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# Affine Connections whose Ricci Tensors are Cyclic Parallels

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**Abstract.** This paper focuses on the situation where $\nabla$ is an affine connection on a smooth manifold $M$. We give necessary and sufficient conditions on the coefficients of the affine connection $\nabla$ on an 3-dimensional affine manifold such that the Ricci tensor of $\nabla$ is cyclic parallel. Examples of three families affine connections which have cyclic parallel Ricci tensors are given on an 3-dimensional affine manifold.

*Key Words:* affine connections, cyclic parallel Ricci tensor, Einstein like manifolds

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## 1 Introduction

A Riemannian manifold with cyclic parallel Ricci tensor is called $\mathcal{A}$-manifold. This is a generalization of Einstein manifolds. There are many compact homogeneous spaces in the class $\mathcal{A}$: Einstein manifolds or locally Riemannian product of Einstein manifolds, compact quotients of naturally reductive homogeneous Riemannian manifolds and nilmanifolds covered by the generalized Heisenberg group (see [10] and references therein for more details). Zborowski [18], constructed example of an $\mathcal{A}$-manifold on a principal torus bundle over a product of Kahler-Einstein manifolds with fiber a torus of arbitrary dimension. Also in [19], Zborowski proved that there exists an $\mathcal{A}$-manifold structure on every $r$-torus bundle over a product of almost Hodge $\mathcal{A}$-manifolds.

The cyclic parallelism of the Ricci tensor is sometimes called the "First Ledger condition". Recall that, the Ledger conditions are an infinite series of curvature conditions derived from the so-called Ledger recurrence formula. The best characterization of D'Atri spaces is given using those conditions of odd order, namely, a $n$-dimensional Riemannian manifold $(M, g)$ is a D'Atri space if and only if it satisfies the series of all odd Ledger conditions $L_{2k+1} =$

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$0, 1 \leq k \leq n-1$. Pedersen and Tod [15] showed that if a smooth Riemannian three-manifold $(M, g)$ which is cyclic parallel Ricci tensor, then it is a locally homogeneous D'Atri space. Szabo [16] proved that a smooth Riemannian manifold satisfying the first Ledger condition is real analytic. Tod [17] used the same condition to characterize the four-dimensional Kahler manifolds which are not Einstein. Arias-Marco and Kowalski [1] classified the 4-dimensional homogeneous Riemannian manifolds whose the Ricci tensor is cyclic parallel and used this result to classify the 4-dimensional homogeneous D'Atri spaces. The authors [13] prove that the Riemann extension of an affine $L_3$-space is also an $L_3$-space. In dimension 2 the converse holds. In higher dimensions, they show that an affine manifold $(M, \nabla)$ is an $L_3$-space if the Ricci tensor of $\nabla$ is symmetric and if the Riemann extension of $(M, \nabla)$ is an $L_3$-space.

The aim of this paper is to extend the definition of Riemannian manifold with cyclic parallel Ricci tensor to the affine case.

Let $(M, \nabla)$ be an affine manifold and Ric be its affine Ricci curvature tensor. We say that the affine Ricci curvature tensor of $(M, \nabla)$ is cyclic parallel, if

$$(\nabla_X \operatorname{Ric})(X, X) = 0, \tag{1}$$

for any tangent vector field $X$ on $M$. Recently, the investigation of affine manifolds with cyclic parallel Ricci tensor has been extremely attractive. In [9], the authors proved, in two dimension that an affine connection $\nabla$ is affine Szabó if and only if the Ricci tensor of $\nabla$ is cyclic parallel while in dimension 3 the concept seems to be very challenging by giving only partial results (see [7, 8] for more information). Using the condition of cyclic parallelism of the Ricci tensor, the authors [2] characterised locally homogeneous $L_3$-affine surfaces. In Riemannian geometry, the cyclic parallelism of the Ricci tensor defines also a new type of manifold called Einstein like of Type $\mathcal{A}$ [11]. In the paper [10], the authors extend the notion of Einstein like of Type $\mathcal{A}$ to the notion of affine $\mathcal{A}$-manifold. As applications, they construct non-trivial examples of pseudo-Riemannian $\mathcal{A}$-manifolds on the cotangent bundle of an affine surface using the deformed Riemannian extension.

Motivated by previous papers, we give the explicit forms of three families of affine connections whose Ricci tensors are cyclic parallel. The paper is organised as follows: in Section 2, we recall some basic definitions and geometric notions, namely, torsion, curvature and Ricci tensor on an affine manifold. In Section 3, we study the cyclic parallel Ricci condition on three particular affine connections on an 3-dimensional affine manifolds. Examples are also given.

## 2 Preliminaries

Throughout the paper, $M$ will denote $n$-dimensional smooth manifold and $\nabla$ will be an arbitrary but fixed affine connection on its tangent bundle.

