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A short course on the Lie theory of semigroups I

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This group of three brief lectures* addresses an audience with some knowledge of the classical theory of Lie groups and Lie algebras, namely, the *Seminar Sophus Lie* which was initiated by the Universities of Erlangen, Greifswald, and Leipzig and the Darmstadt Institute of Technology and which has its first meeting, most fittingly, at the University of Leipzig where Sophus Lie spent the years of 1886 through 1898, the formative years of what we call *Lie theory*. The lectures outline the development, in the last decade of Lie's program for multiplicatively closed subsets of Lie groups. These are called semigroups and were already considered by Sophus Lie himself. He had not much more use for them than demonstrating through their simplest examples that the existence of an identity element and inverses would not follow from the assumptions he made to define what he called a *group* [5]. Today we have several fields of mathematical research in which they occur naturally. As an example we name the areas of (i) nonlinear control (Sussmann theory of controllability on manifolds), (ii) manifolds with a partial order (causality, chronogeometry, geometry), and (iii) representation theory of Lie groups (Ol'shanskiĭ theory).

In the meantime we are interested here in explaining as many aspects as possible on the foundations of a Lie theory of semigroups: How far do *Sophus Lie*'s ideas carry us in this direction? What are the characteristic problems left open at this time? (The monograph [4] represents the status of the theory up to about 1988, the collection [6] goes a bit beyond, and new results have become available since.)

Lie's mechanism: analysis versus algebra

The basis of success for Lie's program of classification of Lie groups is the assignment of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ to a Lie group G; this assignment $\mathfrak{L}$ is an equivalence of categories from the category of simply connected Lie groups to the category of Lie algebras. The exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ is a natural smooth function. For each morphism $f\colon G \to H$ of Lie groups one has $\exp_H \circ \mathfrak{L}(f) = f \circ \exp G$, a commutative diagram. (All objects in sight will be finite dimensional; for an occasionally more general approach see [4].) One has several ways of concretely working with $\mathfrak{g}$.

Number One: We can take for $\mathfrak{g}$ the Lie subalgebra of all left-invariant

* The lectures were given in German

vector fields on G in the Lie algebra $\mathcal{V}(G)$ of all smooth vector fields on G. Evaluation at $\mathbf{1}$ defines an isomorphism of vector spaces $\mathfrak{g} \to T(G)_1$. The curve $t \mapsto \exp t{\cdot}X{:}\,\mathbb{R} \to G$ is the unique trajectory $x{:}\,\mathbb{R} \to G$ such that $\dot{x}(t) = X\big(x(t)\big) \in T(G)_{x(t)}$.

Number Two: We take $\mathfrak{g} = T(G)_{\mathbf{1}}$, the tangent space at the origin; for each $g \in G$, the left tranlsation $\lambda_g{:}\,G \to G$, $\lambda_g(x) = gx$ defines a left invariant vector field $\widetilde{X} \in \mathcal{V}(G)$ via $\widetilde{X}(g) = d\lambda_g(\mathbf{1})(X)$. The assignment $X \mapsto \widetilde{X}$ is an isomorphism of vector spaces of $\mathfrak{g}$ onto the vector space of all left invariant vector fields on G. The Lie algebra structure can then be transported from the range to $\mathfrak{g}$ and the exponential function can be defined as before.

Number Three: We let $\mathfrak{g}$ denote the set $\mathrm{Hom}(\mathbb{R}, G)$ of all morphisms of topological groups $\mathbb{R} \to G$, define $\exp{:}\,\mathfrak{g} \to G$ by $\exp X = X(1)$. Then exp has a local inverse log at $\mathbf{1}$ and $\mathfrak{g}$ carries the structure of a Lie algebra such that $r{\cdot}X = X(r)$,

$$X + Y = \lim_{t\to 0} \frac{1}{t}{\cdot}\log\big((\exp t{\cdot}X)(\exp t{\cdot}Y)\big)$$

and

$$[X, Y] = \lim \frac{1}{t^2}{\cdot}\log[\exp t{\cdot}X, \exp t{\cdot}Y], \quad [g, h] = ghg^{-1}h^{-1} \text{ in } G.$$

