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Boxes and Tangled Tetrahedra

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Abstract. The twin tetrahedron of a given tetrahedron is obtained by circumscribing it by a parallelepiped. However, in general, it is not easy to construct a box that circumscribes a tetrahedron. Actually, constructing a box is equivalent of finding two tangled tetrahedra. We first establish a theorem to construct tangled tetrahedra circumscribed in a box with concurrent diagonals. This generalizes the idea of twin tetrahedra circumscribed in a parallelepiped. And we show that two tetrahedra are twins if and only if they are tangled with concurrent diagonals at the centroid of one of the tetrahedra. We establish a theorem in order to give an alternate proof of this theorem, which we think is a new characterization of the centroid of a tetrahedron. Then we prove that there is a tetrahedron that tangles a reversible tetrahedron with concurrent diagonals such that these two tetrahedra are congruent after relabeling vertices. In addition, both of these tetrahedra can be circumscribed by the same sphere.

Key Words: skew quadrilateral, quadrilateral, tetrahedron, hexahedron with eight vertices, box, tangled tetrahedra, tangled tetrahedra with concurrent diagonals, parallelepiped, twin tetrahedra, isosceles tetrahedron, reversible tetrahedron, trapezoidal box

MSC 2020: 51M04

1 Introduction

There is a parallelepiped that circumscribes a given tetrahedron having each edge of the tetrahedron as diagonals of its six parallelogram faces. By drawing an additional six diagonals on each of these parallelogram faces, we can obtain a so-called twin of the tetrahedron. For a more precise definition of twins, see Definition 4 below. The idea of twin tetrahedra has been around for a long time, and they are an example of two tangled tetrahedra which we will be investigating in this paper. We will be generalizing the notion of twin tetrahedra.

Let A, B, C and D be points in $\mathbb{R}^3$. We denote the line AB by $\overline{AB}$, the line segment AB with the end points A and B is denoted by $[AB]$, the line segment AB without the end points A and B is denoted by (AB), and the length of $[AB]$ is denoted by $|AB|$.

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By joining the points A, B, C, D with line segments $[AB]$, $[BC]$, $[CD]$ and $[DA]$, we obtain a *skew quadrilateral* $ABCD$ if $[AB] \cap [CD] = \emptyset$ and $[AD] \cap [BC] = \emptyset$. If a skew quadrilateral $ABCD$ is *planar*, then we say that the skew quadrilateral $ABCD$ is a *quadrilateral.* If $ABCD$ is a non-planar skew quadrilateral, and if two edges $[AC]$ and $[BD]$ are added to the skew quadrilateral $ABCD$, the resulting solid is a *tetrahedron* and it is denoted by $\nabla ABCD$.

Two tetrahedra $\nabla ABCD$ and $\nabla A'B'C'D'$ are said to be *congruent* if $|AB| = |A'B'|$, $|AC| = |A'C'|$, $|AD| = |A'D'|$, $|BC| = |B'C'|$, $|BD| = |B'D'|$, and $|CD| = |C'D'|$, and we write $\nabla ABCD \cong \nabla A'B'C'D'$. Please be careful that even though $\nabla ABCD \cong \nabla A'B'C'D'$, $\nabla ABCD$ and $\nabla D'C'B'A'$, for example, *may not* be congruent by our definition even though $\nabla A'B'C'D$ and $\nabla D'C'B'A$ are the same tetrahedron.

Let $\{A, B, C, D, A^*, B^*, C^*, D^*\}$ be a set of eight distinct points in $\mathbb{R}^3$. Suppose that AC^*BD^*, AB^*DC^*, C^*DA^*B, BA^*CD^*, D^*CB^*A, and B^*DA^*C are *quadrilaterals* such that no two of their interiors intersect. Then by joining A, B, C, D, A^*, B^*, C^*, D^* with twelve edges to form these quadrilaterals, we obtain a solid called a *hexahedron with eight vertices* and denote it by $\binom{AC^*BD^*}{B^*DA^*C}$.

Cubes, rectangular boxes and parallelepipeds are hexahedra with eight vertices. The word "hexahedron" indicates that the solid has six faces. So, two identical regular tetrahedra glued face to face is an example of a hexahedron with *five* vertices.

In order to shorten "a hexahedron with eight vertices", and in order to avoid the confusion with a hexahedron with five vertices, we call a hexahedron $\binom{AC^*BD^*}{B^*DA^*C}$ with eight vertices a *box.* The notation $\binom{AC^*BD^*}{B^*DA^*C}$ indicates that the quadrilaterals AC^*BD^* and B^*DA^*C are the top and bottom faces of the box, respectively; the vertices A, C^*, B, D^* are connected to vertices B^*, D, A^*, C, respectively, by edges $[AB^*]$, $[C^*D]$, $[BA^*]$, and $[D^*C]$; the segments $[AA^*]$, $[BB^*]$, $[CC^*]$, and $[DD^*]$ are *diagonals* of the box $\binom{AC^*BD^*}{B^*DA^*C}$; and by drawing two diagonals on each face, we obtain two tetrahedron $\nabla ABCD$ and $\nabla A^*B^*C^*D^*$ circumscribed by this box. See Figure 1. This is a motivation for the next definition.

Definition 1. A tetrahedron $\nabla A^*B^*C^*D^*$ is said to *tangle* a tetrahedron $\nabla ABCD$ *at* (E, F, G, H, I, J) if $(AB) \cap (C^*D^*) = \{E\}$, $(AC) \cap (B^*D^*) = \{F\}$, $(AD) \cap (B^*C^*) = \{G\}$, $(BC) \cap (A^*D^*) = \{H\}$, $(BD) \cap (A^*C^*) = \{I\}$, $(CD) \cap (A^*B^*) = \{J\}$. See Figure 1.

Here, by $(AB) \cap (C^*D^*) = \{E\}$, we mean that the edges (AB) and (C^*D^*) intersect at E. The order of the listed vertices in (E, F, G, H, I, J) is to indicate $E \in (AB)$, $F \in (AC)$, $G \in (AD)$, $H \in (BC)$, $I \in (BD)$, $J \in (CD)$. So $\nabla ABCD$ tangles $\nabla A^*B^*C^*D^*$ at (J, I, H, G, F, E). And we say that the box $\binom{AC^*BD^*}{B^*DA^*C}$ *circumscribes* $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$. We also say that $\nabla ABCD$ and $\nabla A^*B^*C^*D^*$ *are tangled at* (E, F, G, H, I, J).

We write $\triangle ABC$ to indicate a triangle ABC, and Ω_{ABC} to indicate the plane containing $\triangle ABC$. We say that points $D' \in \mathbb{R}^3$ and $D \in \mathbb{R}^3$ are on the *same side with respect to* Ω_{ABC} if $[DD'] \cap \Omega_{ABC} = \emptyset$, and points $D' \in \mathbb{R}^3$ and $D \in \mathbb{R}^3$ are on the *opposite sides with respect to* Ω_{ABC} if $(DD') \cap \Omega_{ABC} \neq \emptyset$. If $ABCD$ is a quadrilateral, then Ω_{ABCD} is the plane containing the quadrilateral $ABCD$.

Constructing a box $\binom{AC^*BD^*}{B^*DA^*C}$ is equivalent to finding two tangled tetrahedra $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$. Our first observation is Theorem 1.

Theorem 1. *Let $\nabla ABCD$ be a tetrahedron. Let $E \in (AB)$, $F \in (AC)$, $G \in (AD)$, $H \in (BC)$, $I \in (BD)$, $J \in (CD)$. (See Figure 1.) Then there is a tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$ at (E, F, G, H, I, J) if and only if* *(i)* $\Omega_{EHI}\cap\Omega_{FHJ}\cap\Omega_{GJI}=\{A^*\}$, $\Omega_{EFG}\cap\Omega_{HFJ}\cap\Omega_{IGJ}=\{B^*\}$, $\Omega_{FEG}\cap\Omega_{HEI}\cap\Omega_{JGI}=\{C^*\}$, $\Omega_{GEF}\cap\Omega_{IEH}\cap\Omega_{JFH}=\{D^*\}$, *and*

(ii) the points A^*, B^*, C^*, D^* are on the opposite side of A, B, C, D with respect to Ω_{BCD}, Ω_{ACD}, Ω_{ABD}, Ω_{ABC}, respectively.