Let $M$ be an $n$-dimensional smooth manifold and $\nabla$ be an affine connection on $M$. We consider a system of coordinates $(x_1, x_2, \ldots, x_n)$ in a neighborhood $\mathcal{U}$ of a point $p$ in $M$. In $\mathcal{U}$ the affine connection is given by

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k, \tag{2}$$

where $\left\{\partial_i = \frac{\partial}{\partial x_i}\right\}_{1\leq i\leq n}$ is a basis of the tangent space $T_pM$ and the functions $\Gamma_{ij}^k (i, j, k = 1, 2, \ldots, n)$ are called the coefficients of the affine connection. We shall call the pair $(M, \nabla)$

affine manifold. Some tensor fields associated with the given affine connection $\nabla$ are defined below. The affine torsion tensor denoted as $T$ is defined by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{3}$$

for any vector fields $X$ and $Y$ on $M$. The components of the affine torsion tensor $T$ in local coordinates are:

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k. \tag{4}$$

If the affine torsion tensor of a given affine connection is zero, we say that the affine connection is torsion free. The affine curvature tensor $R$ is defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{5}$$

for any vector fields $X, Y$ and $Z$ on $M$. If the affine connection is without torsion the affine curvature tensor verifies the following properties:

1. $R(X,Y)Z = -R(Y,X)Z$;
2. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$;
3. $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$;

where the covariant derivative of the affine curvature tensor $R$ is given by:

$$(\nabla_X R)(Y,Z)W = \nabla_X R(Y,Z)W - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W. \tag{6}$$

The components in local coordinates of the affine curvature tensor are:

$$R(\partial_k, \partial_l)\partial_j = \sum_{i=1}^{n} R_{jkl}^i \partial_i. \tag{7}$$

We shall assume that $\nabla$ is torsion-free. If $R = 0$, we say that $\nabla$ is flat affine connection. It is known that $\nabla$ is flat if and only if around a point $p$ there exists a local coordinate system such that $\Gamma_{ij}^k = 0$ for all $i, j, k$. We define the affine Ricci tensor Ric by:

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\{Z \mapsto R(Z,X)Y\}. \tag{8}$$

The components in local coordinates of the affine Ricci tensor are given by

$$\mathrm{Ric}(\partial_j, \partial_k) = \sum_i R_{kij}^i. \tag{9}$$

It is known that in Riemannian geometry the Levi-Civita connection of a Riemaniann metric has symmetric Ricci tensor. But this property is due to the fact that the metric induces a canonical isomorphism of $TM \leftrightarrow T^*M$ which carries elements of $so(p,q)$ (acting on $TM$) to a two-form. This combined with first Bianchi identity gives the pairwise exchange symmetry of the Riemann tensor and hence the symmetry of the Ricci tensor.

For an arbitrary affine connection $\nabla$ with zero torsion, the Ricci curvature is not necessarily symmetric. In fact, the property is closely related to the concept of parallel volume element, as we now explain. More precisely, we shall say that an affine connection $\nabla$ is locally equiaffine if around each point $p$ of $M$ there is a parallel volume element, that is, a nonvanishing $n$-form $\omega$ such that $\nabla\omega = 0$.

The importance of local equiaffine structures lies in its relation to symmetries of the Ricci tensor. For an arbitrary pair $(M, \nabla)$ (even when $\nabla$ is, as assumed, torsion-free), the associated Ricci curvature is not necessarily a symmetric tensor.

*Example* 1. Consider the manifold $\mathbb{R}^2$ with the standard coordinates. A torsion-free affine connection is specified by its Christoffel symbols $\Gamma_{ij}^k$ which is required to be symmetric in the lower indices. Consider the case where: $\Gamma_{11}^1 = -x_2$ and $\Gamma_{22}^2 = x_1$, with all other components of the Christoffel symbol vanishing. Using the formula for the Riemann curvature tensor (7) and the Ricci curvature given by (9) , we find $\operatorname{Ric}(\partial_1, \partial_2) = -1$ and $\operatorname{Ric}(\partial_2, \partial_1) = 1$

**Proposition 1** ([14]). *An affine connection $\nabla$ with zero torsion has symmetric Ricci tensor if and only if it is locally equiaffine.*

We give the following definition.

**Definition 1.** By an equiaffine connection $\nabla$ on $M$ we mean a torsion-free affine connection that admits a parallel volume element $\omega$ on $M$. If $\omega$ is a volume element on $M$ such that $\nabla\omega = 0$, then we say that $(\nabla, \omega)$ is an equiaffine structure on $M$.

We note that if $M$ is simply connected, then for any locally equiaffine connection $\nabla$ on $M$ there exists a volume element $\omega$ such that $(\nabla, \omega)$ is an equiaffine structure on $M$. A manifold $M$ with an equiaffine structure is a generalization of the affine space with a fixed determinant function as volume element [14].

The theory of affine connection is a classical topic in differential geometry. It was initially developed to solve pure geometrical problems. It provides an extremely important tool to study geometric structures on manifolds and, as such, has been applied with great success in many different settings. For instance, in [5], the author proved that an affine connection is Osserman if and only if its Ricci tensor is skew-symmetric. The situation is however more involved in higher dimensions where the skew-symmetry of the Ricci tensor is a necessary (but not a sufficient) condition for an affine connection to be Osserman [4, 6]. Connections with skew-symmetric Ricci tensor on surfaces are also studied in [3].

