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Equalizers of Triangles and Isometries

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Abstract. The objective of this article is to determine equalizers of triangles using isometries.

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1 Introduction

The equalizers of a triangle are the lines that bisect both its perimeter and area. In 2024, Abu-Saymeh, in a detailed study, determined the existence of equalizers of triangles as well as their construction with compass and ruler [1]. In this article we show how to determine equalizers of triangles using isometries.

2 How to Determine an Equalizer Using Isometries?

The splitters of a triangle are the lines that bisect its perimeter. In triangle ABC, I is the center of the incircle of ABC and E, G, H the points of tangency of the excircles of ABC with sides AB, BC, CA, respectively, where $AE = GC$, $BG = HA$ and $CH = EB$ (see Figure 1). It is verified that $CA + AE = EB + BC$. It is also true that $\text{area}(CAI) + \text{area}(AEI) = \text{area}(EBI) + \text{area}(BCI)$ because said triangles have heights equal to the radius of the incircle of ABC. So that $\text{area}(CAEI) = \text{area}(EBCI)$.

If we consider the splitter PP' with P in segment CG and P' in segment $EA = CG$ so that $CP = EP'$, it follows that $\text{area}(CPI) = \text{area}(EP'I)$ (see Figure 2). The problem of determining an equalizer could be solved by locating P so that P, I, P' are aligned since that way it would be fulfilled

$$\text{area}(CPI) + \text{area}(CAI) + \text{area}(AEI) = \text{area}(EP'I) + \text{area}(EBI) + \text{area}(BCI).$$

The rays CP and EP' correspond in a rotation of center F (intersection of the perpendicular bisectors of CE, GA and PP') and angle $\pi - m\angle ABC$.

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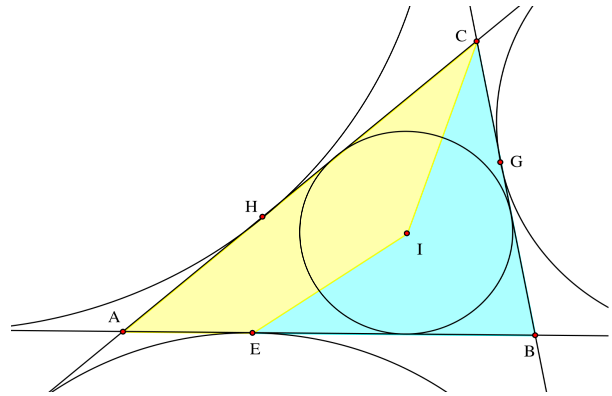

Figure 1: Triangle divided into two quadrilaterals of equal perimeter and equal area

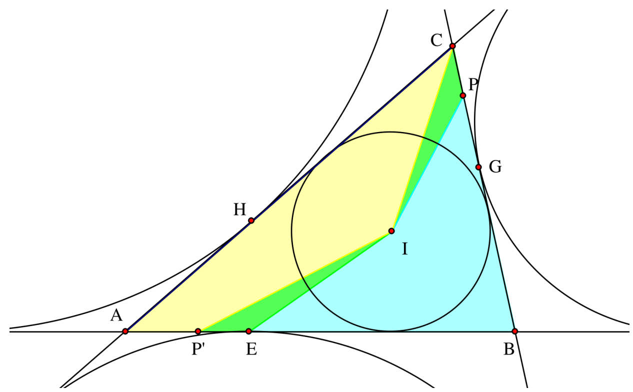

Figure 2: Triangle divided into two quadrilaterals of equal area

The rays CP and EP' also correspond in a glide reflection of axis e (which passes through the midpoints T and U of CE and GA respectively and is parallel to the bisector of the external angle to ABC).

If M is the midpoint of PP', it follows that $m\angle FMP = m\angle FMP' = \pi/2$.

If P, I, P' are aligned, it is also true that $m\angle FMI = \pi/2$, so M belongs to a circle of diameter FI.

M also belongs to the TU-axis of the glide reflection.

From the above we can conclude that M can be constructed as the intersection of the TU-axis of the glide reflection with the circle of diameter FI. In Figure 3 the line M_1I determine the equalizer P_1P_1' with P_1 in segment CG and P_1' in segment EA.

3 Comments to the Previous Method

In Figure 3 the line M_2I determine the segment P_2P_2' that is not an equalizer from CG to EA since P_2 is not in segment CG and P_2' is not in segment EA. So, not all the points of intersection of TU with the circle of diameter FI lead to equalizers.

