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The Euler Line Bisects the Area of only both the Right Triangles and the Isosceles Triangles

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Abstract. It is known that the Euler line bisects the area of both the right triangles and the isosceles triangles. So, to comply with the title, we have proved that the Euler line of any non-right scalene triangle, doesn't bisect its area, intersects the two largest sides of the obtuse triangle, intersects both the smallest and largest sides of the acute triangle, and doesn't pass through a vertex or a midpoint of a side of the triangle. Also, this proof has led to a proof of the well known fact that the three basic centers, the orthocenter H, the centroid G, and the circumcenter O of any non-right scalene $\triangle ABC$ are collinear and $HG = 2GO$.

Key Words: Euler line
MSC 2020: 51M04

1 Introduction

Euler proved in [1] that the basic centers, the centroid, orthocenter, and the circumcenter of a non-equilateral triangle lie on a line called the Euler line. Note that if two of the basic centers of a triangle coincide, then the three centers coincide and the triangle is equilateral. It is known that the Euler line contains both the median to the hypotenuse of any right triangle and contains the axis of symmetry of any isosceles triangle and hence the Euler line bisects the area of both the right triangles and the isosceles triangles. Thus, to prove that the Euler line bisects the area of only both the right triangles and the isosceles triangles, it is sufficient to prove that the Euler line doesn't bisect the area of any non-right scalene triangle. So, we have proved in the Lemma 1 that the three medians of a triangle are the only bisectors of its area that pass through its centroid. Then by the Lemma 1 we have shown in the Main Theorem that the Euler line doesn't bisect the area of any non-right scalene triangle and intersects the two largest sides of the obtuse triangle and intersects both the smallest and largest sides of the acute triangle. Note that the proof of this theorem, as shown in (2) and (3), has led to a proof of the fact that the three basic centers H, G, and O are collinear and $HG = 2GO$.

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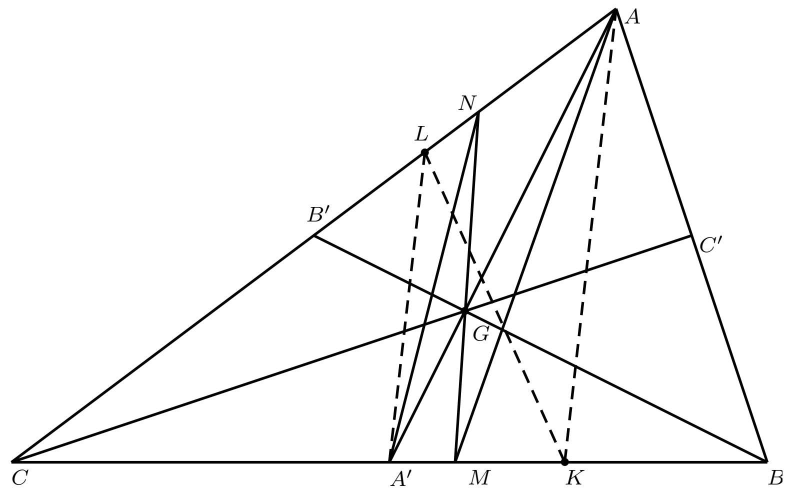

Figure 1: Illustrating proof of Lemma 1

Note that in [2, Theorem 1,p. 133], with a different approach and proof, it is proved that the Euler line bisects the area of only both the right triangles and the isosceles triangles as in (iv) of the conclusion section.

2 Preliminaries

In what follows, let A', B', and C' be the midpoints of the sides BC, CA, and AB respectively, and denote the orthocenter by H, the centroid by G, and the circumcenter by O of any $\triangle ABC$ and let $[ABC]$ stand for the area of any $\triangle ABC$. Also we use the following property of the centroid:

> The centroid G of any triangle divides each median into two segments such that the length of the segment from G to the vertex is twice the length of the segment from G to the midpoint of the opposite side to this vertex. (1)

Next, we prove a lemma which is of interest by itself and is needed to prove the main theorem.