The manifolds equipped with affine connections relaxes the definition of a pseudo-Riemannian manifold by keeping the geodesic structure (hence connection) but discarding the metric. Kobayashi [12] studies the Ricci curvature of affine torsion-free connections which have parallel volume form. The main result is as follows: If $M$ is a compact parallelisable manifold of dimension at least three, then there exists a connection as described above for which the Ricci curvature is positive definite.

## 3 Affine connections whose Ricci tensors are cyclic parallel

Let's recall the following relation: the covariant derivative of the affine Ricci tensor Ric is given by:

$$(\nabla_X \operatorname{Ric})(Z, W) = X(\operatorname{Ric}(Z, W)) - \operatorname{Ric}(\nabla_X Z, W) - \operatorname{Ric}(Z, \nabla_X W). \tag{10}$$

We start with the following Lemma for later use.

**Lemma 1.** *Let $M$ be an 3-dimensional smooth manifold and $X$ be a tangent vector on $M$*

*Put* $(\nabla_X \operatorname{Ric})(X,X) = A$, *then we have:*

$$\begin{aligned} A = {} & \alpha_1^3(\nabla_{\partial_1}\operatorname{Ric})(\partial_1,\partial_1) + \alpha_2^3(\nabla_{\partial_2}\operatorname{Ric})(\partial_2,\partial_2) + \alpha_3^3(\nabla_{\partial_3}\operatorname{Ric})(\partial_3,\partial_3) \\ + {} & \alpha_1^2\alpha_2[(\nabla_{\partial_1}\operatorname{Ric})(\partial_1,\partial_2) + (\nabla_{\partial_1}\operatorname{Ric})(\partial_2,\partial_1) + (\nabla_{\partial_2}\operatorname{Ric})(\partial_1,\partial_1)] \\ + {} & \alpha_1^2\alpha_3[(\nabla_{\partial_1}\operatorname{Ric})(\partial_1,\partial_3) + (\nabla_{\partial_1}\operatorname{Ric})(\partial_3,\partial_1) + (\nabla_{\partial_3}\operatorname{Ric})(\partial_1,\partial_1)] \\ + {} & \alpha_1\alpha_2^2[(\nabla_{\partial_1}\operatorname{Ric})(\partial_2,\partial_2) + (\nabla_{\partial_2}\operatorname{Ric})(\partial_1,\partial_2) + (\nabla_{\partial_2}\operatorname{Ric})(\partial_2,\partial_1)] \\ + {} & \alpha_1\alpha_3^2[(\nabla_{\partial_1}\operatorname{Ric})(\partial_3,\partial_3) + (\nabla_{\partial_3}\operatorname{Ric})(\partial_1,\partial_3) + (\nabla_{\partial_3}\operatorname{Ric})(\partial_3,\partial_1)] \\ + {} & \alpha_2^2\alpha_3[(\nabla_{\partial_2}\operatorname{Ric})(\partial_2,\partial_3) + (\nabla_{\partial_2}\operatorname{Ric})(\partial_3,\partial_2) + (\nabla_{\partial_3}\operatorname{Ric})(\partial_2,\partial_2)] \\ + {} & \alpha_2\alpha_3^2[(\nabla_{\partial_2}\operatorname{Ric})(\partial_3,\partial_3) + (\nabla_{\partial_3}\operatorname{Ric})(\partial_2,\partial_3) + (\nabla_{\partial_3}\operatorname{Ric})(\partial_3,\partial_2)] \\ + {} & \alpha_1\alpha_2\alpha_3[(\nabla_{\partial_1}\operatorname{Ric})(\partial_2,\partial_3) + (\nabla_{\partial_1}\operatorname{Ric})(\partial_3,\partial_2) + (\nabla_{\partial_2}\operatorname{Ric})(\partial_1,\partial_3) \\ + {} & (\nabla_{\partial_2}\operatorname{Ric})(\partial_3,\partial_1) + (\nabla_{\partial_3}\operatorname{Ric})(\partial_1,\partial_2) + (\nabla_{\partial_3}\operatorname{Ric})(\partial_2,\partial_1)]. \end{aligned} \tag{11}$$

Next, we investigate affine connections whose Ricci tensors are cyclic parallel. Note that, an affine connection on an 3-dimensional manifold has $3^3 = 27$ Christofels symbols. Computing all components of the geometric invariants (curvature tensor, Ricci curvature tensor) is a very tedious work. For this reason, in this work, we shall consider three family of affine connections on an 3-dimensional smooth manifold.

