Seminar Sophus Lie
1 (1991) 243–245

On complex Lie algebras with a simple real form

S. S. L. Darmstadt

The following observations, motivated by a question of Lawson's [1], does not seem to be readily accessible in the literature.

Let $\mathfrak{a}$ denote a complex Lie algebra and $\mathfrak{g}$ a real subalgebra. Then $\mathfrak{g}+i\mathfrak{g}$ and $\mathfrak{g}\cap i\mathfrak{g}$ are complex subalgebras. If $X \in \mathfrak{g}$ then $[X, \mathfrak{g}\cap i\mathfrak{g}] \subseteq [X, \mathfrak{g}] \cap i[X, \mathfrak{g}] \subseteq \mathfrak{g} \cap i\mathfrak{g}$. Then $[iX, \mathfrak{g}\cap i\mathfrak{g}] = i[X, \mathfrak{g}\cap i\mathfrak{g}] \subseteq i(\mathfrak{g}\cap i\mathfrak{g}) = \mathfrak{g}\cap i\mathfrak{g}$. Thus $\mathfrak{g}\cap i\mathfrak{g}$ is an ideal of $\mathfrak{g}+i\mathfrak{g}$.

Remark 1. If $\mathfrak{g}$ is simple, then either $\mathfrak{g} = \mathfrak{g}\cap i\mathfrak{g}$, i.e., $i\mathfrak{g} = \mathfrak{g}$ and $\mathfrak{g}$ is a complex subalgebra, or else $\mathfrak{g}\cap i\mathfrak{g} = \{0\}$, i.e. $\mathfrak{g}+i\mathfrak{g} = \mathfrak{g}\oplus i\mathfrak{g} \cong \mathbb{C}\otimes\mathfrak{g}$ is the complexification of $\mathfrak{g}$ embedded into $\mathfrak{a}$. ■

Lemma 2. *Assume $\mathfrak{g}_1\oplus i\mathfrak{g}_1 \cong \mathbb{C}\otimes\mathfrak{g}_1$ and $\mathfrak{g}_2\oplus i\mathfrak{g}_2 \cong \mathbb{C}\otimes\mathfrak{g}_2$, and suppose that there is a morphism $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ of real Lie algebras. Then $F\colon \mathfrak{g}_1\oplus i\mathfrak{g}_1 \to \mathfrak{g}_2\oplus i\mathfrak{g}_2$, $F(X+iY) = f(X)+if(Y)$ is a morphism of complex algebras.*

Proof. Compute or observe that under the isomorphisms $\mathfrak{g}_j + i\mathfrak{g}_j \cong \mathbb{C}\otimes\mathfrak{g}_j$ the map F corresponds to the map $\mathrm{id}_{\mathbb{C}}\otimes f\colon \mathbb{C}\otimes\mathfrak{g}_1 \to \mathbb{C}\otimes\mathfrak{g}_2$! ■

If $\mathfrak{a}$ is the complexification of a real simple Lie algebra $\mathfrak{g}$ which itself is the underlying real vector space of a complex Lie algebra then $\mathfrak{a} = \mathfrak{g}\oplus i\mathfrak{g}$ has a real involution τ given by $\tau(X+iY) = X - iY$ which is a complex conjugate automorphism of $\mathfrak{a}$, satisfying $\tau(iZ) = -i\tau(Z)$ for all $Z \in \mathfrak{a}$. The complex Lie algebra $\mathfrak{a}$ is isomorphic to the direct sum $\mathfrak{g}^+ \oplus \mathfrak{g}^-$ of two simple complex ideals which are conjugate complex (i.e., $g^- = \tau(g^+)$). These may nevertheless be isomorphic as complex algebras. Each real isomorphism $f\colon \mathfrak{g}^+ \to \mathfrak{g}^-$ gives a subalgebra $\mathfrak{g}_f = \{X \oplus f(X)\colon X \in \mathfrak{g}^+\}$ which is isomorphic to $\mathfrak{g}^+ \cong \mathfrak{g}$. We call these *graph subalgebras* of $\mathfrak{a}$. If we define $\sigma\colon \mathfrak{g}^+ \to \mathfrak{g}^-$ by $\sigma(X) = \tau(X)$, i.e., by restricting and corestricting τ, then $\mathfrak{g} = \mathfrak{g}_\sigma$. The graph subalgebra $\mathfrak{g}_f$ is complex if and only if $f\colon \mathfrak{g}^+ \to \mathfrak{g}^-$ is a complex isomorphism. A real automorphism of $\mathfrak{a}$ will be called *special* if it leaves the ideals $\mathfrak{g}^\pm$ invariant.

