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Extension of Positive Definite Functions on Lie Groups

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The aim of this paper is to give a survey of results and questions concerning the problem of extending a continuous positive definite function from an open neighborhood of the neutral element of a Lie group to the whole group retaining positive definiteness. Whereas most positive results concern abelian groups, there are methods which carry over to non-commutative groups and which may prove useful to treat the extension problem in this case.

1. Definitions and Historical Remarks

Let G denote a Lie group, V a symmetric open neighbourhood of the neutral element e of G, H a Hilbert space, and $B(H)$ the Banach space of bounded operators in H. A function $F\colon V^2 = V^{-1}V \to B(H)$ is said to be *positive definite* (p.d.), if

$$\sum_{j,k=1}^{n} \langle F(x_k^{-1}x_j)\xi_j, \xi_k\rangle \geq 0 \tag{1}$$

for all $n \in \mathbb{N}$, $x_1, \ldots x_n$, and $\xi_1, \ldots, \xi_n \in H$. If the Hilbert space H is one-dimensional, we may and shall identify $B(H)$ and $\mathbb{C}$. Thus we have the following definition for positive definiteness of a scalar-valued function f:

$$\sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} f(x_k^{-1}x_j) \geq 0$$

for all $n \in \mathbb{N}$, $x_1, \ldots x_n \in V$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

If G is specified to be the real line $\mathbb{R}$, then $V = (-a, a)$ with some a, $0 < a \leq \infty$, and the positive definiteness of f on $(-2a, 2a)$ means that

$$\sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} f(x_j - x_k) \geq 0$$

for all $n \in \mathbb{N}$, $|x_j| \leq a$, $j = 1, \ldots, n$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

In the following we will be interested mostly in continuous positive definite functions. Therefore we state the following property, which easily may be obtained from the definition with $n = 3$ (cf. e.g. [11]):

Proposition 1. *For a positive definite (operator-valued) function F on the open set $V^{-1}V$ the following are equivalent:*

(i) *The function F is weakly continuous at the neutral element e of G, i.e., the mapping $x \to \langle F(x)\varphi, \psi\rangle$ is continuous at e for all $\varphi, \psi \in H$.*

(ii) *F is uniformly strongly continuous on $V^{-1}V$, i.e., the mapping $x \to F(x)\varphi$ is uniformly continuous for all $\varphi \in H$.* ∎

Thus there is no danger of confusion if we denote a positive definite function satisfying the conditions of this proposition as a *continuous positive definite* (c.p.d.) function.

Examples 2. (i) One can show that the functions f_b, $1 \le b \le \infty$, defined by

$$f_\infty(x) = \max(0, 1-|x|) \text{ and } f_b(x) = \max(0, 1-\min\{|x-bk| : k \in \mathbb{N}\}),\ 1 \le b < \infty$$

are p.d. functions on the real line.

(ii) The function $f(x) = e^{-\frac{x^2}{2}}$ is positive definite. A well known theorem of Bochner states that any c.p.d. function on the real line is the Fourier transform of a non-negative finite Borel measure on $\mathbb{R}$ [7].

(iii) Suppose that U is a strongly continuous unitary representation of the Lie group G in the Hilbert space H. Consider another Hilbert space K and a bounded linear mapping $A\colon K \to H$. Then F defined by $F(x) = A^*U(x)A$ is a c.p.d. function (with values in $B(K)$). Note that any c.p.d. function on G is of that kind. ∎

In 1940, M. G. KREIN [19] posed the question whether any c.p.d. function on an interval $(-2a, 2a)$, $0 < a$, has a c.p.d. extension to the whole real line. He answered it in the affirmative and gave a description of all possible extensions for the case that they are not unique. Example 2(i) above shows that non-uniqueness is possible. Krein used an extension theorem for positive linear functionals in his proof. A direct proof was given by RAIKOV in [26]. ARTEMENKO showed that also discontinuous p.d. functions on an interval admit a p.d. extension to the real line [2], reprinted in [3, 4]. This fact can also be proved by using a Krein-Rutman extension property. The corresponding problem for operator-valued functions seems to be open:

Problem. Suppose that the function $F\colon(-2a, 2a) \to B(H)$ is discontinuous and positive definite, and that $\dim H > 1$. Does there exist a p.d. extension to the entire real line? ∎

Other questions (not being discussed here) concern measurability of p.d. extensions [9, 29].