## 3.1 Family I

Let $M$ be an 3-dimensional smooth manifold and $\nabla$ be an affine torsion free connection. Suppose that the action of the affine connection $\nabla$ on the basis of the tangent space $\{\partial_i\}_{1\le i\le 3}$ is given by [4]:

$$\nabla_{\partial_1}\partial_1 = f_1\partial_3,\quad \nabla_{\partial_2}\partial_2 = f_2\partial_1,\quad \nabla_{\partial_3}\partial_3 = f_3\partial_2, \tag{12}$$

where $f_i = f_i(x_1,x_2,x_3), i = 1,2,3$ are the coefficients of the affine connections. The non zero components of the affine curvature tensor $R$ of the torsion free affine connection $\nabla$ are:

$$\begin{aligned} &R(\partial_1,\partial_2)\partial_1 = -\partial_2 f_1\partial_3, && R(\partial_1,\partial_2)\partial_2 = \partial_1 f_2\partial_1 + f_1 f_2\partial_3; \\ &R(\partial_1,\partial_3)\partial_1 = -f_1 f_3\partial_2 - \partial_3 f_1\partial_3, && R(\partial_1,\partial_3)\partial_3 = \partial_1 f_3\partial_2; \\ &R(\partial_2,\partial_3)\partial_2 = -\partial_3 f_2\partial_1, && R(\partial_2,\partial_3)\partial_3 = f_2 f_3\partial_1 + \partial_2 f_3\partial_2. \end{aligned}$$

The only nonzero components of the affine Ricci tensor of the affine connection (12) are then given by:

$$\operatorname{Ric}(\partial_1,\partial_1) = \partial_3 f_1,\quad \operatorname{Ric}(\partial_2,\partial_2) = \partial_1 f_2,\quad \operatorname{Ric}(\partial_3,\partial_3) = \partial_2 f_3. \tag{13}$$

**Proposition 2.** *The affine Ricci tensor of the affine connection* (12) *is skew symmetric if* $f_1 = f(x_1,x_2), f_2 = f(x_2,x_3)$ *and* $f_3 = f(x_1,x_3)$.

**Lemma 2.** *The non zero components of the covariant derivative of the affine Ricci tensor of* *the affine connection* (12) *are given by:*

$$
\begin{aligned}
&(\nabla_{\partial_1}\operatorname{Ric})(\partial_1,\partial_1)=\partial_1\partial_3 f_1, && (\nabla_{\partial_2}\operatorname{Ric})(\partial_1,\partial_1)=\partial_2\partial_3 f_1,\\
&(\nabla_{\partial_3}\operatorname{Ric})(\partial_1,\partial_1)=\partial_3^2 f_1, && (\nabla_{\partial_2}\operatorname{Ric})(\partial_1,\partial_2)=-f_2\partial_3 f_1,\\
&(\nabla_{\partial_1}\operatorname{Ric})(\partial_1,\partial_3)=-f_1\partial_2 f_3, && (\nabla_{\partial_2}\operatorname{Ric})(\partial_2,\partial_1)=-f_2\partial_3 f_1,\\
&(\nabla_{\partial_1}\operatorname{Ric})(\partial_2,\partial_2)=\partial_1^2 f_2, && (\nabla_{\partial_2}\operatorname{Ric})(\partial_2,\partial_2)=\partial_2\partial_1 f_2,\\
&(\nabla_{\partial_3}\operatorname{Ric})(\partial_2,\partial_2)=\partial_3\partial_1 f_2, && (\nabla_{\partial_3}\operatorname{Ric})(\partial_2,\partial_3)=-f_3\partial_1 f_2,\\
&(\nabla_{\partial_1}\operatorname{Ric})(\partial_3,\partial_1)=-f_1\partial_2 f_3, && (\nabla_{\partial_3}\operatorname{Ric})(\partial_3,\partial_2)=-f_3\partial_1 f_2,\\
&(\nabla_{\partial_1}\operatorname{Ric})(\partial_3,\partial_3)=\partial_1\partial_2 f_3, && (\nabla_{\partial_2}\operatorname{Ric})(\partial_3,\partial_3)=\partial_2^2 f_3,\\
&(\nabla_{\partial_3}\operatorname{Ric})(\partial_3,\partial_3)=\partial_3\partial_2 f_3.
\end{aligned}
$$

*Proof.* The proof is based on the formula (10). □

From the previous proposition, we can give the explicit form of the coefficients $f_1, f_2$ and $f_3$ so that the affine Ricci tensor of the affine connection (12) is cyclic parallel. More precisely:

**Proposition 3.** *The affine Ricci tensor of the affine connection* (12) *is cyclic parallel if and only if:*

$$
\begin{aligned}
f_1&=f_1(x_1,x_2)+c_1,\\
f_2&=f_2(x_2,x_3)+c_2,\\
f_3&=f_3(x_1,x_3)+c_3,
\end{aligned}
\tag{14}
$$

