Journal for Geometry and Graphics
Volume 28 (2024), No. 2, 187–194

Position Vectors of Developable Surfaces According to the Bishop Frame in E^3

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Abstract. In this paper, we prove that ruled surfaces generated by the Bishop frame are naturally developable in Euclidean 3-space. Moreover, the position vectors of such surface are investigated for the first time.

Key Words: ruled surface, developable surface, position vector, Bishop frame, Euclidean 3-space

MSC 2020: 53M04 (primary), 53A05

1 Introduction

In the classical differential geometry [4, 12, 13], a surface is said to be "ruled" if it is generated by moving a straight line continuously along a curve in Euclidean space. These lines are called rulings, and each curve that intersects all the rulings is called a base curve. Ruled surfaces are a prominent topic of surface theory, they are of great interest to many applications and have contributed in several areas, such as mathematical physics, kinematics and Computer Aided Geometric Design (CAGD).

Ruled surfaces which can be transformed into a plane without any deformation and distortion, are called developable surfaces. They form a relatively small subset that contains cylinders, cones, and the tangent surfaces [1, 3, 5, 8, 9]. The notion of developability is one of the most important properties of ruled surfaces.

Our point of interest is study of ruled surfaces according to different frames. In [10], we have defined special couples of ruled surfaces that we called partner ruled surfaces and then studied their simultaneous developability in E^3. One of the moving frames that interests us also is the Bishop frame or parallel transport frame. It is an alternative approach of defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each field of the frame. The parallel transport frame is based on the observation that, when

ISSN 1433-8157/

given the unit vector $\overrightarrow{T}$ for a given curve, we may choose any convenient arbitrary basis $\{\overrightarrow{M}_1, \overrightarrow{M}_2\}$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $\overrightarrow{T}$ at each point. If the derivatives of $\{\overrightarrow{M}_1, \overrightarrow{M}_2\}$ depend only on $\overrightarrow{T}$ and not each other we can make $\overrightarrow{M}_1$ and $\overrightarrow{M}_2$ vary smoothly throughout along the path regardless of the curvature [2].

In this paper, we consider the position vector of a unit speed curve according to its Bishop frame in Euclidean 3-space and then construct the special family of three ruled surfaces generated by Bishop frame vectors, we present our main theorem that gives the position vectors of the well-known ruled surfaces and proves that these are all naturally developable. Finally, we give an example to show that kind of surfaces in $\mathbb{R}^3$.

2 Preliminaries

In the Euclidean 3-space E^3, we consider the usual metric given by

$$\langle , \rangle = \mathrm{d}x_1 + \mathrm{d}x_2 + \mathrm{d}x_3,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3.

Let $\Phi\colon (s, v) \in I \times \mathbb{R} \mapsto \alpha(s) + v\overrightarrow{X}(s)$, be a ruled surface in E^3, where I is an open interval of $\mathbb{R}$.

Let denote by $\overrightarrow{m} = \overrightarrow{m}(s, v)$ the unit normal of the ruled surface Φ at a regular point, so we have

$$\overrightarrow{m} = \frac{\Phi_s \wedge \Phi_v}{\|\Phi_s \wedge \Phi_v\|} = \frac{(\alpha' + v\overrightarrow{X}') \wedge \overrightarrow{X}}{\|(\alpha' + v\overrightarrow{X}') \wedge \overrightarrow{X}\|},$$

where $\Phi_s = \frac{\partial \Phi}{\partial s}$ and $\Phi_v = \frac{\partial \Phi}{\partial v}$.

The first I and the second II fundamental forms of ruled surface Φ at a regular point, are defined respectively by

$$I(\Phi_s\,\mathrm{d}s + \Phi_v\,\mathrm{d}v) = E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}v + G\,\mathrm{d}v^2,$$
$$II(\Phi_s\,\mathrm{d}s + \Phi_v\,\mathrm{d}v) = e\,\mathrm{d}s^2 + 2f\,\mathrm{d}s\,\mathrm{d}v + g\,\mathrm{d}v^2,$$

where

$$E = \|\Phi_s\|^2, \qquad F = \langle \Phi_s, \Phi_v \rangle, \quad G = \|\Phi_v\|^2,$$
$$e = \langle \Phi_{ss}, \overrightarrow{m} \rangle, \quad f = \langle \Phi_{vs}, \overrightarrow{m} \rangle, \quad g = \langle \Phi_{vv}, \overrightarrow{m} \rangle = 0.$$

The Gaussian curvature K of the ruled surface Φ at a regular point is defined by

$$K = -\frac{f^2}{EG - F^2}.$$

Theorem 1 ([11]). *A developable surface is a ruled surface with vanishing Gaussian curvature.*

Definition 1. A tangent surface is a ruled surface whose ruling direction is the unit tangent vector of its base curve.