Whereas p.d. functions on intervals do always admit a p.d. extension, this is not true for p.d. functions given on a square in $\mathbb{R}^2$ [27]. The crucial point in the proof of this fact is that there are nonnegative polynomials in two real variables, which are not representable as sums of squares of polynomials. Later on RUDIN also showed [28] that an extension is possible if the function is given on a disk around zero and if it is rotationally invariant.

As reported in [20], another positive result was already obtained by LIVSHIC in 1945 [21], but was not published in a journal: Any c.p.d. function on $(-2a, 2a) \times \mathbb{R}$ admits a p.d. extension to the plane. The proof uses operator methods. This result was generalized by LEVIN and OVCHARENKO [20] to discontinuous p.d. functions.

AKUTOVICH [1] and DEVINATZ [8] described the uniqueness of the extension of a scalar p.d. function from an interval in terms of the self-adjointness of a certain operator. BEREZANSKIJ and GORBACHUK [6] characterized all extensions of a c.p.d. function on $(-2a, 2a) \times \mathbb{R}$ to $\mathbb{R}^2$ via commutation properties of a certain family of operators. A description of all extensions of operator-valued c.p.d. functions was given in [12].

A nice survey on p.d. functions is [32].

In the following section we will discuss how Hilbert spaces and operators therein are involved in the extension problem.

2. The Gelfand-Naimark-Segal Construction

First we state the fact that a weakly continuous function $F: V^{-1}V \to B(H)$ is p.d. if and only if

$$\int_V \int_V \langle F(x^{-1}y)\varphi(x), \varphi(y)\rangle dx\, dy \geq 0$$

for all continuous functions $\varphi: V \to H$ with compact support in V. Here dx denotes a left invariant Haar measure on G. For scalar valued functions F this property follows from (1) by approximating the measure $\varphi(x)dx$ by measures with finite support in the vague topology. The opposite direction follows from the fact that any measure with finite support can be approximated by measures of the form $\varphi(x)dx$ in the vague topology, where φ is as above. Using a some extra technique, a similar argument can be used in the general case.

Now we construct a Hilbert space K as follows. We start with the linear space $\mathcal{K}$ of all continuous functions $\varphi: V \to H$ with compact support in V. On $\mathcal{K}$ we define a sesquilinear form $[\cdot,\cdot]$ by

$$[\varphi, \psi] = \int_V \int_V \langle F(x^{-1}y)\varphi, \psi\rangle dx\, dy. \tag{2}$$

This sesquilinear form is positive semidefinite because of the positive definiteness of F. Let $\mathcal{N} = \{\varphi \in \mathcal{K} : [\varphi, \varphi] = 0\}$ and let $\mathcal{K}_0$ denote the quotient space $\mathcal{K}/\mathcal{N}$. The Cauchy-Schwarz inequality yields that the sesquilinear form is constant on equivalence classes . Thus it defines an inner product on $\mathcal{K}_0$. Now the Hilbert space K is defined to be the completion of $\mathcal{K}_0$ w.r.t. the corresponding norm. For simplicity of notation we will not distinguish vectors in $\mathcal{K}$ and their equivalence classes. Similarly, we retain the notation for the inner product.

Suppose now that $V = G$. Then the mapping $U(z)$ defined on $\varphi \in \mathcal{K}_0$ by $(U(z)\varphi)(x) = \varphi(z^{-1}x)$, $x \in G$, and then extended to the whole space K by continuity, is unitary because of (2) and the left invariance of Haar measure. Moreover, the continuity of F implies the strong continuity of the mapping $z \to U(z)$, i.e., U is a unitary representation of G.

Let δ denote the unit point measure concentrated at $e \in G$ and $\xi \in H$ an arbitrary vector. If $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence of non-negative functions such that the measures $\varphi(x)dx$ tend to δ, then it follows from (2) that $[U(x)(\varphi(\cdot)\xi), (\varphi(\cdot)\xi)]$ converges to $\langle F(x)\xi, \xi\rangle$, which means that F may be recovered from U.

For $V \neq G$, the above construction is impossible, since the linear space $\mathcal{K}$ is not translation invariant in that case. There remains the possibility of defining an infinitesimal representation ρ of G.

In order to explain this procedure, let $\mathfrak{g}$ denote the Lie algebra of G and $\mathcal{E}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. The enveloping algebra carries a natural involution which is defined by $X^* = -X$, $X \in g$.

