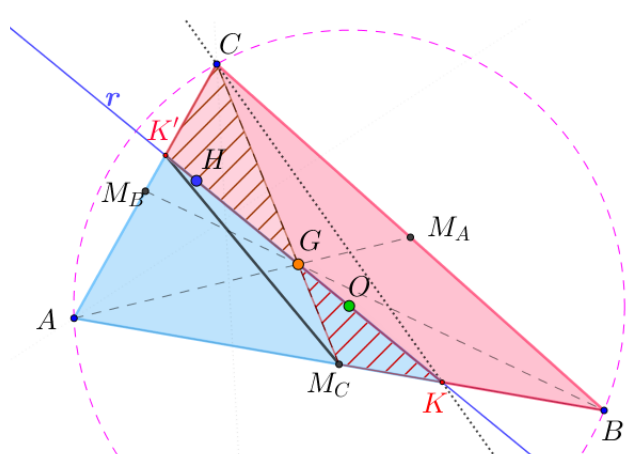

Figure 6: If r bisects T, then K and K' must coincide either with some vertices of T or with a midpoint of some sides of T

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