Moreover, if $X * Y = X + Y + \frac{1}{2}{\cdot}[X, Y] + \cdots + H_n(X, Y) + \cdots$ is the BAKER–CAMPBELL–HAUSDORFF–DYNKIN series then, with this local multiplication $*$, also called CH-*multiplication* $\exp(X * Y) = \exp X \exp Y$ for all X and Y sufficiently close to 0 in $\mathfrak{g}$.

It is not exactly a trivial matter to verify everything needed to establish the equivalence of the third approach with the other two. If the assertions of Number Three are postulated it is not too hard to get to the other two set-ups, the reverse is harder. In the following we shall primarily adopt the second and the third view point. In particular, we shall associate with every subset $S \subseteq G$ clustering at $\mathbf{1}$ a set of *subtangent vectors* $\mathfrak{L}(S)$ *at the origin* via

$$\mathfrak{L}(S) = \{X \in \mathfrak{g} : (\exists X_n \in G)(\exists r_n > 0)\, X = \lim r_n{\cdot}X_n \text{ and } \exp X_n \in S\}.$$

LIE's *machine: From analysis to algebra*

If S is a subgroup of G, then $\mathfrak{L}(S)$ is always a subalgebra of $\mathfrak{g}$. What if S is merely a subsemigroup?

Definition 1.1. (i) A *wedge* or *closed convex cone* in a finite dimensional vector space $\mathfrak{g}$ is a topologically closed subset $\mathfrak{w}$ with $r{\cdot}\mathfrak{w} \in \mathfrak{w}$ for $r \geq 0$ and $\mathfrak{w} + \mathfrak{w} = \mathfrak{w}$. The largest vector subspace $\mathfrak{w} \cap -\mathfrak{w}$ of a wedge will be called *the edge of the wedge* and will be written $\mathfrak{h}(\mathfrak{w})$. We say that $\mathfrak{w}$ is a *pointed cone* if $\mathfrak{h}(\mathfrak{w}) = \{0\}$.

(ii) If x is an element of a wedge $\mathfrak{w}$ we let $\mathfrak{w}_x = \overline{W + \mathbb{R}{\cdot}x}$ denote the set of subtangent vectors to $\mathfrak{w}$ at x and $\mathfrak{t}_x(\mathfrak{w}) = \mathfrak{h}(\mathfrak{w}_x)$ the set of tangent vectors to $\mathfrak{w}$ at x.

(iii) A LIE *wedge* $\mathfrak{w}$ in a LIE algebra $\mathfrak{g}$ is a wedge such that

$$e^{\operatorname{ad} x}\mathfrak{w} = \mathfrak{w} \text{ for all } x \in \mathfrak{h}(\mathfrak{w}).$$

A Lie wedge $\mathfrak{w}$ is said to be *split* if $\mathfrak{g}$ is the direct sum $\mathfrak{h}(\mathfrak{w}) \oplus \mathfrak{q}$ with an $\mathfrak{h}(\mathfrak{w})$-submodule $\mathfrak{q}$ (i. e., a vector subspace satisfying $[\mathfrak{h}(\mathfrak{w}), \mathfrak{q}] \subseteq \mathfrak{q}$). ■

One observes at once that *a split Lie wedge* is a direct sum $\mathfrak{w} = \mathfrak{h}(\mathfrak{w}) \oplus \mathfrak{c}$ where $\mathfrak{c} = \mathfrak{q} \cap \mathfrak{w}$ is a pointed cone. The name "Lie wedge" is justified by

Proposition 1.2. *If S is a subsemigroup of G with $\mathbf{1} \in \overline{S}$, then $\mathfrak{L}(S)$ is a Lie wedge.* ■

(See e. g. [4].)