Now we define a $*$-representation ρ of $\mathcal{E}(\mathfrak{g})$ by operators on a certain vector subspace $D(\rho)$ of K as follows: The vector space $D(\rho)$ shall consist of (the classes of) all infinitely differentiable H-valued functions on V with compact support. We define

$$(*) \qquad (\rho(X)\varphi)(x) = \frac{d}{dt}\varphi(\exp(-tX)x)\Big|_{t=0} \quad \text{for } X \in \mathfrak{g} \text{ and } \varphi \in D(\rho).$$

The mapping ρ preserves brackets and thus extends uniquely to a $*$-representation of the $*$-algebra $\mathcal{E}(\mathfrak{g})$ by the universal property of $\mathcal{E}(\mathfrak{g})$.

If $V = G$ and U is the unitary representation defined above, then U and ρ are connected via

$$\rho(X)\varphi = \frac{d}{dt}U(\exp(tX))\varphi\Big|_{t=0}, \quad X \in \mathfrak{g},\ \varphi \in D(\rho).$$

If a unitary representation U and a $*$-representation ρ are connected as in (3), we call ρ the *derived representation* of U and write $\rho = dU$. We will say that ρ is *extendible*, if it can be represented as a restriction of a derived representation in a possibly larger Hilbert space. The action of any such extension (if there is one) in the original Hilbert space is completely determined by ρ, i.e., their differences appear only in the extension space (cf. [12]). Thus we have the following situation (cf. e.g. [18]):

Proposition 3. *A c.p.d. function $F\colon V^{-1}V \to B(H)$ has a p.d. extension to the whole group if and only if ρ is extendible.* ∎

Criteria for extendibility of $*$-representations were given e.g. in [25] and [16].

Thus we have arrived at the more general problem of extendibility of $*$-representations of enveloping algebras. There seem to be essentially two methods to treat this situation.

The first one establishes an equivalence of extendibility and a certain positivity of ρ, called *complete strong positivity*, which was considered first by POWERS in [25]. In fact POWERS considered extendibility within the same Hilbert space. The extendibility problem in the sense defined above was treated by Jørgensen [16]; see also [31], Chapter 11). To translate these results into the extension problem for p.d. functions, we have to carry over the corresponding definitions which mean a stronger positivity property for F.

A matrix $(E_{ij})_{i,j=1}^{\infty}$ of infinitely differentiable functions on G is said to be of *positive type*, if

$$\sum_{i,j=1}^{n}\sum_{k,l=1}^{m}\lambda_{ik}\overline{\lambda}_{jl}E_{ij}(x_{ik}^{-1}x_{jl})\geq 0$$

for all natural numbers n, m, all $x_{ik}\in G$, and all complex numbers λ_{ik}, $i=1,\dots,n$, $k=1,\dots,m$.

Suppose that $\pi(a)$ is the differential operator corresponding to $a\in\mathcal{E}(g)$ and acting on differentiable functions on G. Note that π is defined as an algebra homomorphism and by

$$(\pi(X)h)(x)=\frac{d}{dt}h(\exp(-tX)x)\bigg|_{t=0},\quad X\in g.$$

A matrix $(a_{ij})_{ij=1}^{\infty}$ with entries in $\mathcal{E}(g)$ (only finitely many of them being different from zero) is said to be *strongly positive*, if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\pi(a_{ij})E_{ij})(e)\geq 0$$

for all matrices $(E_{ij})_{i,j=1}^{\infty}$ of positive type.

Consider a scalar p.d. function $F=f$ on $V^{-1}V$. It is said to be *completely strongly positive*, if

$$\sum_{i,j=1}^{\infty}[\pi(a_{ij})\varphi_i,\varphi_j]\geq 0$$

for all infinitely differentiable functions φ_i, $i=1,2,\dots$ with compact support in V. The form [.,.] is as in (2).

Now we are ready to state the following characterization of extendibility of (scalar) p.d. functions which is nice but seems not to be appropriate for the solution of concrete extension problems.

Proposition 4. *The p.d. function f is extendible if and only if it is completely strongly positive.* ■

A second method for extending positive definite functions is based on the following theorem (cf. [16, 30]).