Even more, in Figure 4 the TU-axis intersects the circle of diameter FI at two points M_1, M_2 and the lines M_1I, M_2I are both not equalizers of triangle ABC from CG to EA and in fact there are no equalizers cutting the smallest two sides AB, BC. (see [1])

The line TU meets the lines BC at K and AB at L. K and L correspond in the glide reflection that transforms the ray CG into the ray EA and since they are in the TU-axis, the

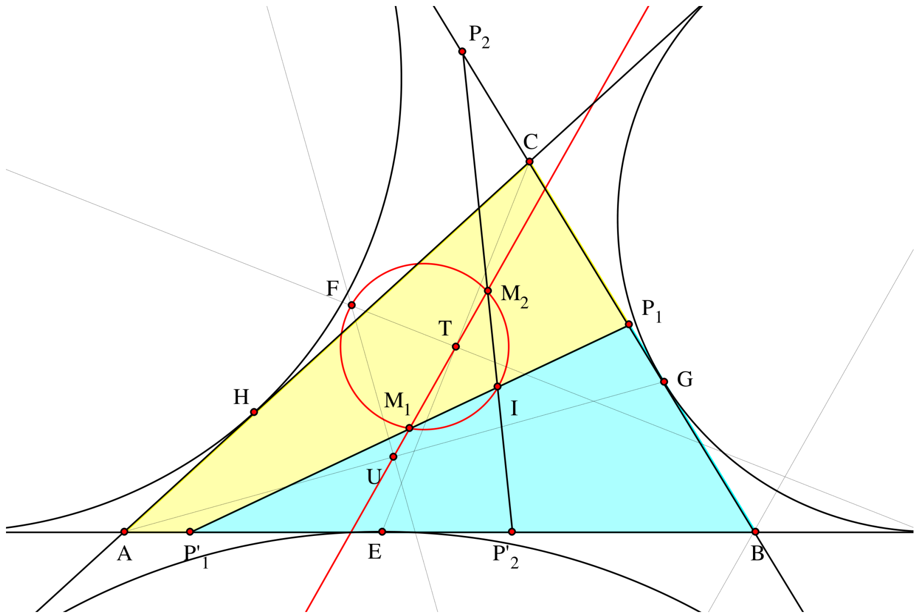

Figure 3: Equalizer from CG to EA determined using isometries

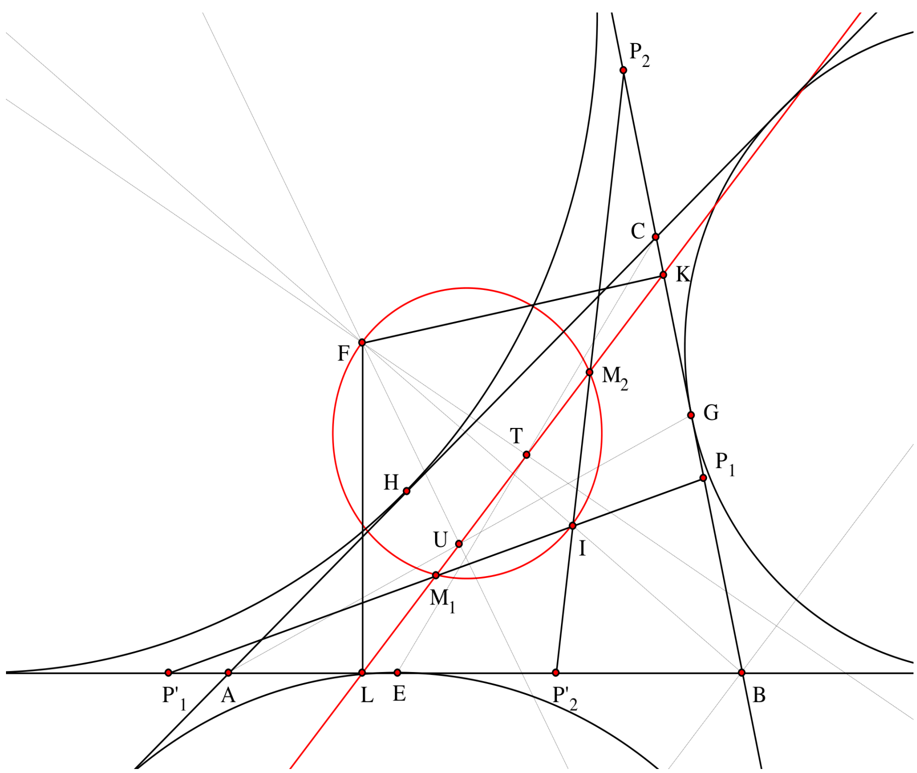

Figure 4: TU intersects the circle at two points but no equalizers from CG to EA

distance KL is the minimum between a point of ray CG and its image in EA. K and L too correspond in the rotation of center F that transforms the ray CG into the ray EA and since the distance KL is the minimum between a point of ray CG and its image in EA, it is fulfilled that $FK \perp BC$ and $FL \perp AB$ for all possible positions of KL. The perpendicular bisector of KL passes through F and coincides with the ray BI, the angle bisector of B. The points M_1 and M_2 of intersection of the line TU and the circle of diameter FI are symmetrical with respect to FI. The line KL is parallel to the exterior bisector of the angle B. Note that the angle bisector of B, perpendicular bisector of CE, and the perpendicular bisector of KL coincide if CE is parallel to the exterior bisector of angle B.