Theorem 2. *A tangent surface is developable.*

When $\alpha = \alpha(s)$ is a unit speed curve with non-vanishing second derivative $\alpha''(s) \neq 0$, there exists the Frenet-Serret frame denoted by $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$, where $\vec{T}(s) = \alpha'(s)$ is the unit tangent, $\vec{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ is the unit principal normal and $\vec{B}(s) = \vec{T}(s) \wedge \vec{N}(s)$ is the binormal vector. The derivative formulas of Frenet-Serret frame are expressed as:

$$\begin{pmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{pmatrix},$$

where κ is the curvature and τ is the torsion of the curve α [6, 7].

In the case when a curve α has vanishing second derivative, we can still consider "the Bishop frame" or "the parallel transport frame", as an alternative approach to define a moving frame. One can express parallel transport of an orthonormal frame along a curve simply by performing the parallel transport of each field of the frame. Therefore, in such a frame, the tangent vector T and any convenient fields $\{\vec{M}_1, \vec{M}_2\}$ that make an orthonormal basis can be used. Therefore, the derivative formulas of Bishop frame $\{\vec{T}, \vec{M}_1, \vec{M}_2\}$, are expressed as:

$$\begin{pmatrix} \vec{T}'(s) \\ \vec{M}_1'(s) \\ \vec{M}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{M}_1(s) \\ \vec{M}_2(s) \end{pmatrix}, \tag{1}$$

where $k_1(s)$ and $k_2(s)$ are called the first and the second Bishop curvatures, respectively [2].

The relation between the Frenet frame, the Bishop frame is given as follows:

$$\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{M}_1 \\ \vec{M}_2 \end{pmatrix},$$

or

$$\begin{pmatrix} \vec{T} \\ \vec{M}_1 \\ \vec{M}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}, \tag{2}$$

where θ, k_1, k_2 are such that $\theta = \arctan\left(\frac{k_2}{k_1}\right)$ can be determined from $\theta = \int \tau(s)\,\mathrm{d}s$, $k_1 = \kappa\cos\theta$, $k_2 = \kappa\sin\theta$ or, vice versa, $\kappa = \sqrt{k_1^2 + k_2^2}$, $\tau = \frac{\mathrm{d}\theta}{\mathrm{d}s}$.

3 Position vectors of developable ruled surfaces according to the Bishop frame in $\mathbf{E}^3$

In this main section, we investigate new theorem which states that a family of three ruled surfaces generated by Bishop frame are developable. Moreover, the theorem gives the position vector of such a special surfaces.

Theorem 3. *Ruled surfaces generated by Bishop frame are developable and their position vectors are given by*

$$\begin{cases} {}^0\Phi(s,v) = [\alpha_0(s) + v]\vec{T}(s) - [\int \alpha_0(s)k_1(s)\,\mathrm{d}s + c_0]\vec{M}_1(s) - [\int \alpha_0(s)k_2(s)\,\mathrm{d}s + d_0]\vec{M}_2(s), \\ {}^1\Phi(s,v) = \alpha_0(s)\vec{T}(s) - [\int \alpha_0(s)k_1(s)\,\mathrm{d}s + c_1 + v]\vec{M}_1(s) - [\int \alpha_0(s)k_2(s)\,\mathrm{d}s + d_1]\vec{M}_2(s), \\ {}^2\Phi(s,v) = \alpha_0(s)\vec{T}(s) - [\int \alpha_0(s)k_1(s)\,\mathrm{d}s + c_2]\vec{M}_1(s) - [\int \alpha_0(s)k_2(s)\,\mathrm{d}s + d_2 + v]\vec{M}_2(s), \end{cases}$$

where α_0 is a C^2-differentiable function of $s \in I \subset \mathbb{R}$ and c_0, c_1, c_2, d_0, d_1 and d_2 are constants.