The representation theoretical condition 1.1(iii) for a Lie wedge can be transformed in a geometrical one:

Proposition 1.3. *A wedge $\mathfrak{w}$ in a Lie algebra $\mathfrak{g}$ is a Lie wedge if and only if*

$$[x, \mathfrak{h}(\mathfrak{w})] \subseteq \mathfrak{t}_x(\mathfrak{w}) \textit{ for all } x \in \mathfrak{w}. \quad ■$$

Thus LIE's machine produces, from the input of a semigroup in the group, not only a purely algebraic object but one in which both algebra and convex geometry play a role.

Of course, every subalgebra of $\mathfrak{g}$ is a LIE wedge. A little less trivially, but still clearly, a LIE wedge which is also a vector space is a subalgebra.

For the present discussion these cases are degenerate and we turn to genuine wedges.[1] On the other hand, there are the following extreme cases among the genuine wedges:

Extreme Case 1. $\mathfrak{w}$ is a pointed cone in a Lie algebra $\mathfrak{g}$. Every such cone is a LIE wedge by default.

Extreme Case 2. $\mathfrak{w}$ is a half-space (bounded by a hyperplane). Such a wedge is a LIE wedge if and only if the boundary $\mathfrak{h}(\mathfrak{w})$ is a subalgebra. Conversely, if $\mathfrak{h}$ is any hyperplane subalgebra of $\mathfrak{g}$ then the two half spaces bounded by $\mathfrak{h}$ are half-space LIE wedges.

Both of these extreme cases are illustrations of what might occur. There is no restriction on pointed cones whatsoever. Half-space LIE wedges occur as frequently as hyperplane subalgebras. One can classify these in every given LIE algebra $\mathfrak{g}$. Their intersection is a characteristic ideal $\Delta(\mathfrak{g})$ so that every hyperplane subalgebra $\mathfrak{h}$ contains $\Delta(\mathfrak{g})$ and $\mathfrak{h} \mapsto \mathfrak{h}/\Delta(\mathfrak{g})$ is a bijection between the respective sets of hyperplane subalgebras. The factor algebra $\mathfrak{g}/\Delta(\mathfrak{g})$ is a direct sum of a certain metabelian LIE algebra $\mathfrak{r}$ and a finite number of copies

[1] The distinction is substantial from a topological point of view: If one intersects an n-dimensional vector subspace with the unit sphere (with respect to some norm) one obtains a sphere $\mathbb{S}^{n-1}$. If one intersects a genuine n-dimensional wedge with the unit sphere one obtains an $n-1$ cell homeomorphic to $\mathbb{I}^n$, $\mathbb{I}=[0,1]$.

of $\mathfrak{sl}(2,\mathbb{R})$. The radical $\mathfrak{r}$ has a structure which one can describe in terms of the so-called *base roots*, i. e. nonzero linear functionals $\alpha\colon \mathfrak{g} \to \mathbb{R}$ vanishing on $\mathfrak{r}'$. Indeed if B is the finite set of all of these then $r = \mathfrak{h} \oplus r'$ where the Cartan algebra $\mathfrak{h}$ of r is abelian and the commutator algebra $\mathfrak{r}'$ is the direct sum $\bigoplus_{\alpha\in B} \mathfrak{m}_\alpha$ of abelian ideals $\mathfrak{m}_\alpha$ such that $[x,m] = \langle\alpha,x\rangle{\cdot}m$ for all $x \in \mathfrak{r}$ and $m \in \mathfrak{m}_\alpha$. A hyperplane subalgebra of $\mathfrak{r}$ either contains $\mathfrak{r}'$ or is any hyperplane meeting $\mathfrak{m}_\alpha$ in a hyperplane for some α and contains $\mathfrak{m}_\beta$ for $\beta \neq \alpha$. A hyperplane subalgebra of $\mathfrak{g}/\Delta(\mathfrak{g})$ meets $\mathfrak{r}$ in a hyperplane subalgebra and contains all the $\mathfrak{sl}(2)$–summands, or else it contains $\mathfrak{r}$ and all but one $\mathfrak{sl}(2)$ summands, meeting this last one in one of the well-known plane-subalgebras of $\mathfrak{sl}(2,\mathbb{R})$. (These are the planes which are tangent to the double cone of the vectors $\kappa(X,X) \le 0$ with the CARTAN-KILLING form κ of $(2,\mathbb{R})$.) The simplest kind of Lie algebra of the type of $\mathfrak{r}$ consist of the block matrices