Proof. Suppose $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ at (E,F,G,H,I,J). Then $\Omega_{EHI}=\Omega_{A^*C^*D^*}$, $\Omega_{FHJ}=\Omega_{A^*B^*D^*}$, and $\Omega_{GJI}=\Omega_{A^*B^*C^*}$. Hence, $\Omega_{EHI}\cap\Omega_{FHJ}\cap\Omega_{GJI}=\{A^*\}$. Similarly, $\Omega_{EFG}\cap\Omega_{HFJ}\cap\Omega_{IGJ}=\{B^*\}$, $\Omega_{FEG}\cap\Omega_{HEI}\cap\Omega_{JGI}=\{C^*\}$, and $\Omega_{GEF}\cap\Omega_{IEH}\cap\Omega_{JFH}=\{D^*\}$. And the points A^*, B^*, C^*, D^* are on the opposite side of A, B, C, D with respect to Ω_{BCD}, Ω_{ACD}, Ω_{ABD}, Ω_{ABC}, respectively.

Conversely, suppose (i) and (ii) are satisfied. Then $\Omega_{EHI}=\Omega_{A^*C^*D^*}$, $\Omega_{FHJ}=\Omega_{A^*B^*D^*}$, and $\Omega_{GJI}=\Omega_{A^*B^*C^*}$, and $\Omega_{EFG}=\Omega_{B^*C^*D^*}$. Thus, the line $\overline{C^*D^*}=\Omega_{A^*C^*D^*}\cap\Omega_{B^*C^*D^*}=\Omega_{EHI}\cap\Omega_{EFG}$. Hence, E is on the edge $[C^*D^*]$. That is, $[AB]\cap[C^*D^*]=\{E\}$. Similarly, $[AC]\cap[B^*D^*]=\{F\}$, $[AD]\cap[B^*C^*]=\{G\}$, $[BC]\cap[A^*D^*]=\{H\}$, $[BD]\cap[A^*C^*]=\{I\}$, and $[CD]\cap[A^*B^*]=\{J\}$. Therefore, $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ at (E,F,G,H,I,J). □

In Theorem 1, choosing E, F, G, H, I, J randomly on the edges of $\nabla ABCD$ may not result in obtaining a tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$ at (E,F,G,H,I,J). So we will consider a more specific way of finding points E, F, G, H, I, J in order to find a tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$.

Definition 2. Suppose $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$. The diagonals $[AA^*]$, $[BB^*]$, $[CC^*]$, $[DD^*]$ of the circumscribing box $\binom{AC^*BD^*}{B^*DA^*C}$ are called *diagonals* of the tangled tetrahedra $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$.

If the diagonals $[AA^*]$, $[BB^*]$, $[CC^*]$, $[DD^*]$ concur at a point P, we say that $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ *are tangled with concurrent diagonals at P*. We also say that $\nabla A^*B^*C^*D^*$ *tangles $\nabla ABCD$ with concurrent diagonals at P*. See Figure 2.

Remark 1. Every parallelepiped has concurrent diagonals. We are not used to thinking of a box without concurrent diagonals. But there are many boxes without concurrent diagonals. Figure 4 in Section 4 is an example of a box without concurrent diagonals.

Definition 3. Let P be a point inside of $\nabla ABCD$. Let $\{A'\}=\overline{AP}\cap\Omega_{BCD}$, $\{B'\}=\overline{BP}\cap\Omega_{ACD}$, $\{C'\}=\overline{CP}\cap\Omega_{ABD}$, and $\{D'\}=\overline{DP}\cap\Omega_{ABC}$. The points P is said to be a *deep interior point* of a tetrahedron $\nabla ABCD$ if $|PA|>|PA'|$, $|PB|>|PB'|$, $|PC|>|PC'|$, and $|PD|>|PD'|$.

Remark 2. Let $\nabla ABCD$ be a tetrahedron. See Figure 3. Let E, F, G, H, I, J be the midpoints of the edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, $[CD]$, respectively. Let P be a point inside of $\nabla ABCD$. Let $\{A'\}=\overline{AP}\cap\Omega_{BCD}$. Then $|PA|>|PA'|$ if and only if $P\notin\nabla AEFG$. Hence, the point P is a deep interior point of $\nabla ABCD$ if and only if $P\in\nabla ABCD-(\nabla AEFG\cup\nabla BEHI\cup\nabla CFHJ\cup\nabla DGJI)$. That is, the point P is a deep interior point of $\nabla ABCD$ if and only if P is in the interior of the octahedron $EFGHIJ$. For example, since $[EJ]$ is a diagonal of the octahedron $EFGHIJ$, any point on the segment (EJ) is a deep interior point of $\nabla ABCD$.

In Theorem 2 of Section 2, we will show that, for any deep interior point P of any tetrahedron $\nabla ABCD$, there exists a unique tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$ with concurrent diagonals at P. Theorem 2 is a consequence of the results in [5], and the basis of this paper.

Definition 4. Suppose a box $\binom{AC^*BD^*}{B^*DA^*C}$ that circumscribes a tetrahedron $\nabla ABCD$ is a parallelepiped having $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, $[CD]$ as diagonals of its six parallelogram faces. So, for example, the face AC^*BD^* contains the edge $[AB]$ of $\nabla ABCD$ and is parallel to $\overline{CD}$. Such a parallelepiped always exists for any tetrahedron $\nabla ABCD$. In this case, the tetrahedron $\nabla A^*B^*C^*D^*$ tangles and is congruent to $\nabla ABCD$, and the tetrahedra $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ are said to be *twin tetrahedra* of simply *twins.* See [1, Page 58] for additional information on twins.

In Theorem 3 of Section 3, we will show that the two tetrahedra are twins if and only if they are tangled tetrahedra with concurrent diagonals at the centroid of one of the tetrahedra. We will give two proofs of Theorem 3. The alternate proof of Theorem 3 uses Theorem 4. Theorem 4 gives a characterization of the centroid of a tetrahedron that we think is new.

Definition 5. A tetrahedron $\nabla ABCD$ such that $|AB| = |CD|$, $|AC| = |BD|$, and $|AD| = |BC|$ is said to be an *isosceles tetrahedron.*

The parallelepiped $\binom{AC^*BD^*}{B^*DA^*C}$ that circumscribes an isosceles tetrahedron $\nabla ABCD$ is a rectangular box since $\nabla ABCD \cong \nabla A^*B^*C^*D^*$ implies that the two diagonals of the face AC^*BD^*, for example, is of the same length, i.e., AC^*BD^* is a rectangle. For additional basic properties of an isosceles tetrahedron, see [1, Pages 94–102]. The next definition gives a weaker version of an isosceles tetrahedron.

Definition 6. A tetrahedron $\nabla ABCD$ is *reversible* if ($|AB| = |CD|$ and $|AC| = |BD|$), or ($|AC| = |BD|$ and $|AD| = |BC|$), or ($|AB| = |CD|$ and $|AD| = |BC|$). Hence, a tetrahedron is reversible if and only if its faces can be labeled f_1, f_2, f_3, f_4 so that (f_1 is congruent to f_2) and (f_3 is congruent to f_4).

Remark 3. We believe that a reversible tetrahedron is a notion introduced by D. A. Klain. Using the notations in the above definition, Klain in [6] proved that a tetrahedron is reversible if and only if $area(f_1) = area(f_2)$ and $area(f_3) = area(f_4)$.

We make the following convention.