Lemma 1. *There are infinite number of lines that bisect the area of a triangle and the three medians of $\triangle ABC$ are the only bisectors of its area that pass through the centroid G as shown in Figure 1.*

Proof. Let MN be a bisector of the area of $\triangle ABC$ that passes through the centroid G and is not a median. Then $[ANA'] = [MA'N]$ and hence $AM \parallel NA'$. So, by (1) we get $\frac{MG}{GN} = \frac{AG}{GA'} = 2 = \frac{BG}{GB'}$. Thus $\frac{BG}{GB'} = \frac{MG}{GN}$. Therefore $\triangle BGM \sim \triangle B'GN$ and hence $BM \parallel NB'$ contradicting that BM and NB' produced meet at C. Thus MN doesn't bisect the area of $\triangle ABC$ as required.

To prove that there are infinite number of lines that bisect the area of a triangle, let K be any point between A' and B and since $A'B' \parallel AB$, there is a point L between A and B' such that $AK \parallel A'L$. Then $[ALA'] = [KA'L]$. Thus $[KCL] = [AA'C]$ and hence KL bisects the area of $\triangle ABC$ into two equal parts. So, L moves from A to B' as K moves from A' to B and similarly if K is a point of the segment $C'A$ or the segment $B'C$. Thus there are infinite number of lines KL that bisect the $[ABC]$ and doesn't pass through the centroid G. □

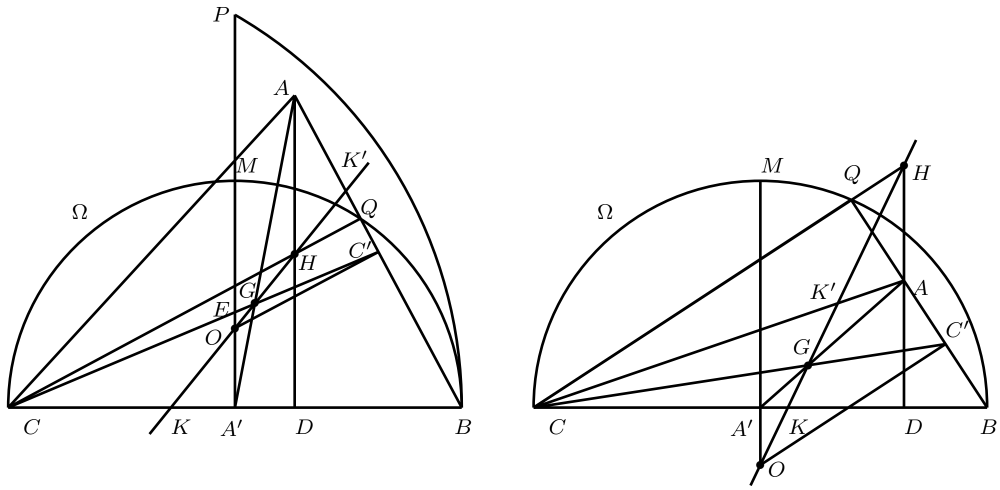

Figure 2(a) Figure 2(b)

Figure 2: Illustrating the proof of Theorem 1

In the next theorem, we prove some general properties of the Euler line of non-right scalene triangles and this proof, as shown in (2) and (3), has led to proving that the three basic centers H, G, and O of any non-right scalene $\triangle ABC$ are collinear and that $HG = 2GO$.

3 The Main Theorem

Theorem 1. *Let $\triangle ABC$ be any non-right scalene triangle. Then the Euler line of $\triangle ABC$, doesn't divide the area of the triangle, doesn't pass through a vertex or a midpoint of a side of the triangle, intersects both the smallest and largest sides of the acute triangle, and intersects the two largest sides of the obtuse triangle.*

Proof. Let ABC be a scalene triangle with side lengths $a = BC,\ b = AC,\ c = AB$ and such that $a > b > c$ and the $\angle A$ is not a right angle. Thus we distinguish two cases.