*where $c_1$, $c_2$, $c_3$ are constants.*

*Proof.* Using the Lemma 1, we have:

$$
\begin{aligned}
A=\;&\alpha_1^3(\partial_1\partial_3 f_1)+\alpha_2^3(\partial_2\partial_1 f_2)+\alpha_3^3(\partial_3\partial_2 f_3)\\
&+\alpha_1^2\alpha_2[\partial_2\partial_3 f_1]+\alpha_1\alpha_3^2[\partial_1\partial_2 f_3]+\alpha_2^2\alpha_3[\partial_3\partial_1 f_2]\\
&+\alpha_1^2\alpha_3[\partial_3^2 f_1-2f_1\partial_2 f_3]\\
&+\alpha_1\alpha_2^2[\partial_1^2 f_2-2f_2\partial_3 f_1]\\
&+\alpha_2\alpha_3^2[\partial_2^2 f_3-2f_3\partial_1 f_2].
\end{aligned}
\tag{15}
$$

Since, $\alpha_i\neq 0, i=1,2,3$, we can deduce the following system:

$$
\begin{aligned}
&\partial_1\partial_3 f_1=0, && \partial_2\partial_3 f_1=0, && \partial_3^2 f_1-2f_1\partial_2 f_3=0,\\
&\partial_2\partial_1 f_2=0, && \partial_3\partial_1 f_2=0, && \partial_1^2 f_2-2f_2\partial_3 f_1=0,\\
&\partial_1\partial_2 f_3=0, && \partial_3\partial_2 f_3=0, && \partial_2^2 f_3-2f_3\partial_1 f_2=0.
\end{aligned}
$$

By simple integration, we find the explicit formulas of the coefficients $f_1$, $f_2$ and $f_3$ of the affine connection. □

**Corollary 1.** *The affine Ricci tensor of the affine connection* (12) *is cyclic parallel if and only if it is skew symmetric.*

*Example* 2. Using the affine connection described by (12), one can construct examples of affine connections whose the affine Ricci tensor are cyclic parallel. Consider the following connections on $\mathbb{R}^3$ given by:

1. $f_1=x_1e^{x_2}$, $f_2=\frac{x_3}{x_2}$, $f_3=0$.
2. $f_1=0$, $f_2=x_2(1+x_3^2)$, $f_3=x_1x_3$.
3. $f_1=\frac{x_2}{x_1}$, $f_2=0$, $f_3=\frac{1}{x_1x_3}$.

## 3.2 Family II

Let $M$ be an 3-dimensional smooth manifold and $\nabla$ be an affine torsion free connection. Suppose that the action of the affine connection $\nabla$ on the basis of the tangent space $\{\partial_i\}_{1\leq i\leq 3}$ is given by [4]:

$$\nabla_{\partial_1}\partial_1 = f_1\partial_2, \quad \nabla_{\partial_2}\partial_2 = f_2\partial_3, \quad \nabla_{\partial_3}\partial_3 = f_3\partial_1, \tag{16}$$

where $f_i = f_i(x_1, x_2, x_3)$, $i = 1, 2, 3$ are the coefficients of the affine connections. The non-zero components of the curvature tensor $R$ of the affine connection (16) are given by:

$$\begin{aligned}
&R(\partial_1,\partial_2)\partial_1 = -\partial_2 f_1\partial_2 - f_1 f_2\partial_3, && R(\partial_1,\partial_2)\partial_2 = \partial_1 f_2\partial_3;\\
&R(\partial_1,\partial_3)\partial_1 = -\partial_3 f_1\partial_2, && R(\partial_1,\partial_3)\partial_3 = \partial_1 f_3\partial_1 + f_3 f_1\partial_2;\\
&R(\partial_2,\partial_3)\partial_2 = -\partial_3 f_2\partial_3 - f_2 f_3\partial_1, && R(\partial_2,\partial_3)\partial_3 = \partial_2 f_3\partial_1.
\end{aligned}$$

The non-zero components of the affine Ricci tensor Ric of the affine connection (16) are given by:

$$\mathrm{Ric}(\partial_1,\partial_1) = \partial_2 f_1, \quad \mathrm{Ric}(\partial_2,\partial_2) = \partial_3 f_2, \quad \mathrm{Ric}(\partial_3,\partial_3) = \partial_1 f_3. \tag{17}$$

**Proposition 4.** *The affine Ricci tensor of the affine connection* (16) *is skew symmetric if* $f_1 = f(x_1, x_3)$, $f_2 = f(x_1, x_2)$ *and* $f_3 = f(x_2, x_3)$.