Convention. We will identify a reversible tetrahedron $\nabla ABCD$ such that $|AC| = |BD|$ and $|AD| = |BC|$ by saying that $\nabla ABCD$ is *reversible with* $|AB| \leq |CD|$ or $\frac{|CD|}{|AB|} = k \geq 1$.

Suppose $\nabla ABCD$ is a reversible tetrahedron with $|AB| \leq |CD|$. In Theorem 5 of Section 4, we will prove that there is a tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$ with concurrent diagonals such that $\nabla D^*C^*B^*A^* \cong \nabla ABCD$. Here, please note that we are not talking about the twin tetrahedron $\nabla A^*B^*C^*D^*$ of $\nabla ABCD$, and the box $\binom{AC^*BD^*}{B^*DA^*C}$ circumscribing $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ is trapezoidal (see Definition 7 below) rather than a parallelepiped.

2 Two Tangled Tetrahedra with Concurrent Diagonals

Lemma 1 is from [4, Theorem 2]. We will give it a new proof different from the one in [4].

Lemma 1. *Let P be a point inside of $\nabla ABCD$. Then there are unique points E, F, G, H, I, J on the edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, $[CD]$, respectively, such that the segments $[EJ]$, $[FI]$ and $[GH]$ concur at P. (See Figure 2 for points E, J, and P.)*

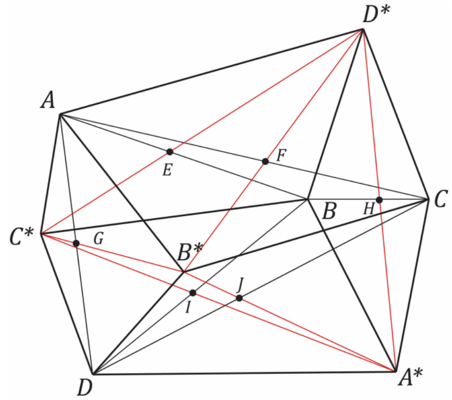

Figure 1: A box $\binom{AC^*BD^*}{B^*DA^*C}$ circumscribing tetrahedron $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$. The points D and D^*, for example, are on the opposite sides of the plane Ω_{ABC}. The tetrahedron $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ at (E, F, G, H, I, J).

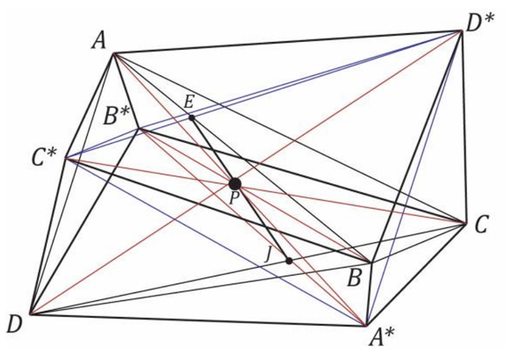

Figure 2

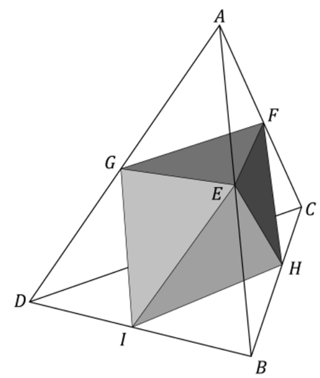

Figure 3

Figure 2: Two tetrahedra $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ that are tangled with concurrent diagonals at P. The point P is the intersection of the diagonals $[AA^*]$, $[BB^*]$, $[CC^*]$, $[DD^*]$, of the box $\binom{AC^*BD^*}{B^*DA^*C}$. The points E and J are the intersections of ($[AB]$ and $[C^*D^*]$) and ($[CD]$ and $[A^*B^*]$), respectively. The segment $[EJ]$ is shown to contain the point P. The segments $[FI]$ and $[GH]$ are not drawn here, but they also contain P.

Figure 3: A tetrahedron $\nabla ABCD$ and the octahedron $EFGHIJ$ are the ones mentioned in Remark 2. Interior of the octahedron $EFGHIJ$ is the deep interior of the tetrahedron $\nabla ABCD$. The point J is hidden in Figure 3.

Proof. Let E be the intersection of the edge $[AB]$ and the plane Ω_{PCD}. Then the line $\overline{EP}$ intersects the edge $[CD]$ at a unique point. Let J be the intersection of the edge $[CD]$ and the plane Ω_{PAB}. The line $\overline{JP}$ intersect the edge $[AB]$ at a unique point. But then the lines $\overline{EP}$ and $\overline{JP}$ must be the intersection of the planes Ω_{PCD} and Ω_{PAB} so that $\overline{EP} = \overline{JP}$. Hence, the points E, P and J are collinear. And therefore, E and J are the unique points on the segments $[AB]$ and $[CD]$, respectively, such that P is on the segment $[EJ]$.

Similarly, let F, G, H, I be the intersections of ($[AC]$ and Ω_{PBD}), ($[AD]$ and Ω_{PBC}), ($[BC]$ and Ω_{PAD}), and ($[BD]$ and Ω_{PAC}), respectively. Then these points are the unique points on the edges $[AC]$, $[AD]$, $[BC]$, and $[BD]$, respectively such that $P \in [FI]$ and $P \in [GH]$. Hence, the segments $[EJ]$, $[FI]$ and $[GH]$ concur at P. $\square$

We combined Theorem 1 and Corollary 1 of [5] in the next lemma.

Lemma 2. *Let P be a point inside of $\nabla ABCD$. Let $E \in (AB)$, $F \in (AC)$, $G \in (AD)$, $H \in (BC)$, $I \in (BD)$, $J \in (CD)$ such that $[EJ] \cap [FI] \cap [GH] = \{P\}$. Let $A' = \overline{AP} \cap \Omega_{BCD}$. Then we have the following:*

(a) If $|PA| = |PA'|$, then the planes Ω_{EHI}, Ω_{FHJ} and Ω_{GJI} do not intersect at a point.

(b) If $|PA| < |PA'|$, then Ω_{EHI}, Ω_{FHJ} and Ω_{GJI} intersect at a point. Their intersection A^* is on the same side of A with respect to the plane Ω_{BCD}.

(c) If $|PA| > |PA'|$, then Ω_{EHI}, Ω_{FHJ} and Ω_{GJI} also intersect at a point. Their intersection A^* is on the opposite sides of A with respect to the plane Ω_{BCD}.

In either case (b) or (c), if $\Omega_{EHI} \cap \Omega_{FHJ} \cap \Omega_{GJI} = \{A^*\}$, the point A^* is on the line $\overline{AP}$. (See Figure 2 for the statement (c).)

Lemma 3. *Let P be a deep interior point of $\nabla ABCD$. Let $E \in (AB)$, $F \in (AC)$, $G \in (AD)$, $H \in (BC)$, $I \in (BD)$, $J \in (CD)$ such that $[EJ] \cap [FI] \cap [GH] = \{P\}$. Then $\Omega_{EHI} \cap \Omega_{FHJ} \cap \Omega_{GJI} \neq \emptyset$, $\Omega_{EFG} \cap \Omega_{HFJ} \cap \Omega_{IGJ} \neq \emptyset$, $\Omega_{FEG} \cap \Omega_{HEI} \cap \Omega_{JGI} \neq \emptyset$, and $\Omega_{GEF} \cap \Omega_{IEH} \cap \Omega_{JFH} \neq \emptyset$.*

Proof. This is a consequence of Lemma 2(c). □

Theorem 2. *Let P be a deep interior point of a tetrahedron $\nabla ABCD$. Let $E \in (AB)$, $F \in (AC)$, $G \in (AD)$, $H \in (BC)$, $I \in (BD)$, $J \in (CD)$ such that $[EJ] \cap [FI] \cap [GH] = \{P\}$. Let $\Omega_{EHI} \cap \Omega_{FHJ} \cap \Omega_{GJI} = \{A^*\}$, $\Omega_{EFG} \cap \Omega_{HFJ} \cap \Omega_{IGJ} = \{B^*\}$, $\Omega_{FEG} \cap \Omega_{HEI} \cap \Omega_{JGI} = \{C^*\}$, and $\Omega_{GEF} \cap \Omega_{IEH} \cap \Omega_{JFH} = \{D^*\}$. (See Figure 2.) Then*

(1) the tetrahedron $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ at (E, F, G, H, I, J),

(2) $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ are tangled with concurrent diagonals at P, and

(3) the tetrahedron $\nabla A^*B^*C^*D^*$ is the only tetrahedron that tangles $\nabla ABCD$ with concurrent diagonals at P. In other words, the box $\binom{AC^*BD^*}{B^*DA^*C}$ is the only box that circumscribes $\nabla ABCD$ with concurrent diagonals concurring at P.