Case (a) Referring to Figure 2(a), let $60° < \angle A < 90°$, Ω the semicircle with diameter BC, and the perpendicular bisector PA' of BC intersects BC at A', the circular arc $\widehat{BP}$ with center C at P, and Ω at M. So, A is any interior point of the region bounded by the circular arcs $\widehat{BP}$, $\widehat{BM}$, and MP. Since the tangent to Ω is also tangent to $\widehat{BP}$ and $\angle B$ is acute, it follows that AB intersects Ω at Q and CQ is perpendicular to AB. But $AC < BC$ and by Pythagorean theorem we have $(AC)^2 - (CQ)^2 = (AQ)^2, (BC)^2 - (CQ)^2 = (BQ)^2$. Therefore $AQ < BQ$ and hence the median from C meets AB at C' between B and Q, the median AA' at the centroid G, and MA' at E. Since $AC > AB$, the perpendicular from A meets CQ at the orthocenter H and meets also, by Pythagorean theorem, CB at D between A' and B.

Case (b) Referring to Figure 2(b), let $\angle A > 90°$, Ω the semicircle with diameter BC and the perpendicular bisector MA' of BC intersect Ω at M and BC at A'. So, A is an interior point of the region bounded by the circular arc $\widehat{BM}$, MA', and $A'B$. Let the ray $\overrightarrow{BA}$ meet Ω at Q and let AD be the perpendicular from A to BC that meet BC at D between A' and B. Then the ray $\overrightarrow{CQ}$ meets the ray $\overrightarrow{DA}$ at the orthocenter H of $\triangle ABC$ and H is exterior to Ω. Let the median CC' intersect the median AA' at the centroid G.

Since $AD \parallel MA'$ in both cases as seen from Figures 2(a) and 2(b), it follows that the line HG in case (a) meets MA' at O between E and A', $A'C$ at K, and AQ at K' and the line HG in case (b) meets BA' at K, AC at K' and extension of MA' at O.

Next, we prove that $HG = 2OG$, O is the circumcenter of $\triangle ABC$ and the basic centers, H, G,and O are collinear in both cases.

Since $AH \parallel A'O$ in both cases and by (1) we have $\frac{AG}{A'G} = 2$, we get in both cases

$$\triangle AHG \sim \triangle A'OG. \quad \text{Thus,} \quad \frac{HG}{OG} = \frac{AG}{A'G} = 2 \quad \text{and hence} \quad HG = 2OG. \tag{2}$$

So, by (1) and (2) we have in both cases

$\frac{HG}{OG} = 2 = \frac{CG}{C'G}$. Thus $\triangle CHG \sim \triangle C'OG$ and hence $\angle HCG \cong \angle OC'G$. Therefore $\angle QCC' \cong \angle OC'C$ So, $CQ \parallel OC'$. But $\angle CQ \perp AB$. Hence $OC' \perp AB$ and OC' is the perpendicular bisector of AB. So, O is the circumcenter of $\triangle ABC$ and H, G, O are collinear as wanted. (3)

Thus we conclude, in both cases, that the Euler line doesn't pass through a vertex or a midpoint of a side of $\triangle ABC$. But the Euler line passes through the centroid of $\triangle ABC$ and since, by Lemma 1, the three medians are the only lines that pass through the centroid and bisect the area of $\triangle ABC$, it follows that the Euler line of a non-right scalene triangle doesn't bisect the area of the $\triangle ABC$. Also, as seen in Figures 2(a) and 2(b), the Euler line intersects both the smallest and largest sides of the acute triangle and intersects the two largest sides of the obtuse triangle as required. □

4 Conclusion

We conclude from all of the above the following:

(i) The three medians of $\triangle ABC$ are the only bisectors of its area that pass through the centroid G.

(ii) The Euler line of a non-right scalene triangle doesn't bisect its area and doesn't pass through a vertex or a midpoint of a side of this triangle.

(iii) The Euler line contains the median to the hypotenuse of any right triangle and contains the axis of symmetry of any isosceles triangle.

(iv) The Euler line bisects the area of only both the right triangles and the isosceles triangles.

(v) The Euler line intersects both the smallest and largest sides of any acute non-right scalene triangle and intersects the two largest sides of any obtuse non-right scalene triangle.

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