**Lemma 3.** *The non zero components of covariant derivative of the affine Ricci tensor of the affine connection* (16) *are given by:*

$$\begin{aligned}
&(\nabla_{\partial_1}\mathrm{Ric})(\partial_1,\partial_1) = \partial_1\partial_2 f_1, && (\nabla_{\partial_2}\mathrm{Ric})(\partial_1,\partial_1) = \partial_2^2 f_1,\\
&(\nabla_{\partial_3}\mathrm{Ric})(\partial_1,\partial_1) = \partial_3\partial_2 f_1, && (\nabla_{\partial_1}\mathrm{Ric})(\partial_1,\partial_2) = -f_1\partial_3 f_2,\\
&(\nabla_{\partial_3}\mathrm{Ric})(\partial_1,\partial_3) = -f_3\partial_2 f_1; && (\nabla_{\partial_3}\mathrm{Ric})(\partial_3,\partial_3) = \partial_3\partial_1 f_3,\\
&(\nabla_{\partial_1}\mathrm{Ric})(\partial_2,\partial_1) = -f_1\partial_3 f_2, && (\nabla_{\partial_1}\mathrm{Ric})(\partial_2,\partial_2) = \partial_1\partial_3 f_2,\\
&(\nabla_{\partial_2}\mathrm{Ric})(\partial_2,\partial_2) = -\partial_2\partial_3 f_2, && (\nabla_{\partial_3}\mathrm{Ric})(\partial_2,\partial_2) = \partial_3^2 f_2,\\
&(\nabla_{\partial_2}\mathrm{Ric})(\partial_2,\partial_3) = -f_2\partial_1 f_3, && (\nabla_{\partial_3}\mathrm{Ric})(\partial_3,\partial_1) = -f_3\partial_2 f_1,\\
&(\nabla_{\partial_2}\mathrm{Ric})(\partial_3,\partial_2) = -f_2\partial_1 f_3, && (\nabla_{\partial_1}\mathrm{Ric})(\partial_3,\partial_3) = \partial_1^2 f_3,\\
&(\nabla_{\partial_2}\mathrm{Ric})(\partial_3,\partial_3) = \partial_2\partial_1 f_3.
\end{aligned}$$

*Proof.* The proof is based on the formula (10). □

From the previous proposition, we can give the explicit form of the coefficients $f_1$, $f_2$ and $f_3$ so that the affine Ricci tensor of the affine connection (16) is cyclic parallel. More precisely:

**Proposition 5.** *The affine Ricci tensor of the affine connection* (16) *is cyclic parallel if and only if:*

$$\begin{aligned}
f_1 &= f_1(x_1, x_3) + d_1,\\
f_2 &= f_2(x_1, x_2) + d_2,\\
f_3 &= f_3(x_2, x_3) + d_3,
\end{aligned} \tag{18}$$

*where* $d_1$, $d_2$, $d_3$ *are constants.*

*Proof.* Using the Lemma 1, we have:

$$\begin{aligned}A &= \alpha_1^3(\partial_1\partial_2 f_1) + \alpha_2^3(-\partial_2\partial_3 f_2) + \alpha_3^3(\partial_3\partial_1 f_3)\\ &+ \alpha_1^2\alpha_2[\partial_2^2 f_1 - 2f_1\partial_3 f_2]\\ &+ \alpha_1\alpha_3^2[\partial_1^2 f_3 - 2f_3\partial_2 f_1]\\ &+ \alpha_2^2\alpha_3[\partial_3^2 f_2 - 2f_2\partial_1 f_3]\\ &+ \alpha_1^2\alpha_3[\partial_3\partial_2 f_1] + \alpha_1\alpha_2^2[\partial_1\partial_3 f_2] + \alpha_2\alpha_3^2[\partial_2\partial_1 f_3].\end{aligned}$$

Since, $\alpha_i \neq 0$, $i = 1, 2, 3$, we can deduce the following system:

$$\begin{aligned}&\partial_1\partial_2 f_1 = 0, \quad \partial_3\partial_2 f_1 = 0, \quad \partial_2^2 f_1 - 2f_1\partial_3 f_2 = 0;\\ &\partial_1\partial_3 f_2 = 0, \quad \partial_2\partial_3 f_2 = 0, \quad \partial_3^2 f_2 - 2f_2\partial_1 f_3 = 0;\\ &\partial_2\partial_1 f_3 = 0, \quad \partial_3\partial_1 f_3 = 0, \quad \partial_1^2 f_3 - 2f_3\partial_2 f_1 = 0\end{aligned}$$

By simple integration, we find the explicit formulas of the coefficients $f_1$, $f_2$ and $f_3$ of the affine connection. □

**Corollary 2.** *The affine Ricci tensor of the affine connection* (16) *is cyclic parallel if and only if it is skew symmetric.*

*Example* 3. Using the affine connection described by (16), one can construct examples of affine connections whose the affine Ricci tensor are cyclic parallel. Consider the following connections on $\mathbb{R}^3$ given by:

1. $f_1 = x_1^2 x_3,\ f_2 = x_1 + x_2,\ f_3 = 0.$
2. $f_1 = 0,\ f_2 = x_1 x_2,\ f_3 = x_2 x_3^3.$
3. $f_1 = x_1 x_3,\ f_2 = 0,\ f_3 = x_2 e^{x_3}.$