Proof. We know the existence of the points A^*, B^*, C^*, D^* by Lemma 3. By Lemma 2(c), since P is a deep interior point of $\nabla ABCD$, the points A^*, B^*, C^*, D^* are on the opposite side of A, B, C, D with respect to Ω_{BCD}, Ω_{ACD}, Ω_{ABD}, Ω_{ABC}, respectively. Therefore, by Theorem 1, $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ at (E, F, G, H, I, J). Moreover, we have $[AA^*] \cap [BB^*] \cap [CC^*] \cap [DD^*] = \{P\}$ by Lemma 2. Hence, $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ are tangled with concurrent diagonals at P.

By Lemma 1, points E, F, G, H, I, J are the uniquely determined points on the edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, $[CD]$, respectively, such that the segments $[EJ]$, $[FI]$ and $[GH]$ concur at P. Hence, the tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$ at (E, F, G, H, I, J) must be unique. Therefore, the tetrahedron $\nabla A^*B^*C^*D^*$ is the only tetrahedron that tangles $\nabla ABCD$ with concurrent diagonals at P. □

3 Centroids and Twin Tetrahedra

Let $\nabla ABCD$ be a tetrahedron. The point $P \in \mathbb{R}^3$ such that $\overrightarrow{AP} = \frac{1}{4}(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD})$ is the *centroid* of the tetrahedron $\nabla ABCD$, and it is inside of this tetrahedron.

The next lemma is Theorem 198 and Corollary 199 on [1, Pages 59–60] expressed slightly differently with our terminology.

Lemma 4. *Let $\binom{AC^*BD^*}{B^*DA^*C}$ be a parallelepiped and let $[AA^*] \cap [BB^*] \cap [CC^*] \cap [DD^*] = \{P\}$. Then the twin tetrahedron $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ are tangled tetrahedra with concurrent diagonals at P, and P is the centroid of both $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$.*

Lemma 5. *If $E \in (AB)$, $F \in (AC)$, $G \in (AD)$, $H \in (BC)$, $I \in (BD)$, $J \in (CD)$ are the midpoints of these edges of a tetrahedron $\nabla ABCD$, then $[EJ], [FI]$ and $[GH]$ concur at their midpoints, say at P; and the points P is the centroid of $\nabla ABCD$.*

Proof. See Theorem 152 on page 48 of [1], for example. □

Lemma 6. *Let P be the centroid of a tetrahedron $\nabla ABCD$. Then there is a tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$ with the concurrent diagonals at P.*

Proof. Let the points E and J be the midpoints of the edges $[AB]$ and $[CD]$, respectively. Then $P \in (EJ)$, and P is a deep interior point of $\nabla ABCD$. By Theorem 2, there is a unique tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$ with the concurrent diagonals at P. □

Lemmas 4 and 6 lead us to the following question:

Question 1. Is the tetrahedron $\nabla A^*B^*C^*D^*$ in Lemma 6 the twin of $\nabla ABCD$?

The answer is in the next theorem.

Theorem 3. *Two tetrahedra are twins if and only if they are tangled with concurrent diagonals at P, where P is the centroid of* one *of the tetrahedra.*

Proof. By Lemma 4, we only prove that if two tetrahedra are tangled with concurrent diagonals at P, where P is the centroid of *one* of the tetrahedra, then they are twins. Suppose P is the centroid of a tetrahedron $\nabla ABCD$, and suppose the tetrahedron $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ with concurrent diagonals at P. Let $\nabla A'B'C'D'$ be the twin tetrahedron of $\nabla ABCD$. Then $\nabla A'B'C'D'$ tangles $\nabla ABCD$ with concurrent diagonals at P by Lemma 4. But by Theorem 2, the tetrahedron that tangles $\nabla ABCD$ with concurrent diagonals at P is unique. Hence, $\nabla A'B'C'D'$ and $\nabla A^*B^*C^*D^*$ must be identical. This proves that $\nabla A^*B^*C^*D^*$ is the twin of $\nabla ABCD$. □

Perhaps, the above proof of Theorem 3 is too short to be convincing. By considering an alternate proof of Theorem 3, we discovered Theorem 4 about the centroid of a tetrahedron that we think is new. Theorem 4 below is a stronger version of Lemma 6. And we use it to give an alternate proof of Theorem 3. Ceva's theorem is used to prove Theorem 4.

Theorem (Ceva's Theorem; see [2, Pages 4–5])**.** *Let $\triangle ABC$ be a triangle. Let A', B', C' be points on the edges $[BC]$, $[CA]$, $[AB]$, respectively. Then $[AA']$, $[BB']$ and $[CC]$ concur if and only if $\frac{|AC'|}{|C'B|} \cdot \frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} = 1$.*

Theorem 4. *Let $\nabla ABCD$ be a tetrahedron. Let $E \in (AB)$, $F \in (AC)$, $G \in (AD)$, $H \in (BC)$, $I \in (BD)$, $J \in (CD)$. If $[EJ] \cap [FI] \cap [GH] = \{P\}$, and if P is the midpoint of the segments $[EJ]$, $[FI]$, and $[GH]$, then P is the* centroid *of $\nabla ABCD$.*

Proof. We prove this using vectors. Let $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AC} = \vec{c}$, $\overrightarrow{AD} = \vec{d}$. In order to show that P is the centroid, we prove that $\overrightarrow{AP} = \frac{1}{4}(\vec{b} + \vec{c} + \vec{d})$.

Note that $\overrightarrow{AG} = g\overrightarrow{AD} = g\vec{d}$ and $\overrightarrow{BH} = h\overrightarrow{BC} = h(\vec{c} - \vec{b})$ for some $0 < g,\ h < 1$. Since P is the midpoint of the segment $[GH]$, we have

$$\overrightarrow{AP} = \frac{1}{2}\left(\overrightarrow{AG} + \overrightarrow{AH}\right) = \frac{1}{2}\left(\overrightarrow{AG} + \overrightarrow{AB} + \overrightarrow{BH}\right) = \frac{(1-h)}{2}\vec{b} + \frac{h}{2}\vec{c} + \frac{g}{2}\vec{d}.$$

Since $J \in \Omega_{APB}$, we have $\overrightarrow{AJ} = s\overrightarrow{AB} + t\overrightarrow{AP}$ for some s, $t \in \mathbb{R}$. Hence,

$$\overrightarrow{AJ} = s\,\vec{b} + \left(\frac{t(1-h)}{2}\vec{b} + \frac{th}{2}\vec{c} + \frac{tg}{2}\vec{d}\right) = \left(s + \frac{t(1-h)}{2}\right)\vec{b} + \frac{th}{2}\vec{c} + \frac{tg}{2}\vec{d}.$$

Since J is a point on the edge $[CD]$, we have

$$\overrightarrow{CJ} = u\overrightarrow{CD} \quad \text{for some} \quad 0 < u < 1.$$

Since $\overrightarrow{AJ} = \overrightarrow{AC} + \overrightarrow{CJ} = \overrightarrow{AC} + u\overrightarrow{CD}$, we have

$$\overrightarrow{AJ} = \vec{c} + u\big(-\vec{c} + \vec{d}\big) = (1-u)\,\vec{c} + u\,\vec{d}.$$

Hence, we must have $\left(s + \frac{t(1-h)}{2}\right)\vec{b} + \frac{th}{2}\vec{c} + \frac{tg}{2}\vec{d} = \overrightarrow{AJ} = (1-u)\,\vec{c} + u\,\vec{d}$.