$$\begin{pmatrix} r{\cdot}E_{n-1} & \begin{pmatrix} r_1, \\ \vdots \\ r_{n-1} \end{pmatrix} \\ 0 & 0 \end{pmatrix}, \qquad r_1, \ldots, r_{n-1}, r \in \mathbb{R}.$$

together with abelian algebras these algebras are called *almost abelian* algebras.

The intersection of a family of LIE wedges is again a LIE wedge. Hence intersecting a bunch of half-space LIE wedges is again a LIE wedge; this supplies us with a class of LIE wedges which, accordingly, we consider as well-understood. Let us call them *special* LIE *wedges* for the sake of the argument. (They are also called *intersection algebras.* Of course, all wedges containing the commutator algebra $[\mathfrak{g},\mathfrak{g}]$ are special; in fact we call them *trivial* LIE *wedges.* In any almost abelian Lie algebra *every* wedge is special.

Why are special LIE wedges special? On a LIE algebra $\mathfrak{g}$ we have an open convex symmetric neighborhood B of zero on which the CH-multiplication $*\colon B\times B \to \mathfrak{g}$ is defined. If $\mathfrak{h}$ is a Lie subalgebra, then $(\mathfrak{h}\cap B)*(\mathfrak{h}\cap B) \subseteq \mathfrak{h}$. Is this still true when we take a LIE wedge $\mathfrak{w}$ instead of a LIE algebra $\mathfrak{h}$? The answer is yes (for any B on which we can $*$-multiply) if we take a half-space LIE wedge, and consequently this remains true for all special LIE wedges. However, if one begins to inspect pointed cones $\mathfrak{w}$ in a LIE algebra $\mathfrak{g}$ one recognizes quickly, that it is very rarely the case that $\mathfrak{w}$ is closed under the local $*$-multiplication. The simplest non-abelian Lie algebra which is not almost abelian is the HEISENBERG algebra $\mathfrak{g} = \mathrm{span}(P,Q,E)$ with $[P,Q] = E$ (and zero brackets otherwise). Let G denote the LIE group defined on $\mathfrak{g}$ by the globally defined CH-multiplication $(X,Y) \mapsto X + Y + \frac{1}{2}[X,Y]$. The identity function $\mathfrak{g} \to G$ is the exponential function. The set $S = \{x{\cdot}P + y{\cdot}Q + z{\cdot}E : 0 \le x,\ y,\ |z| \le \frac{1}{2}xy\}$ is a subsemigroup of G with $\mathfrak{L}(S) = \mathbb{R}^+{\cdot}P + \mathbb{R}^+{\cdot}Q$, $\mathbb{R}^+ = [0,\infty[$. Because of $x{\cdot}P * y{\cdot}Q = x{\cdot}P + y{\cdot}Q + \frac{1}{2}xy{\cdot}E$ every element of S is a product of two elements of $\mathfrak{L}(S)$. Thus $(S\cap B)*(S\cap B) \subseteq \mathfrak{L}(S)$ for no neighborhood B of 0 and $\mathfrak{L}(S)$ is not special. (This example and numerous others are discussed in [4].) The special LIE wedges suggest a class of LIE wedges which are described through the following definition

Definition 1.4. A wedge $\mathfrak{w}$ in a LIE algebra $\mathfrak{g}$ is called a LIE *semialgebra* if there is a neighborhood B of 0 in $\mathfrak{g}$ such that the multiplication $*$ is defined on $B \times B$ and that $(\mathfrak{w} \cap B) * (\mathfrak{w} \cap B) \subseteq \mathfrak{w}$. ■

For LIE semialgebras, too a geometric description is available.