Proof. Let $\alpha\colon s \in I \subset \mathbb{R} \mapsto \alpha(s)$ be a C^2-differentiable unit speed curve whose Bishop frame is given by $\{\overrightarrow{T}(s), \overrightarrow{M}_1(s), \overrightarrow{M}_2(s)\}$. Then the position vector of the curve $\alpha(s)$ according to its Bishop frame is expressed as follows

$$\alpha(s) = \alpha_0(s)\overrightarrow{T}(s) + \alpha_1(s)\overrightarrow{M}_1(s) + \alpha_2(s)\overrightarrow{M}_2(s), \tag{3}$$

where α_0, α_1, α_2 are the differentiable functions of $s \in I$, defined respectively by

$$\alpha_0(s) = \langle \alpha(s), \overrightarrow{T}(s)\rangle, \quad \alpha_1(s) = \langle \alpha(s), \overrightarrow{M}_1(s)\rangle, \quad \alpha_2(s) = \langle \alpha(s), \overrightarrow{M}_2(s)\rangle.$$

Differentiating (3) with respect to s and using Bishop frame formulas (1), we get

$$\begin{aligned}\alpha'(s) = \alpha_0'(s)\overrightarrow{T}(s) + \alpha_1'(s)\overrightarrow{M}_1(s) + \alpha_2'(s)\overrightarrow{M}_2(s) + \alpha_0(s)[k_1(s)\overrightarrow{M}_1(s) + k_2(s)\overrightarrow{M}_2(s)] \\ - \alpha_1(s)k_1(s)\overrightarrow{T}(s) - \alpha_2(s)k_2(s)\overrightarrow{T}(s).\end{aligned}$$

By replacing $\alpha'(s)$ with $\overrightarrow{T}(s)$, we get

$$\begin{aligned}\overrightarrow{T}(s) = [\alpha_0'(s) - \alpha_1(s)k_1(s) - \alpha_2(s)k_2(s)]\overrightarrow{T}(s) + [\alpha_1'(s) + \alpha_0(s)k_1(s)]\overrightarrow{M}_1(s) \\ + [\alpha_2'(s) + \alpha_0(s)k_2(s)]\overrightarrow{M}_2(s),\end{aligned}$$

by identification, we get

$$\begin{cases} \alpha_0'(s) - \alpha_1(s)k_1(s) - \alpha_2(s)k_2(s) = 1, \\ \alpha_1'(s) + \alpha_0(s)k_1(s) = 0, \\ \alpha_2'(s) + \alpha_0(s)k_2(s) = 0. \end{cases} \tag{4}$$

The family of three ruled surfaces generated by Bishop frame of the curve $\alpha(s)$ are given by:

$$\begin{cases} {}^0\Phi(s,v) = (\alpha_0(s)\overrightarrow{T}(s) + \alpha_1(s)\overrightarrow{M}_1(s) + \alpha_2(s)\overrightarrow{M}_2(s)) + v\overrightarrow{T}(s), \\ {}^1\Phi(s,v) = (\alpha_0(s)\overrightarrow{T}(s) + \alpha_1(s)\overrightarrow{M}_1(s) + \alpha_2(s)\overrightarrow{M}_2(s)) + v\overrightarrow{M}_1(s), \\ {}^2\Phi(s,v) = (\alpha_0(s)\overrightarrow{T}(s) + \alpha_1(s)\overrightarrow{M}_1(s) + \alpha_2(s)\overrightarrow{M}_2(s)) + v\overrightarrow{M}_2(s). \end{cases} \tag{5}$$

It is clear that the first ruled surface ${}^0\Phi(s,v)$ is developable because it is a tangent surface. Then, by using the properties (4), we get the position vector of such a developable ruled surface as follows

$${}^0\Phi(s,v) = (\alpha_0(s) + v)\overrightarrow{T}(s) - \Big(\int \alpha_0(s)k_1(s)\,\mathrm{d}s + c_0\Big)\overrightarrow{M}_1(s) - \Big(\int \alpha_0(s)k_2(s)\,\mathrm{d}s + d_0\Big)\overrightarrow{M}_2(s).$$

Now, let us search under which conditions our second ruled surface ${}^1\Phi(s,v)$ could be developable.