4 Conclusion

In [1] it is shown that:

1. Every triangle can have either one, two, or three equalizers.
2. For equilateral triangles the three medians are the equalizers.
3. For isosceles triangles, the median from the vertex of equal sides to the opposite side is an equalizer. There are a maximum of two equal equalizers joining the largest two sides of a triangle having two equal largest sides. These two equal equalizers are symmetric with respect to the median from the vertex of equal sides to the opposite side. There is a special class of similar isosceles triangles where the line that passes through the incenter, and normal to the bisector of the smallest angle, is the single equalizer joining the two largest equal sides.
4. For scalene triangles, (i) there are no equalizers joining the smallest two sides (see Figure 4), (ii) there is one equalizer joining the smallest and largest sides (see Figure 3), and there are a maximum of two equalizers joining the largest two sides (see Figure 5). There is a special class of scalene triangles where the line that passes through the incenter, and normal to the bisector of the smallest angle, is the single equalizer joining the two largest sides. For more details, necessary and sufficient conditions for the existence of equalizers depending on the side measurements of the triangle are given in [1].

Taking into account the above, to determine equalizers from CG to EA of a triangle ABC we can:

(i) Construct the point F (center of rotation that transforms CG into EA) as the intersection of the perpendicular bisectors of CE and GA.

(ii) Construct the TU-axis (axis of the glide reflection that transforms CG into EA) being T and U the midpoints of CE and GA, respectively.

(iii) The points M can be constructed as the intersection of TU with the circle of diameter FI. If the intersection is empty there is no equalizer. If they are tangent, then $M = I$ and TU is the single equalizer joining the two largest sides and normal to the bisector of the smallest angle. If they are secant, there may be two (see Figure 5), one (see Figure 3), or no equalizer (see Figure 4), depending on whether both lines MI cut the segment CG, if only one cuts the segment CG or if neither cuts the segment CG, respectively.

Similarly to determine equalizers from GB to AH and from HC to BE.

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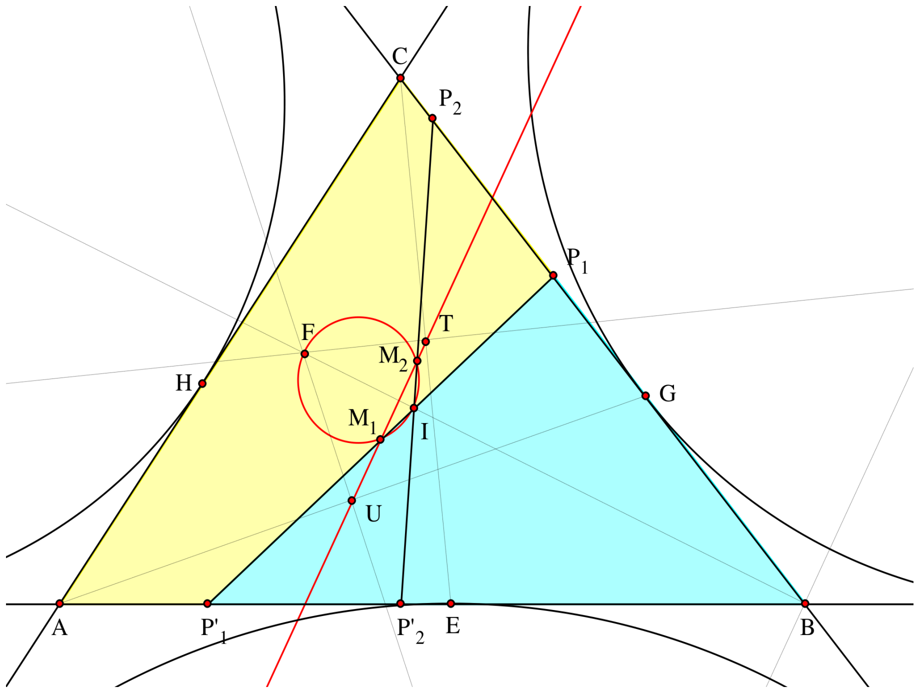

Figure 5: TU intersects the circle at two points and there are two equalizers from CG to EA

References

[1] S. Abu-Saymeh: *The Splitters and Equalizers of Triangles.* Journal for Geometry and Graphics **28**(1), 41–54, 2024.

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