## 3.3 Family III

Let $M$ be an 3-dimensional smooth manifold and $\nabla$ be an affine torsion free connection. Suppose that the action of the affine connection $\nabla$ on the basis of the tangent space $\{\partial_i\}_{1\leq i\leq 3}$ is given by [6]:

$$\nabla_{\partial_1}\partial_1 = f_1\partial_1, \quad \nabla_{\partial_2}\partial_2 = f_2\partial_2, \quad \nabla_{\partial_3}\partial_3 = f_3\partial_3, \tag{19}$$

where $f_i = f_i(x_1, x_2, x_3)$, $i = 1, 2, 3$ are the coefficients of the affine connections. The non-zero components of the curvature tensor $R$ of the affine connection (19) are given by:

$$\begin{aligned}&R(\partial_1,\partial_2)\partial_1 = -\partial_2 f_1\partial_1, \quad R(\partial_1,\partial_2)\partial_2 = \partial_1 f_2\partial_2;\\ &R(\partial_1,\partial_3)\partial_1 = -\partial_3 f_1\partial_1, \quad R(\partial_1,\partial_3)\partial_3 = \partial_1 f_3\partial_3;\\ &R(\partial_2,\partial_3)\partial_2 = -\partial_3 f_2\partial_2, \quad R(\partial_2,\partial_3)\partial_3 = \partial_2 f_3\partial_3.\end{aligned}$$

The non-zero components of the affine Ricci tensor Ric of the affine connection (19) are given by:

$$\begin{aligned}&\mathrm{Ric}(\partial_1,\partial_2) = -\partial_1 f_2, \quad \mathrm{Ric}(\partial_1,\partial_3) = -\partial_1 f_3,\\ &\mathrm{Ric}(\partial_2,\partial_3) = -\partial_2 f_3, \quad \mathrm{Ric}(\partial_2,\partial_1) = -\partial_2 f_1,\\ &\mathrm{Ric}(\partial_3,\partial_1) = -\partial_3 f_1, \quad \mathrm{Ric}(\partial_3,\partial_2) = -\partial_3 f_2,.\end{aligned} \tag{20}$$

From the above relation (20), the affine Ricci tensor is skew symmetric if and only if:

$$\partial_1 f_2 + \partial_2 f_1 = 0,\ \partial_1 f_3 + \partial_3 f_1 = 0,\ \partial_2 f_3 + \partial_3 f_2 = 0. \tag{21}$$

**Lemma 4.** *The non zero components of covariant derivative of the affine Ricci tensor of the affine connection* (19) *are given by:*

$$\begin{aligned}
(\nabla_{\partial_1}\operatorname{Ric})(\partial_1,\partial_2) &= f_1\partial_1 f_2 - \partial_1^2 f_2, & (\nabla_{\partial_3}\operatorname{Ric})(\partial_1,\partial_2) &= -\partial_3\partial_1 f_2,\\
(\nabla_{\partial_2}\operatorname{Ric})(\partial_1,\partial_2) &= f_2\partial_1 f_2 - \partial_2\partial_1 f_2, & (\nabla_{\partial_2}\operatorname{Ric})(\partial_1,\partial_3) &= -\partial_2\partial_1 f_3,\\
(\nabla_{\partial_1}\operatorname{Ric})(\partial_1,\partial_3) &= f_1\partial_1 f_3 - \partial_1^2 f_3, & (\nabla_{\partial_3}\operatorname{Ric})(\partial_2,\partial_1) &= -\partial_3\partial_2 f_1,\\
(\nabla_{\partial_3}\operatorname{Ric})(\partial_1,\partial_3) &= f_3\partial_1 f_3 - \partial_3\partial_1 f_3, & (\nabla_{\partial_2}\operatorname{Ric})(\partial_3,\partial_1) &= -\partial_2\partial_3 f_1,\\
(\nabla_{\partial_1}\operatorname{Ric})(\partial_2,\partial_1) &= f_1\partial_2 f_1 - \partial_1\partial_2 f_1, & (\nabla_{\partial_1}\operatorname{Ric})(\partial_2,\partial_3) &= -\partial_1\partial_2 f_3,\\
(\nabla_{\partial_2}\operatorname{Ric})(\partial_2,\partial_1) &= f_2\partial_2 f_1 - \partial_2^2 f_1, & (\nabla_{\partial_1}\operatorname{Ric})(\partial_3,\partial_2) &= -\partial_1\partial_3 f_2,\\
(\nabla_{\partial_1}\operatorname{Ric})(\partial_3,\partial_1) &= f_1\partial_3 f_1 - \partial_1\partial_3 f_1, & &\\
(\nabla_{\partial_3}\operatorname{Ric})(\partial_3,\partial_1) &= f_3\partial_3 f_1 - \partial_3^2 f_1, & &\\
(\nabla_{\partial_2}\operatorname{Ric})(\partial_2,\partial_3) &= f_2\partial_2 f_3 - \partial_2^2 f_3, & &\\
(\nabla_{\partial_3}\operatorname{Ric})(\partial_2,\partial_3) &= f_3\partial_2 f_3 - \partial_3\partial_2 f_3, & &\\
(\nabla_{\partial_2}\operatorname{Ric})(\partial_3,\partial_2) &= f_2\partial_3 f_2 - \partial_2\partial_3 f_2, & &\\
(\nabla_{\partial_3}\operatorname{Ric})(\partial_3,\partial_2) &= f_3\partial_3 f_2 - \partial_3^2 f_2. & &
\end{aligned}$$