Differentiating the second line of (5) with respect to s and v, respectively, and using the Bishop frame formulas (1), we get

$$\begin{cases} {}^1\Phi_s = (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)\overrightarrow{T} + (\alpha_1' + \alpha_0 k_1)\overrightarrow{M}_1 + (\alpha_2' + \alpha_0 k_2)\overrightarrow{M}_2, \\ {}^1\Phi_v = \overrightarrow{M}_1. \end{cases} \tag{6}$$

By taking the cross product of both vectors ${}^1\Phi_s$ and ${}^1\Phi_v$, we get the normal vector on the ruled surface ${}^1\Phi(s,v)$:

$$ {}^1\Phi_s \times {}^1\Phi_v = -(\alpha_2' + \alpha_0 k_2)\overrightarrow{T} + (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)\overrightarrow{M}_2, $$

so under regularity condition, the unit normal vector takes the following form:

$$ \frac{{}^1\Phi_s \times {}^1\Phi_v}{\|{}^1\Phi_s \times {}^1\Phi_v\|} = \frac{-(\alpha_2' + \alpha_0 k_2)\overrightarrow{T} + (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)\overrightarrow{M}_2}{\sqrt{(\alpha_2' + \alpha_0 k_2)^2 + (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)^2}}. \tag{7} $$

From (6) and (7), we get the components of the first fundamental form of the ruled surface ${}^1\Phi(s,v)$, at regular points, as follows

$$ \begin{cases} {}^1E = (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)^2 + (\alpha_1' + \alpha_0 k_1)^2 + (\alpha_2' + \alpha_0 k_2)^2 \\ {}^1F = \alpha_1' + \alpha_0 k_1 \\ {}^1G = 1. \end{cases} \tag{8} $$

On the other hand, differentiating ${}^1\Phi_s$ and ${}^1\Phi_v$ with respect to s and v, respectively and using the Bishop formulas (1), we get

$$ \begin{cases} {}^1\Phi_{ss} = [\alpha_0'' - \alpha_1 k_1' - \alpha_2 k_2' - 2(\alpha_1' k_1 + \alpha_2' k_2) - \alpha_0(k_1^2 + k_2^2) - v k_1']\overrightarrow{T} \\ \qquad\qquad + [\alpha_1'' + 2\alpha_0' k_1 + \alpha_0 k_1' - k_1(\alpha_1 k_1 + \alpha_2 k_2) - v k_1^2]\overrightarrow{M}_1 \\ \qquad\qquad + \alpha_2'' + 2\alpha_0' k_2 + \alpha_0 k_2' - k_2(\alpha_1 k_1 + \alpha_2 k_2) - v k_1 k_2]\overrightarrow{M}_2, \\ {}^1\Phi_{vs} = -k_1\overrightarrow{T}, \\ {}^1\Phi_{vv} = 0. \end{cases} \tag{9} $$

Hence, from (7) and (9), we get the components of the second fundamental form of ${}^1\Phi(s,v)$ at regular points:

$$ \begin{cases} {}^1e = \dfrac{-(\alpha_2' + \alpha_0 k_2)[\alpha_0'' - \alpha_1 k_1' - \alpha_2 k_2' - 2(\alpha_1' k_1 + \alpha_2' k_2) - \alpha_0(k_1^2 + k_2^2) - v k_1']}{\sqrt{(\alpha_2' + \alpha_0 k_2)^2 + (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)^2}} \\ \qquad + \dfrac{(\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)[\alpha_2'' + 2\alpha_0' k_2 + \alpha_0 k_2' - k_2(\alpha_1 k_1 - \alpha_2 k_2) - v k_1 k_2]}{\sqrt{(\alpha_2' + \alpha_0 k_2)^2 + (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)^2}}, \\ {}^1f = \dfrac{k_1(\alpha_2' + \alpha_0 k_2)}{\sqrt{(\alpha_2' + \alpha_0 k_2)^2 + (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)^2}}, \\ {}^1g = 0. \end{cases} \tag{10} $$

From (8) and (10), we get the Gaussian curvature of the ruled surface ${}^1\Phi(s,v)$ at regular points as follows

$$ {}^1K = -\Big[\frac{k_1(\alpha_2' + \alpha_0 k_2)}{(\alpha_2' + \alpha_0 k_2)^2 + (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_1)^2}\Big]^2. \tag{11} $$

We deduce that the second ruled surface ${}^1\Phi(s,v)$ is developable if and only if $k_1 = 0$ or $\alpha_2' + \alpha_0 k_2 = 0$. The second equation is satisfied because of the third property of (4).