Lemma 3. *For two graph subalgebras $\mathfrak{g}_f$ and $\mathfrak{g}_g$ there is a special real automorphism F of $\mathfrak{a}$ with $F(\mathfrak{g}_f) = \mathfrak{g}_g$. If $f = \sigma$ and $\mathfrak{g}_g$ is complex, then F is neither complex nor complex conjugate.*

Proof. We note that $gf^{-1}\colon \mathfrak{g}^- \to \mathfrak{g}^-$ is an automorphism α of $\mathfrak{g}^-$, and $g = \alpha\circ f$. We define a special automorphism $F\colon \mathfrak{a} \to \mathfrak{a}$ by $F(X\oplus Y) = X\oplus\alpha(Y)$. Now we have $\mathfrak{g}_g = \{X \oplus g(X)\colon X \in \mathfrak{g}^+\} = \{X \oplus \alpha(f(X))\colon X \in \mathfrak{g}^+\} =$

$F\{X \oplus f(X)\colon X \in \mathfrak{g}\} = F(\mathfrak{g}_f)$. Suppose that $f = \sigma$ and that g is complex, then $F|\mathfrak{g}^+ = \mathrm{id}_{\mathfrak{g}^+}$ is complex and $F|\mathfrak{g}^- = \alpha$ is complex conjugate. Thus F is neither complex nor complex conjugate. ∎

Proposition 4. *Let $\mathfrak{a}$ be a complex Lie algebra with a simple real form $\mathfrak{g}$ and a real subalgebra $\mathfrak{g}^*$ which is isomorphic to $\mathfrak{g}$. Then one of the following mutually exclusive cases occurs:*

(A) *There is a complex automorphism $F\colon \mathfrak{a} \to \mathfrak{a}$ with $F(\mathfrak{g}) = \mathfrak{g}^*$.*

(B) *$\mathfrak{a}$ is not simple, and there is a special real autmorphism $F\colon \mathfrak{a} \to \mathfrak{a}$ which is neither complex nor complex conjugate with $F(\mathfrak{g}) = \mathfrak{g}^*$.*

(C) *$\mathfrak{a}$ is not simple, and $\mathfrak{g}^*$ is one of the two unique ideals $\mathfrak{g}^+$ or $\mathfrak{g}^-$ with $\mathfrak{a} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. There is no isomorphism of $\mathfrak{a}$ mapping $\mathfrak{g}$ to $\mathfrak{g}^*$, but the projection onto $\mathfrak{g}^*$ is a complex endomorphism of $\mathfrak{a}$ mapping $\mathfrak{g}$ isomorphically onto $\mathfrak{g}^*$.*

Proof. Since $\mathfrak{g}^* \cong \mathfrak{g}$, the real subalgebra $\mathfrak{g}^*$ is simple. By Remark 1, we have two cases: Case (i): $\mathfrak{g}^* \oplus i\mathfrak{g}^*$. Case (ii): $i\mathfrak{g}^* = \mathfrak{g}^*$.

In Case (i), Lemma 2 shows the existence of an endomorphism F of $\mathfrak{a}$ with $F(\mathfrak{g}) = \mathfrak{g}^*$. If $\mathfrak{a}$ is simple, then F must be an automorphism. If $\mathfrak{a} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ then the complex endomorphism F must respect the two summands. If it were zero on one of the summands, then $\mathfrak{g}$ would have to be contained in the other, but $\mathfrak{g}$ is contained in neither. Hence F is an automorphism and we are in Case (A).