This shows that $s + \frac{t(1-h)}{2} = 0$, $\frac{th}{2} = 1-u$, and $\frac{tg}{2} = u$. From $\frac{tg}{2} = u$, we have $\frac{t}{2} = \frac{u}{g}$.

Substituting this into $\frac{th}{2} = 1-u$, we have $\frac{uh}{g} = 1-u$. Hence, we have $u = \frac{g}{g+h}$. This gives us

$$\overrightarrow{AJ} = \frac{h}{g+h}\vec{c} + \frac{g}{g+h}\vec{d}.$$

From this, we have

$$\overrightarrow{JP} = \overrightarrow{AP} - \overrightarrow{AJ} = \left\{\frac{(1-h)}{2}\vec{b} + \frac{h}{2}\vec{c} + \frac{g}{2}\vec{d}\right\} - \left\{\frac{h}{g+h}\vec{c} + \frac{g}{g+h}\vec{d}\right\}$$
$$= \frac{1-h}{2}\vec{b} + \left(\frac{h}{2} - \frac{h}{g+h}\right)\vec{c} + \left(\frac{g}{2} - \frac{g}{g+h}\right)\vec{d}.$$

Since P is the midpoint of $[EJ]$, we have

$$\overrightarrow{AE} = \overrightarrow{AJ} + 2\overrightarrow{JP} = \left\{\frac{h}{g+h}\vec{c} + \frac{g}{g+h}\vec{d}\right\} + 2\left\{\frac{1-h}{2}\vec{b} + \left(\frac{h}{2} - \frac{h}{g+h}\right)\vec{c} + \left(\frac{g}{2} - \frac{g}{g+h}\right)\vec{d}\right\}$$
$$= (1-h)\,\vec{b} + h\left(1 - \frac{1}{g+h}\right)\vec{c} + g\left(1 - \frac{1}{g+h}\right)\vec{d}.$$

Since E is on the edge AB, we must have $\overrightarrow{AE} = (1-h)\,\vec{b}$. Hence, $1 - \frac{1}{g+h} = 0$ so that

$$h = 1-g \quad \text{and} \quad u = g.$$

Summarizing, we have

$$\overrightarrow{AG} = g\overrightarrow{AD}, \quad \overrightarrow{BH} = (1-g)\overrightarrow{BC}, \quad \overrightarrow{CJ} = u\overrightarrow{CD} = g\overrightarrow{CD}, \quad \text{and} \quad \overrightarrow{AE} = g\overrightarrow{AB}.$$

Since $\Omega_{DEP} \cap \Omega_{DFP} \cap \Omega_{DHP} = \overline{DP}$, the line $\overline{DP}$ intersects the triangular face $\triangle ABC$, say at D'. So, the segments $[AH]$, $[BF]$, $[CE]$ must concur at D'. By Ceva's theorem applied to $\triangle ABC$, we must have

$$\frac{|AE|}{|EB|} \cdot \frac{|BH|}{|HC|} \cdot \frac{|CF|}{|FA|} = 1,$$

or $\frac{g}{1-g} \cdot \frac{1-g}{g} \cdot \frac{|CF|}{|FA|} = 1$. Hence, $|CF| = |FA|$, or F is the midpoint of $[AC]$. We have $\overrightarrow{AF} = \frac{1}{2}\vec{c}$.

Similarly, since $\Omega_{AHP} \cap \Omega_{AIP} \cap \Omega_{AJP} = \overline{AP}$, the line $\overline{AP}$ intersects the triangular face $\triangle BCD$, say at A'. So, the segments $[BJ]$, $[CI]$, $[DH]$ must concur at A'. By Ceva's theorem applied to $\triangle BCD$, we must have

$$\frac{|BH|}{|HC|} \cdot \frac{|CJ|}{|JD|} \cdot \frac{|DI|}{|IB|} = 1,$$

or $\frac{1-g}{g}\cdot\frac{g}{1-g}\cdot\frac{|DI|}{|IB|}=1$. Hence, $|DI|=|IB|$, or I is the midpoint of BD. We have $\overrightarrow{AI}=\frac{1}{2}(\vec{b}+\vec{d})$.

Now, since P is the midpoint of the segment $[FI]$, we have

$$\overrightarrow{AP}=\frac{1}{2}\Big(\overrightarrow{AF}+\overrightarrow{AI}\Big)=\frac{1}{2}\Big(\frac{1}{2}\vec{c}+\frac{1}{2}\big(\vec{b}+\vec{d}\big)\Big)=\frac{1}{4}\big(\vec{b}+\vec{c}+\vec{d}\big).$$

This proves that P must be the centroid of $\nabla ABCD$. □

An alternate proof of Theorem 3. We only prove that if two tetrahedra are tangled with concurrent diagonals at P, where P is the centroid of *one* of the tetrahedra, then they are twins. Suppose P is the centroid of a tetrahedron $\nabla ABCD$. Let $\nabla A^*B^*C^*D^*$ be the tetrahedron that tangles $\nabla ABCD$ with the concurrent diagonals at P. The existence of $\nabla A^*B^*C^*D^*$ is proved in Lemma 6. We will prove that $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ are twin tetrahedra by showing the box $\binom{AC^*BD^*}{B^*DA^*C}$ is a parallelepiped.

Let $E\in(AB)$, $F\in(AC)$, $G\in(AD)$, $H\in(BC)$, $I\in(BD)$, $J\in(CD)$ be the midpoints, respectively. Then $[EJ]\cap[FI]\cap[GH]=\{P\}$; P is the midpoint of the segments $[EJ]$, $[FI]$, and $[GH]$; and P is the centroid of a tetrahedron $\nabla ABCD$ by Lemma 5. On the other hand, since $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ at (E,F,G,H,I,J) we have $E\in(C^*D^*)$, $F\in(B^*D^*)$, $G\in(B^*C^*)$, $H\in(A^*D^*)$, $I\in(A^*C^*)$, $J\in(A^*B^*)$, and since P is the midpoint of the segments $[EJ]$, $[FI]$, and $[GH]$, we know that P is also the centroid of $\nabla A^*B^*C^*D^*$ by Theorem 4. The point P being the centroid $\nabla A^*B^*C^*D^*$ implies that the points E,F,G,H,I,J are the midpoints of the edges $[C^*D^*]$, $[B^*D^*]$, $[B^*C^*]$, $[A^*D^*]$, $[A^*C^*]$, and $[A^*B^*]$, respectively. But the points E, F, G, H, I, J are chosen to be the midpoints of $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, $[CD]$, respectively. In particular, the diagonals $[AB]$ and $[C^*D^*]$ of the quadrilateral AC^*BD^* intersect at their midpoint E. Therefore, the quadrilateral AC^*BD^* is a parallelogram. Similarly, all the remaining five quadrilateral faces of $\binom{AC^*BD^*}{B^*DA^*C}$ are parallelograms. Therefore, this proves that the box $\binom{AC^*BD^*}{B^*DA^*C}$ is a parallelepiped. This proves that $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ are twins. □

4 Reversible Tetrahedra

The next question is, perhaps, a natural consequence of Theorem 3.

Question 2. Are there tetrahedra $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ that are tangled with the concurrent diagonals at P such that the circumscribing box $\binom{AC^*BD^*}{B^*DA^*C}$ may not be a parallelepiped, yet for some permutation (A',B',C',D') of (A^*,B^*,C^*,D^*), the tetrahedron $\nabla A'B'C'D'$ is congruent to $\nabla ABCD$?

We will answer this YES in Theorem 5 using a reversible tetrahedron.

Definition 7. A quadrilateral is a *trapezoid* if two opposing edges are parallel.