Consequently, from the properties (4), we conclude that the ruled surface ${}^{1}\Phi(s,v)$ is developable and its position vector is expressed as follows

$$ {}^{1}\Phi(s,v) = \alpha_0(s)\vec{T}(s) - \Big(\int \alpha_0(s)k_1(s)\,\mathrm{d}s + c_1 + v\Big)\vec{M}_1(s) - \Big(\int \alpha_0(s)k_2(s)\,\mathrm{d}s + d_1\Big)\vec{M}_2(s). $$

Now, let us consider the third ruled surface ${}^{2}\Phi(s,v)$, but above all, we can remark that this one plays a similar role as the second one, so by replacing the index 1 by the index 2 in the expression (11), we get the Gaussian curvature of ruled surface ${}^{2}\Phi(s,v)$ as follows

$$ {}^{2}K = -\Big[\frac{k_2(\alpha_1' + \alpha_0 k_1)}{(\alpha_1' + \alpha_0 k_1)^2 + (\alpha_0' - \alpha_1 k_1 - \alpha_2 k_2 - v k_2)^2}\Big]^2. $$

Similarly, we deduce that ruled surface ${}^{2}\Phi(s,v)$ is developable if and only if $k_2 = 0$ or $\alpha_1' + \alpha_0 k_1 = 0$, which is already satisfied due to the second property of (4).

Consequently, from the properties (4), we conclude that the ruled surface ${}^{2}\Phi(s,v)$ is also developable and its position vector is expressed as follows

$$ {}^{2}\Phi(s,v) = \alpha_0(s)\vec{T}(s) - \Big(\int \alpha_0(s)k_1(s)\,\mathrm{d}s + c_2\Big)\vec{M}_1(s) - \Big(\int \alpha_0(s)k_2(s)\,\mathrm{d}s + d_2 + v\Big)\vec{M}_2(s). \quad \square $$

Example 1. Let consider the unit speed curve $\alpha(s) = \frac{1}{\sqrt{2}}\Big(\frac{\sin(\sqrt{2}s)}{\sqrt{2}}, -\frac{\cos(\sqrt{2}s)}{\sqrt{2}}, s\Big)$. The Frenet frame of this curve is

$$ \vec{T}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos(\sqrt{2}s)\\ \sin(\sqrt{2}s)\\ 1\end{pmatrix}; \quad \vec{N}(s) = \begin{pmatrix}-\sin(\sqrt{2}s)\\ \cos(\sqrt{2}s)\\ 0\end{pmatrix}; \quad \vec{B}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix}-\cos(\sqrt{2}s)\\ -\sin(\sqrt{2}s)\\ 1\end{pmatrix}, $$
$$ \kappa(s) = \tau(s) = 1. $$

Then, from (2), the Bishop frame is given by

$$ \vec{T}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos(\sqrt{2}s)\\ \sin(\sqrt{2}s)\\ 1\end{pmatrix}, \quad \vec{M}_1(s) = \begin{pmatrix}-\cos(s)\sin(\sqrt{2}s) + \frac{1}{\sqrt{2}}\sin(s)\cos(\sqrt{2}s)\\ \cos(s)\cos(\sqrt{2}s) + \frac{1}{\sqrt{2}}\sin(s)\sin(\sqrt{2}s)\\ -\frac{1}{\sqrt{2}}\sin(s)\end{pmatrix}, $$
$$ \vec{M}_2(s) = \begin{pmatrix}-\sin(s)\sin(\sqrt{2}s) - \frac{1}{\sqrt{2}}\cos(s)\cos(\sqrt{2}s)\\ \sin(s)\cos(\sqrt{2}s) - \frac{1}{\sqrt{2}}\cos(s)\sin(\sqrt{2}s)\\ \frac{1}{\sqrt{2}}\cos(s)\end{pmatrix}, $$

$\theta(s) = \int \tau(s)\,\mathrm{d}s = s + c$ where c is a constant taken here as $c = 0$, $k_1 = \cos(s)$, $k_2 = \sin(s)$.