In Case (ii), because of $\mathfrak{g} \cong \mathfrak{g}^*$, the algebra $\mathfrak{a}$ is the complexification of the underlying real algebra of a complex simple Lie algebra and is therefore of the form $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ according to remarks preceding Lemma 3. The following mutually exclusive cases occur: (a) $\mathfrak{g}^* = \mathfrak{g}^+$. (b) $\mathfrak{g}^* = \mathfrak{g}^-$. (c) $\mathfrak{g}^*$ is a complex graph subalgebra. In Cases (a) and (b) we have Case (C) of the Proposition. Then $\mathfrak{g}^*$ is an ideal while $\mathfrak{g}$ is not. Hence no real or complex automorphism of $\mathfrak{a}$ maps $\mathfrak{g}$ onto $\mathfrak{g}^*$. In Case (c) we apply Lemma 3 to the two graph subalgebras $\mathfrak{g} = \mathfrak{g}_\sigma$ and $\mathfrak{g}^*$ and find ourselves in Case (B) of the Proposition. ∎

Corollary 5. *If $\mathfrak{a}$ is a simple complex Lie algebra and $\mathfrak{g}$ a real form, and if $\mathfrak{g}^*$ is a real subalgebra of $\mathfrak{a}$ which is isomorphic to $\mathfrak{g}$ then there is a complex automorphism of $\mathfrak{a}$ mapping $\mathfrak{g}$ onto $\mathfrak{g}^*$.* ∎

We remark, that the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of a complex simple Lie algebra is a semidirect product of the normal connected subgroup $\mathrm{Int}(\mathfrak{g}) = \langle e^{\mathrm{ad}\,X}\colon X \in \mathfrak{g}\rangle$ of inner automorphisms by a finite group E, namely, the automorphism group of a basis of a root system. The group E is of order 2 for A_n, $n > 1$, D_n, $n \neq 4$, E_6 and of order 3 for D_4, and it is trivial in all other cases. (See [3] or [2], notably p. 298.)

Corollary 6. *Let A be a complex simple and simply connected Lie group with complex conjugation κ, and let G denote the real form of elements fixed under κ. Let H denote any real analytic subgroup of A whose Lie algebra is isomorphic* *to* $\mathfrak{g} = L(G)$. *Then* H *is conjugate to* G *under a complex automorphism of* A. *In particular,* A *is closed and isomorphic to* H.

Proof. The Lie algebra $\mathfrak{a}$ of A is the complexification of $\mathfrak{g}$ and is simple. Hence we are in Case (A) of Proposition 4 and find a complex automorphism $F: \mathfrak{a} \to \mathfrak{a}$ with $F(g) = \mathfrak{h}$. We have $\operatorname{Aut}\mathfrak{g} = (\operatorname{Inn}\mathfrak{g}) \rtimes E = \operatorname{Ad}(G) \rtimes E$ with a finite group E or order 1,2, or 3 according to the list above. Since A is simply connected, there is a unique automorphism $f: A \to A$ inducing F. The group H is connected (see e.g. [2], p. 214, Theorem 9). The analytic groups H and $f(G)$ have the same Lie algebra and are both connected. Hence they agree. ■

If the group of automorphisms of $\mathfrak{a}$ fixing $\mathfrak{g}$ as a whole contains an automorphism which is not inner, or if there are no outer automorphisms of $\mathfrak{a}$ then the group of inner automorphisms acts transitively on the set of subalgebras isomorphic to $\mathfrak{g}$. The algebra $\mathfrak{sl}(n\mathbb{C})$ (realizing type A_{n-1}) for instance admits a non-inner involution of the form $X \mapsto -X^\top$ which leaves $\mathfrak{sl}(n, \mathbb{R})$ invariant. A similar statement holds for D_n, n odd, and E_6 (see [2], 297f.) In such cases, in Corollary 6, the groups G and H are conjugate under an inner automorphism.

References

[1] Lawson, J. D., *A question on the simple real forms of the Lie algebra* sp(2, $\mathbb{C}$), e-mail message, October 1991.

[2] Onishchik, A. L., and E. B. Vinberg, "Lie Groups and Algebraic Groups", Springer-Verlag Berlin etc., 1990.

[3] Tits, J., "Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen", (Springer) Lecture Notes in Mathematics **40** (1967).

Karl H. Hofmann
Fachbereich Mathematik
Technische Hochschule Darmstadt
Schloßgartenstraße 7
D-6100 Darmstadt, Germany

Received October 10, 1991