Proposition 1.5. *A wedge $\mathfrak{w}$ in a* LIE *algebra $\mathfrak{g}$ is a* LIE *semialgebra if and only if*

$$[x, \mathfrak{t}_x(\mathfrak{w})] \subseteq \mathfrak{t}_x(\mathfrak{w}) \text{ for all } x \in \mathfrak{w}. \qquad ■$$

It is a remarkable fact that LIE semialgebras can be classified, whereas we are far away from a classification of LIE wedges in general. More about this in the lecture by EGGERT!

Finally LIE's machine yields from the input of a normal subgroup N of a LIE group G an ideal $\mathfrak{L}(N)$ of $\mathfrak{L}(G)$. One way of describing ideals is saying that a vector subspace $\mathfrak{i}$ of a LIE algebra $\mathfrak{g}$ is an ideal if and only if it is invariant under all inner automorphisms:

$$e^{\operatorname{ad} X}\mathfrak{i} = \mathfrak{i} \qquad \text{for all } X \in G.$$

If we are given a semigroup $S \subseteq G$ which is invariant under all inner automorphisms of G, then $\mathfrak{w} = \mathfrak{L}(S)$ is invariant under all inner automorphisms $e^{\operatorname{ad} X}$ of $\mathfrak{g} = \mathfrak{L}(G)$. This calls for a definition.

Definition 1.6. A wedge $\mathfrak{w}$ in a LIE algebra $\mathfrak{g}$ is called*invariant* if

$$e^{\operatorname{ad} X}\mathfrak{w} = \mathfrak{w} \text{ for all } X \in \mathfrak{g}. \qquad ■$$

The geometric characterisation of invariant wedges is as follows:

Proposition 1.7. *A wedge $\mathfrak{w}$ in a* LIE *algebra $\mathfrak{g}$ is invariant if and only if*

$$[x, \mathfrak{g}] \subseteq \mathfrak{t}_x(\mathfrak{w}) \text{ for all } x \in \mathfrak{w} \qquad ■$$

A vector space is an invariant wedge if and only if it is an ideal. Invariant wedges are, in essence classified; more on this in the lectures by NEEB and ZIMMERMANN! (See also [4].)