If a box $\binom{AC^*BD^*}{B^*DA^*C}$ has two rectangular faces AC^*BD^* and B^*DA^*C with centers E and J, respectively, such that the line $\overline{EJ}$ is perpendicular to both $\Omega_{AC^*BD^*}$ and $\Omega_{B^*DA^*C}$; (the edges $[AC^*]$, $[A^*C]$, $[BD^*]$, $[B^*D]$ are parallel); and (the edges $[C^*B]$, $[CB^*]$, $[D^*A]$ and $[DA^*]$ are parallel); then the box $\binom{AC^*BD^*}{B^*DA^*C}$ is said to be *trapezoidal*. See Figures 4 and 5 for examples of trapezoidal boxes. The next lemma gives a motivation for this naming.

Remark 4. In the above definition, the condition (the edges $[AC^*]$, $[A^*C]$, $[BD^*]$, $[B^*D]$ are parallel), for example, is necessary in order for $\binom{AC^*BD^*}{B^*DA^*C}$ to be a box.

Lemma 7. *Suppose* $\binom{AC^*BD^*}{B^*DA^*C}$ *is a trapezoidal box. Then the lateral faces* AB^*DC^*, C^*DA^*B, BA^*CD^*, *and* D^*CB^*A *are trapezoids such that* $|AB^*| = |C^*D| = |BA^*| = |D^*C|$; $|AD| = |B^*C^*| = |BC| = |A^*D^*|$; *and* $|BD| = |A^*C^*| = |AC| = |B^*D^*|$. *Hence,* $\nabla D^*C^*B^*A^* \cong \nabla ABCD$ *and these tetrahedra can be circumscribed by the same sphere.*

Proof. By the definition of a trapezoidal box $\binom{AC^*BD^*}{B^*DA^*C}$, we can embed it in $\mathbb{R}^3$ by letting $A = (-a, -b, c)$, $C^* = (a, -b, c)$, $B = (a, b, c)$, $D^* = (-a, b, c)$, and $B^* = (-\alpha, \beta, 0)$, $D = (\alpha, -\beta, 0)$, $A^* = (\alpha, \beta, 0)$, $C = (-\alpha, \beta, 0)$ for some a, b, c, α, $\beta > 0$. Hence, $|AB^*| = |C^*D| = |BA^*| = |D^*C| = \sqrt{(a-\alpha)^2 + (b-\beta)^2 + c^2}$. Hence, AB^*DC^* and BA^*CD^* are congruent trapezoids, and C^*DA^*B and D^*CB^*A are congruent trapezoids. Hence, $|AD| = |B^*C^*| = |BC| = |A^*D^*|$ and $|BD| = |A^*C^*| = |AC| = |B^*D^*|$. Moreover, since the faces AC^*BD^* and B^*DA^*C are rectangles, we have $|AB| = |D^*C^*|$ and $|CD| = |B^*A^*|$. Therefore, $\nabla D^*C^*B^*A^* \cong \nabla ABCD$.

Next let $Q = (0, 0, q)$, where $q = \frac{1}{2c}\{(a^2 + b^2) - (\alpha^2 + \beta^2) + c^2\}$. We will show that Q is the center of the sphere circumscribing the trapezoidal box $\binom{AC^*BD^*}{B^*DA^*C}$. Since $|QA| = |QB| = |QC^*| = |QD^*|$ and $|QA^*| = |QB^*| = |QC| = |QD|$, we only have to shown that $|QA| = |QA^*|$.

$$\begin{aligned}
4c^2|QA|^2 &= 4c^2(a^2+b^2) + (2c^2 - 2cq)^2 \\
&= 4c^2(a^2+b^2) + \{c^2 - (a^2+b^2) + (\alpha^2+\beta^2)\}^2 \\
&= 4c^2(a^2+b^2) + c^4 + (a^2+b^2)^2 + (\alpha^2+\beta^2)^2 - 2c^2(a^2+b^2) + 2(\alpha^2+\beta^2)c^2 \\
&\qquad - 2(a^2+b^2)(\alpha^2+\beta^2) \\
&= c^4 + (a^2+b^2)^2 + (\alpha^2+\beta^2)^2 + 2c^2(a^2+b^2) + 2(\alpha^2+\beta^2)c^2 - 2(a^2+b^2)(\alpha^2+\beta^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
4c^2|QA^*|^2 &= 4c^2(\alpha^2+\beta^2) + \{c^2 + (a^2+b^2) - (\alpha^2+\beta^2)\}^2 \\
&= 4c^2(\alpha^2+\beta^2) + c^4 + (a^2+b^2)^2 + (\alpha^2+\beta^2)^2 + 2c^2(a^2+b^2) - 2(\alpha^2+\beta^2)c^2 \\
&\qquad - 2(a^2+b^2)(\alpha^2+\beta^2) \\
&= c^4 + (a^2+b^2)^2 + (\alpha^2+\beta^2)^2 + 2c^2(a^2+b^2) + 2(\alpha^2+\beta^2)c^2 - 2(a^2+b^2)(\alpha^2+\beta^2).
\end{aligned}$$

Hence, $|QA| = |QA^*|$. This proves that $\nabla D^*C^*B^*A^*$ and $\nabla ABCD$ are circumscribed by the same sphere. $\square$

Lemma 8. *A trapezoidal box* $\binom{AC^*BD^*}{B^*DA^*C}$ *has four concurrent diagonals if and only if* $\frac{|B^*D|}{|AC^*|} = \frac{|DA^*|}{|C^*B|}$.

Proof. As in the proof of Lemma 7, since $\binom{AC^*BD^*}{B^*DA^*C}$ is a trapezoidal box, we let $A = (-a, -b, c)$, $C^* = (a, -b, c)$, $B = (a, b, c)$, $D^* = (-a, b, c)$, and $B^* = (-\alpha, -\beta, 0)$, $D = (\alpha, -\beta, 0)$, $A^* = (\alpha, \beta, 0)$, $C = (-\alpha, \beta, 0)$ for some a, b, c, α, $\beta > 0$.

Suppose the trapezoidal box $\binom{AC^*BD^*}{B^*DA^*C}$ has four concurrent diagonals. Then $\overrightarrow{AA^*} = \langle \alpha + a, \beta + b, -c\rangle$ so that the vector equation of $\overline{AA^*}$ is

$$\langle x, y, z\rangle = \langle -a + u(\alpha + a), -b + u(\beta + b), c - cu\rangle, \quad u \in \mathbb{R}.$$

Similarly, the vector equation of $\overline{BB^*}$ is

$$\langle x, y, z\rangle = \langle a - v(\alpha + a), b - v(\beta + b), c - cv\rangle, \quad v \in \mathbb{R}.$$

Since $\overline{AA^*}$ and $\overline{BB^*}$ intersect, we must have

$$\langle -a+u(\alpha+a), -b+u(\beta+b), c-cu\rangle = \langle a-v(\alpha+a), b-v(\beta+b), c-cv\rangle$$

for some u, $v \in \mathbb{R}$. From $c-cu=c-cv$, we must have $u=v$.

The equation $-a+u(\alpha+a)=a-u(\alpha+a)$ gives us $u=\frac{a}{\alpha+a}$.

The equation $-b+u(\beta+b)=b-u(\beta+b)$ gives us $u=\frac{b}{\beta+b}$.

Hence, $\frac{a}{\alpha+a}=\frac{b}{\beta+b}$, which implies that $\frac{\alpha}{a}=\frac{\beta}{b}$. Since $\frac{|A^*D|}{|AC^*|}=\frac{\alpha}{a}$ and $\frac{|DA^*|}{|C^*B|}=\frac{\beta}{b}$, we have $\frac{|B^*D|}{|AC^*|}=\frac{|DA^*|}{|C^*B|}$.