Proposition 5. *Suppose that* $\mathfrak{g}$ *is abelian and admits a decomposition* $\mathfrak{g} = \mathrm{span}(X) \oplus \mathfrak{g}_0$, $X \in \mathfrak{g}$ *such that the restriction of the* $*$*-representation* ρ *of* $\mathcal{E}(\mathfrak{g})$ *to* $\mathcal{E}(\mathfrak{g}_0)$ *is integrable. Then* ρ *is extendible.* ■

This result may be applied to p.d. functions to give the following result [14]:

Proposition 6. *If* G *is an abelian topological group and* F *an operator-valued c.p.d. function on* $(-2a, 2a) \times G$, *then* F *is extendible to the whole group* $\mathbb{R} \times G$. ■

This result can also be obtained from SZ.-NAGY's dilation theory, e.g. from Prop.9.2, Chapter I, in [33] (see [14]), which also leads to uniqueness of the extension under certain additional assumptions.

3. Integral Representations

A generalized version of BOCHNER's theorem says that any c.p.d. (scalar) function f on a locally compact abelian group G admits an integral representation of the form

$$f(x) = \int_\Gamma [\gamma, x] \mu(d\gamma), \tag{5}$$

where Γ is the dual group of G, $[\gamma, x]$ the value of the character γ at $x \in G$, and μ a non-negative finite Borel measure on Γ. This is the origin of another idea for the extension of c.p.d. functions. If G is an additively written locally compact abelian group, and if $f\colon V - V \to \mathbb{C}$ is c.p.d., then we try to find an integral representation for f as in (5). As a by-product, we obtain an extension of f *defined* by (5) outside $V - V$. A particular result in NUSSBAUM's papers [23] and [24] is that any c.p.d. rotation-invariant function on the unit ball in $\mathbb{R}^n$ has such an integral representation. In particular, this gives another proof of RUDIN's result in [28]. With a suitable generalization of the technique we obtained the following result [11]:

Proposition 7. *Let* G_0 *denote a locally compact abelian group,* G *the group* $\mathbb{R}^n \times G_0$, *and* $V = \{s \in \mathbb{R}^n : \|s\| < a\} \times G_0$. *Then a c.p.d. operator-valued function* F *on* $V - V$ *has a c.p.d. extension to the whole group* G, *if n=1 or if* F *is invariant w.r.t. rotations in* $\mathbb{R}^n$. ■

We want to sketch the idea of the proof in the case of a c.p.d. function $f\colon (-2a, 2a) \to \mathbb{C}$. First we apply the GNS-construction as described in Section 2. If X is a basis vector of the one-dimensional Lie algebra $\mathfrak{g}$, the operator $A = i\rho(X)$, acting by $A\varphi = -i\varphi'$ is symmetric. Let $\mathcal{A}$ denote any self-adjoint extension of it in a possibly larger Hilbert space $\mathcal{K} \supseteq K$. Then $\mathcal{K}$ decomposes into a direct integral consistent with the action of $\mathcal{A}$:

$$\mathcal{K} = \int_R^{\oplus} \mathcal{K}(\lambda)\mu(d\lambda)$$
$$\mathcal{A}\varphi = \int_R^{\oplus} \lambda\varphi(\lambda)\mu(d\lambda).$$

Since a δ-sequence of functions with compact support in $(-a,a)$ converges in K (due to continuity of f), we may consider the δ-distribution as a vector in K. Thus we obtain a positive definite extension via

$$f(t) = \langle \exp(-it\mathcal{A})\delta, \delta \rangle = \int_R e^{-it\lambda} |\delta(\lambda)|^2 \mu(d\lambda).$$

For the technically rather complicated matter of direct integrals of Hilbert spaces we refer to [10]. To make the steps correctly, one needs essentially a certain continuity of the generalized eigenfunction decomposition $\varphi \to \varphi(\lambda)$, which is stated in the so-called nuclear spectral theorem (cf. [31], Theorem 12.2.1). (The first paper in that direction is probably [15], see also [22] and [5].)

This method of integral decomposition seems to be the only one up to now which leads to positive results in the non-commutative case, too. At the end of this paper we shall briefly describe corresponding results from [17]. We consider the Heisenberg group $G = \mathbb{C} \times \mathbb{R}$ with the composition law

$$(z,x)(w,y) = (z+w, x+y+2\mathrm{Im}(z\overline{w})).$$

Let $V = \{(z,x) \in G : |z| < \frac{1}{2}\}$ and consider a scalar c.p.d. function f on $V^{-1}V = 2V$ which is invariant w.r.t. rotations $(z,x) \to (\alpha z, x)$, $|\alpha| = 1$.

Proposition 8. *Under these assumptions there is a p.d. kernel k on G such that*

$$f(x^{-1}y) = k(x,y), \quad x,y \in V.$$

If k may be chosen to be analytic, $k(e,\cdot)$ is a c.p.d. function extending f to the whole group. ■

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