*Proof.* The proof is based on the formula (10). □

**Proposition 6.** *If the affine Ricci tensor of the affine connection* (19) *is skew symmetric. then it is also cyclic parallel.*

*Proof.* Using the Lemma (10), we have:

$$\begin{aligned}
A = {} & \alpha_1^2\alpha_2[f_1\partial_1 f_2 - \partial_1^2 f_2 + f_1\partial_2 f_1 - \partial_1\partial_2 f_1]\\
& + \alpha_1^2\alpha_3[f_1\partial_1 f_3 - \partial_1^2 f_3 + f_1\partial_3 f_1 - \partial_1\partial_3 f_1]\\
& + \alpha_1\alpha_2^2[f_2\partial_1 f_2 - \partial_2\partial_1 f_2 + f_2\partial_2 f_1 - \partial_2^2 f_1]\\
& + \alpha_1\alpha_3^2[f_3\partial_1 f_3 - \partial_3\partial_1 f_3 + f_3\partial_3 f_1 - \partial_3^2 f_1]\\
& + \alpha_2^2\alpha_3[f_2\partial_2 f_3 - \partial_2^2 f_3 + f_2\partial_3 f_2 - \partial_2\partial_3 f_2]\\
& + \alpha_2\alpha_3^2[f_3\partial_2 f_3 - \partial_3\partial_2 f_3 + f_3\partial_3 f_2 - \partial_3^2 f_2]\\
& + \alpha_1\alpha_2\alpha_3[-\partial_1\partial_2 f_3 - \partial_1\partial_3 f_2 - \partial_2\partial_1 f_3\\
& - \partial_2\partial_3 f_1 - \partial_3\partial_1 f_2 - \partial_3\partial_2 f_1].
\end{aligned}$$

Since, $\alpha_i \neq 0$, $i = 1, 2, 3$, we can deduce the following system:

$$\begin{aligned}
f_1(\partial_1 f_2 + \partial_2 f_1) - (\partial_1^2 f_2 + \partial_1\partial_2 f_1) &= 0,\\
f_1(\partial_1 f_3 + \partial_3 f_1) - (\partial_1^2 f_3 + \partial_1\partial_3 f_1) &= 0,\\
f_2(\partial_1 f_2 + \partial_2 f_1) - (\partial_2\partial_1 f_2 + \partial_2^2 f_1) &= 0,\\
f_2(\partial_2 f_3 + \partial_3 f_2) - (\partial_2^2 f_3 + \partial_2\partial_3 f_2) &= 0,\\
f_3(\partial_1 f_3 + \partial_3 f_1) - (\partial_3\partial_1 f_3 + \partial_3^2 f_1) &= 0,\\
f_3(\partial_2 f_3 + \partial_3 f_2) - (\partial_3\partial_2 f_3 + \partial_3^2 f_2) &= 0,\\
(\partial_1\partial_2 f_3 + \partial_1\partial_3 f_2) + (\partial_2\partial_1 f_3 + \partial_2\partial_3 f_1) + (\partial_3\partial_1 f_2 + \partial_3\partial_2 f_1) &= 0.
\end{aligned}$$

Assume that, the affine Ricci tensor of the affine connection (19) is skew symmetric, that is, the relation (21) holds. Then the above system is reduced to zero. □

**Theorem 1.** *Let $M$ be a 3-dimensional smooth manifold and $\nabla$ be an affine torsion free connection. Suppose that the action of the affine connection $\nabla$ on the basis of the tangent space $\{\partial_i\}_{1\leq i\leq 3}$ is given by:*

1. $\nabla_{\partial_1}\partial_1 = f_1\partial_3$, $\nabla_{\partial_2}\partial_2 = f_2\partial_1$, $\nabla_{\partial_3}\partial_3 = f_3\partial_2$, *or*
2. $\nabla_{\partial_1}\partial_1 = f_1\partial_2$, $\nabla_{\partial_2}\partial_2 = f_2\partial_3$, *or* $\nabla_{\partial_3}\partial_3 = f_3\partial_1$,
3. $\nabla_{\partial_1}\partial_1 = f_1\partial_1$, $\nabla_{\partial_2}\partial_2 = f_2\partial_2$, $\nabla_{\partial_3}\partial_3 = f_3\partial_3$.

*where $f_i = f_i(x_1, x_2, x_3)$, $i = 1, 2, 3$ are the coefficients of the affine connections. If the affine Ricci tensor of the affine connection is skew symmetric, then it is also cyclic parallel. Moreover, for the connections* (1) *and* (2)*, the converse is also true, i.e., cyclic parallelity implies skew symmetry.*

*Remark* 1. It is uncertain whether there exist any solutions to the system of differential equations given in the proof of Proposition 6 other than the one determined by (21). Consequently, it is possible that for the connection (3), cyclic parallelity also implies skew symmetry.

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