On the other hand, suppose $\frac{|B^*D|}{|AC^*|}=\frac{|DA^*|}{|C^*B|}$. Since $\frac{\alpha}{a}=\frac{|B^*D|}{|AC^*|}=\frac{|DA^*|}{|C^*B|}=\frac{\beta}{b}$, let $\frac{\alpha}{a}=\frac{\beta}{b}=k$. Then $\alpha=ka$ and $\beta=kb$. The vector equation of $\overline{AA^*}$ is

$$\langle x,y,z\rangle = \langle -a+ua(k+1), -b+ub(k+1), c-cu\rangle, \quad u\in\mathbb{R}.$$

Then vector equation of $\overline{BB^*}$ is

$$\langle x,y,z\rangle = \langle a-va(k+1), b-vb(k+1), c-cv\rangle, \quad v\in\mathbb{R}.$$

Let $u=\frac{1}{k+1}=v$. Then we see that the point $P=(0,0,\frac{ck}{k+1})$ is a point on $[AA^*]$ and $[BB^*]$.

The vector equation of $\overline{CC^*}$ is

$$\langle x,y,z\rangle = \langle -ka-sa(k+1), kb+ub(k+1), cs\rangle, \quad s\in\mathbb{R}.$$

The vector equation of $\overline{DD^*}$ is

$$\langle x,y,z\rangle = \langle ka+ta(k+1), -kb-tb(k+1), ct\rangle, \quad t\in\mathbb{R}.$$

Let $s=-\frac{k}{k+1}=t$. Then we see that the point $P=(0,0,\frac{ck}{k+1})$ is also a point on $[CC^*]$ and $[DD^*]$. Therefore, the diagonals $[AA^*]$, $[BB^*]$, $[CC^*]$, and $[DD^*]$ concur at P. □

Lemma 9 is a corollary of Lemma 8.

Lemma 9. *Suppose* $\binom{AC^*BD^*}{B^*DA^*C}$ *is a trapezoidal box such that* $\frac{|B^*D|}{|AC^*|}=\frac{|DA^*|}{|C^*B|}$ *so that its four diagonals concur, say at* P. *Let* $\frac{|B^*D|}{|AC^*|}=\frac{|DA^*|}{|C^*B|}=k$ *for some* $k>0$. *Then* $\frac{|CD|}{|AB|}=k$. *Moreover, if we let* E *and* J *be the midpoints of the edges* $[AB]$ *and* $[CD]$, *respectively, then* $P\in(EJ)$ *and* $\frac{|PJ|}{|EP|}=k$.

Proof. Let $A=(-a,-b,c)$, $C^*=(a,-b,c)$, $B=(a,b,c)$, $D^*=(-a,b,c)$, and $B^*=(-\alpha,-\beta,0)$, $D=(\alpha,-\beta,0)$, $A^*=(\alpha,\beta,0)$, $C=(-\alpha,\beta,0)$ for some a, b, c, α, $\beta>0$. Since $\frac{|B^*D|}{|AC^*|}=\frac{|DA^*|}{|C^*B|}$, we have $\frac{\alpha}{a}=\frac{|B^*D|}{|AC^*|}=k=\frac{|DA^*|}{|C^*B|}=\frac{\beta}{b}$ so that $\alpha=ka$, and $\beta=kb$. Now $\overrightarrow{AB}=\langle 2a,2b,0\rangle$ so that $|AB|=2\sqrt{a^2+b^2}$. Hence, $\frac{|CD|}{|AB|}=k$.

Let E and J be the midpoints of the edges $[AB]$ and $[CD]$, respectively. We have $E=(0,0,c)$ and $J=(0,0,0)$. From the second half of the proof of Lemma 8, we have $P=(0,0,\frac{ck}{k+1})$ so that $|EP|=c-\frac{ck}{k+1}=\frac{c}{k+1}$, and $|PJ|=\frac{ck}{k+1}$. Therefore, we have $P\in(EJ)$ and $\frac{|PJ|}{|EP|}=k$. □

Lemma 10 (Crelle's Theorem)**.** *If* $\nabla ABCD$ *is a tetrahedron, then there is a triangle having edges of lengths* $|AB|\cdot|CD|$, $|AC|\cdot|BD|$, *and* $|AD|\cdot|BC|$.

Proof. See [3, 7, 8]. □

Lemma 11. *Let $\nabla ABCD$ be a reversible tetrahedron such that $\frac{|CD|}{|AB|} = k \geq 1$. Then the tetrahedron $\nabla ABCD$ can be embedded in $\mathbb{R}^3$ so that $A = (-a, -b, c)$, $B = (a, b, c)$, $C = (-ka, kb, 0)$, and $D = (ka, -kb, 0)$, for some a, b, $c > 0$.*

Proof. Since $\nabla ABCD$ is reversible, we let $|AC| = |BD| = x$, $|AD| = |BC| = y$, and $\frac{1}{k}|CD| = |AB| = 2z$ for some x, y, $z > 0$. Then

$$|AB| \cdot |CD| = 4kz^2, \quad |AC| \cdot |BD| = x^2, \quad \text{and} \quad |AD| \cdot |BC| = y^2.$$

Let $a = \frac{1}{\sqrt{2}}\Big(\frac{y^2-x^2}{4k} + z^2\Big)^{1/2}$, $b = \frac{1}{\sqrt{2}}\Big(\frac{x^2-y^2}{4k} + z^2\Big)^{1/2}$, and $c = \frac{1}{\sqrt{2}}\{(x^2+y^2) - (2kz^2 + 2z^2)\}^{1/2}$.

Since there is a triangle of sides $|AB|\cdot|CD|$, $|AC|\cdot|BD|$ and $|AD|\cdot|BC|$ by Lemma 10, we must have the triangle inequality $|AC|\cdot|BD|+|AB|\cdot|CD| > |AD|\cdot|BC|$ so that $x^2+4kz^2 > y^2$, or $\frac{x^2-y^2}{4k} + z^2 > 0$. Since $\frac{x^2-y^2}{4k} + z^2 > 0$, the definition of the number b makes sense.

Similarly, since $|AD|\cdot|BC|+|AB|\cdot|CD| > |AC|\cdot|BD|$ by Lemma 10 so that $\frac{y^2-x^2}{4k}+z^2 > 0$, the definition of the number a makes sense.

Again, we have $|AC| \cdot |BD| + |AD| \cdot |BC| > |AB| \cdot |CD|$ by Lemma 10. So we have $x^2 + y^2 > 4kz^2$. Note that $k \geq 1$ implies that $4kz^2 \geq 2kz^2 + 2z^2 = 2z^2(k+1)$ so that $(x^2 + y^2) - 2z^2(k+1) > 0$. Hence, the definition of the number c makes sense.

Now,

$$|AB|^2 = 4(a^2+b^2) = 2\Big(\frac{y^2-x^2}{4k} + z^2\Big) + 2\Big(\frac{x^2-y^2}{4k} + z^2\Big) = 4z^2$$

so that $|AB| = 2z$. Next,

$$\begin{aligned}
|AC|^2 &= a^2(k-1)^2 + b^2(k+1)^2 + c^2 \\
&= \frac{1}{2}\Big(\frac{(y^2 - x^2(k-1)^2)}{4k} + z^2(k-1)^2\Big) + \frac{1}{2}\Big(\frac{(x^2-y^2)(k+1)^2}{4k} + z^2(k+1)^2\Big) + c^2 \\
&= \frac{-2(y^2-x^2)k + 2(x^2-y^2)k}{2\cdot 4k} + \frac{1}{2}z^2(k-1)^2 + \frac{1}{2}z^2(k+1)^2 + c^2 \\
&= \frac{(x^2-y^2)}{2} + z^2(k^2+1) + \frac{1}{2}\{(x^2+y^2 - 2z^2(k^2+1)\} = x^2.
\end{aligned}$$