In this case, it is simple to find that

$$ \alpha_0 = \frac{s}{2}; \quad \alpha_1 = -\frac{1}{2}(\cos(s) + s\sin(s)); \quad \alpha_2 = \frac{1}{2}(-\sin(s) + s\cos(s)). $$

One can discover the shape of developable surfaces according to Bishop frame of such a curve: Figure 1 is the $\vec{T}-$developable surface, Figure 2 is the $\vec{M_1}-$ developable surface and Figure 3 is the $\vec{M_2}-$developable surface.

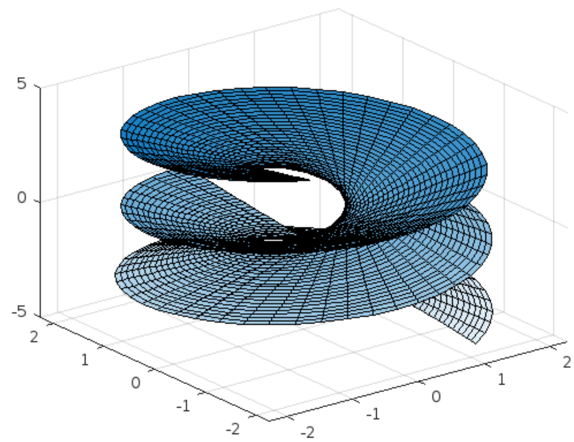

Figure 1: $\overrightarrow{T}$-developable surface

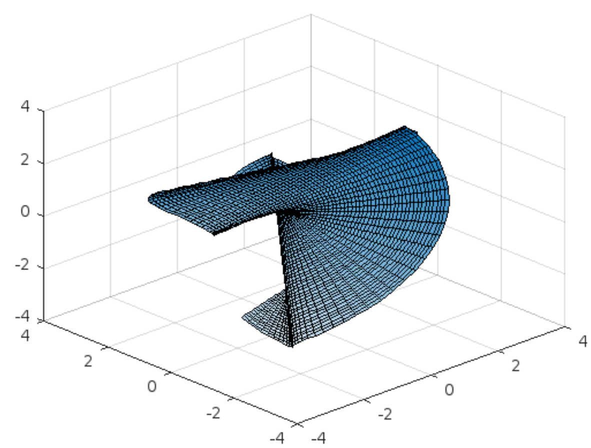

Figure 2: $\overrightarrow{M}_1$-developable surface

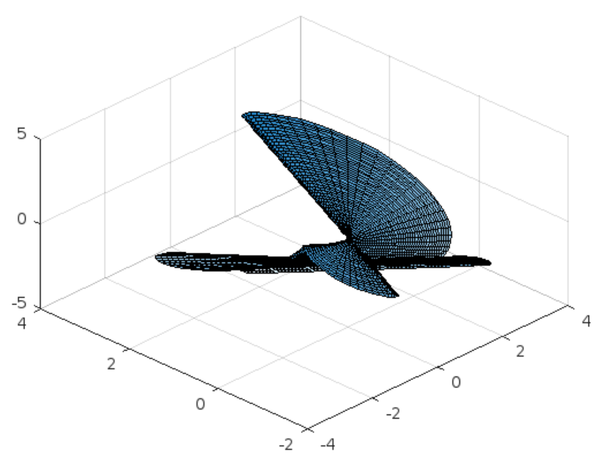

Figure 3: $\overrightarrow{M}_2$-developable surface

Conclusion

Ruled surfaces generated by the Bishop frame in Euclidean 3-space are developable. Furthermore, their position vectors are described by the means of Bishop frame and Bishop curvatures of their base curve.

Our results could be applied in various fields of mathematics, in mechanics, robotics, architecture and similar.

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Received December 15, 2024; final form February 5, 2025.