And

$$\begin{aligned}
|AD|^2 &= a^2(k+1)^2 + b^2(k-1)^2 + c^2 \\
&= \frac{1}{2}\Big(\frac{(y^2-x^2)(k+1)^2}{4k} + z^2(k+1)^2\Big) + \frac{1}{2}\Big(\frac{(x^2-y^2)(k-1)^2}{4k} + z^2(k-1)^2\Big) + c^2 \\
&= \frac{2(y^2-x^2)k - 2(x^2-y^2)k}{2\cdot 4k} + z^2(k+1)^2 + z^2(k-1)^2 + c^2 \\
&= \frac{2(y^2-x^2)k - 2(x^2-y^2)k}{2\cdot 4k} + \frac{1}{2}z^2(k+1)^2 + \frac{1}{2}z^2(k-1)^2 + c^2 \\
&= \frac{(-x^2+y^2)}{2} + z^2(k^2+1) + \frac{1}{2}\{x^2+y^2 - 2z^2(k^2+1)\} = y^2.
\end{aligned}$$

This proves Lemma 11. □

Theorem 5. *Let $\nabla ABCD$ be a reversible tetrahedron such that $\frac{|CD|}{|AB|} = k \geq 1$. Let E and J be the midpoints of the edges $[AB]$ and $[CD]$, respectively. Let $P \in (EJ)$ such that $\frac{|PJ|}{|EP|} = k$. Let $\nabla A^*B^*C^*D^*$ be the tetrahedron that tangles $\nabla ABCD$ with the concurrent diagonals at P. Then $\nabla D^*C^*B^*A^* \cong \nabla ABCD$, and these tetrahedra can be circumscribed by the same sphere. Moreover, $\binom{AC^*BD^*}{B^*DA^*C}$ is a trapezoidal box such that $\frac{|B^*D|}{|AC^*|} = \frac{|DA^*|}{|C^*B|} = k$.*

Proof. Since $P \in (EJ)$, the point P is a deep interior point of $\nabla ABCD$. Hence, there exists a unique tetrahedron $\nabla A^*B^*C^*D^*$ that tangles $\nabla ABCD$ with the concurrent diagonals at P by Theorem 2. Since $\nabla ABCD$ is a reversible tetrahedron such that $\frac{|CD|}{|AB|} = k$, we can let $A = (-a, -b, c)$, $B = (a, b, c)$, $C = (-ka, -kb, 0)$, and $D = (ka, -kb, 0)$, for some a, b, $c > 0$, by Lemma 11. Let $A' = (ka, kb, 0)$, $B' = (-ka, -kb, 0)$, $C' = (a, -b, c)$, and $D' = (-a, b, c)$. Then it is not difficult to check that $\binom{AC'BD'}{B'DA'C}$ is a trapezoidal box such that $\frac{|B'D|}{|AC'|} = \frac{|DA'|}{|C'B|} = k$. This implies that the trapezoidal box $\binom{AC'BD'}{B'DA'C}$ has four concurring diagonals by Lemma 8, say at P'. By Lemma 7, $\nabla D'C'B'A' \cong \nabla ABCD$ and these tetrahedra can be circumscribed by the same sphere. By Lemma 9, $P' \in (EJ)$ such that $\frac{|P'J|}{|EP'|} = k$. However, since $P \in (EJ)$ such that $\frac{|PJ|}{|EP|} = k$, we must have $P' = P$. Also, $\nabla A^*B^*C^*D^*$ is the unique tetrahedron that tangles $\nabla ABCD$ with the concurrent diagonals at P. Therefore, $\nabla A^*B^*C^*D^* = \nabla A'B'C'D'$ by Theorem 2. This prove this theorem. $\square$

Remark 5. Suppose $\nabla ABCD$ is an isosceles tetrahedron. Then $\nabla ABCD$ is a reversible tetrahedron such that $\frac{|CD|}{|AB|} = 1$. Let E and J be the midpoints of the edges $[AB]$ and $[CD]$, respectively, and let $P \in [EJ]$ such that $\frac{|PJ|}{|EP|} = 1$. Then P is the centroid of $\nabla ABCD$. Then the trapezoidal box $\binom{AC^*BD^*}{B^*DA^*C}$ guaranteed by Theorem 5 is a parallelepiped by Theorem 3. That is, $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ are twins. Since the diagonals of the faces of this parallelepiped are pairwise the same length, $\binom{AC^*BD^*}{B^*DA^*C}$ is a rectangular box. Hence, $\nabla ABCD \cong \nabla D^*C^*B^*A^* \cong \nabla A^*B^*C^*D^*$.

On the other hand, suppose $\nabla ABCD$ is a reversible tetrahedron such that $\frac{|CD|}{|AB|} = k > 1$. Then $\nabla ABCD$ is not isosceles. Let $\nabla A^*B^*C^*D^*$ be the tetrahedron that tangles $\nabla ABCD$ with concurrent diagonals such that $\nabla D^*C^*B^*A^* \cong \nabla ABCD$ in Theorem 5. Figure 5 is showing this situation. Since $|A^*B^*| = |CD| \neq |AB|$, $\nabla A^*B^*C^*D^*$ is *not* the twin of $\nabla ABCD$.

Remark 6. The condition that (the tetrahedron $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ with the concurrent diagonals at P) in Theorem 5 is important since without it, there is the other possibility that a tetrahedron $\nabla A^*B^*C^*D^*$ tangles $\nabla ABCD$ and $\nabla D^*C^*B^*A^* \cong \nabla ABCD$. For example, let $\nabla A^*B^*C^*D^*$ and $\nabla ABCD$ be the tetrahedra circumscribed by the box in Figure 4. Then $\nabla ABCD$ is an isosceles tetrahedron, and $\nabla ABCD \cong \nabla A^*B^*C^*D^* \cong \nabla D^*C^*B^*A^*$.

Acknowledgement

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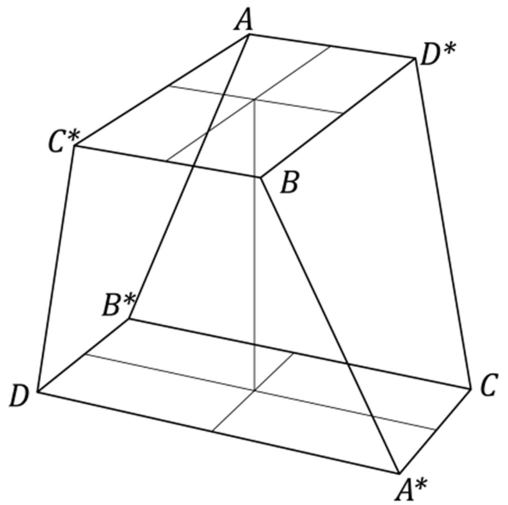

Figure 4

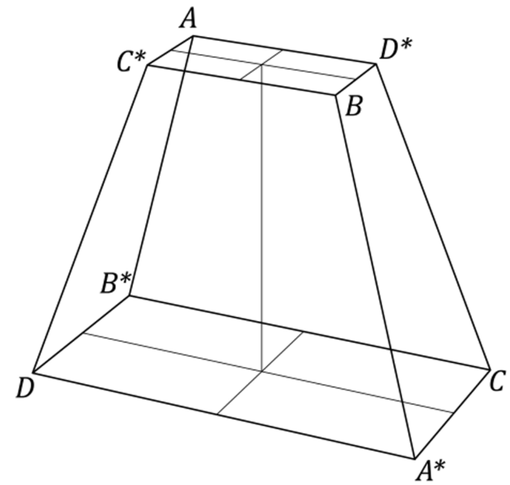

Figure 5

These two figures are examples of trapezoidal boxes. Figure 4 is drawn so that $\frac{|B^*D|}{|AC^*|} = 2$ and $\frac{|A^*D|}{|BC^*|} = \frac{1}{2}$ so that the diagonals of this trapezoidal box do not intersect by Lemma 8. Figure 5 is drawn so that $\frac{|B^*D|}{|AC^*|} = 2 = \frac{|A^*D|}{|BC^*|}$ so that the diagonals of this trapezoidal box intersect by Lemma 8.

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