LIE*'s machine: From algebra and convex geometry to analysis*

Even in classical Lie theory, the reverse operation is harder. In essence this is LIE's Third Fundamental Theorem: *For every* LIE *algebra $\mathfrak{g}$ there is a Lie group G such that $\mathfrak{L}(G) \cong \mathfrak{g}$.* This, in particular, yields the Corollary: *If G is a Lie group and $\mathfrak{h}$ a subalgebra of $\mathfrak{g} = \mathfrak{L}(G)$, then there is an analytic subgroup H of G with $\mathfrak{L}(H) = \mathfrak{h}$.* In fact, the most efficient proof today leads via ADO's Theorem—saying that every Lie algebra has a faithful representation in $\mathfrak{g}l(n) = \mathfrak{L}\big(\operatorname{Gl}(n)\big)$—and via the Corollary to LIE's Third Theorem. A LIE wedge $\mathfrak{w}$ has, in and as of itself no independent existence. It exists in a Lie algebra $\mathfrak{g}$. The best we can do in removing redundancy is to require that $\mathfrak{g}$ is generated as a LIE algebra by $\mathfrak{w}$. Thus we consider pairs $(\mathfrak{g}, \mathfrak{w})$ consisting of a LIE algebra $\mathfrak{g}$ and a LIE wedge $\mathfrak{w}$ generating $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{L}(G)$ with a Lie group G and with a subsemigroup S such that $\mathfrak{w} = \mathfrak{L}(S)$ and $\exp \mathfrak{w} \subseteq S$ then S has dense interior $\operatorname{int} S$ which satisfies $S(\operatorname{int} S) \cup (\operatorname{int} S)S \subseteq S$ an—important result which essentially comes from control theory. We aim for what would be a natural generalisation of the Corollary to LIE's Third Theorem: *Given a* LIE *wedge* $\mathfrak{w}$ *generating the* LIE *algebra* $\mathfrak{g}$ *and a* LIE *group* G *with* $\mathfrak{L}(G) = \mathfrak{g}$ *then we find a subsemigroup* S *of* G *such that* $\mathfrak{L}(S) = \mathfrak{w}$. This is false most of the time as one recognizes most strikingly for any compact semisimple LIE algebra $\mathfrak{g}$ and any pointed (hence LIE) wedge with inner points in $\mathfrak{g}$. Any connected LIE group G with $\mathfrak{L}(G) \cong \mathfrak{g}$ is compact. If S is a subsemigroup of G with $\mathfrak{L}(S) = \mathfrak{w}$ then $\overline{S}$ is closed subsemigroup of a compact group and therefore is a group; since it has nonempty interior it agrees with G. Hence S is dense in G whence $\mathfrak{L}(S) = \mathfrak{g}$. But even in the LIE algebra $\mathfrak{g}$ of LIE groups diffeomorphic to $\mathbb{R}^n$ such as the HEISENBERG group G there are pointed wedges $\mathfrak{w}$ for which there is no semigroup S in G such that $\mathfrak{L}(S) = \mathfrak{w}$ (see [4].) The frame of mind in which these phenomena are to be treated is that of nonlinear control theory. Before we go on to formulate what should be the right version of LIE's Third Theorem for semigroups we record that a local version is correct for semigroups. Historically, LIE's own formulation had to be a mere local version, too, because his theory was local while his examples were global.

Proposition 1.5. *If* $\mathfrak{w}$ *is a Lie wedge in the* LIE *algebra* $\mathfrak{g}$ *of a Lie group* G, *then there is an open neighborhood* U *of* $\mathbf{1}$ *in* G *and a subset* $S \subseteq U$ *satisfying the following conditions:*

(i) $1 \in \overline{S}$ *and* $SS \cap U \subseteq S$

(ii) $\mathfrak{L}(S) = \mathfrak{w}$. ∎

(For a proof see e. g. [4].)

Definition 1.6. A set $S \subseteq G$ satisfying (i) and (ii) is called *a local subsemigroup of* G *with respect to* U.

The following conjecture is consistent with what we know today:

Conjecture 1.6. For every pair $(\mathfrak{g}, \mathfrak{w})$ consisting of a LIE algebra and a LIE wedge $\mathfrak{w}$ generating $\mathfrak{g}$ as a Lie algebra there are

(i) a (simply connected) Lie group G with $\mathfrak{L}(G) = \mathfrak{g}$,

(ii) a pathwise connected (and simply connected) topological semigroup S, and

(iii) a continuous morphism of semigroups with identity $p: S \to G$ and

(iv) open identity neighborhoods U and V of S and G, respectively, such that

(a) $p|U: U \to p(U)$ is a homeomorphism and $p(U)$ is a local subsemigroup of G with respect to V.

(b) $\mathfrak{L}\big(p(U)\big) = \mathfrak{w}$. ∎

Proposition 1.7. (W. WEISS) *If* $\mathfrak{w}$ *is pointed then* Conjecture 1.6 *is correct.* ■

Very little is known beyond this general result. Better progress was made regarding the question when for a given LIE group G, and for a given LIE wedge $\mathfrak{w}$ in $\mathfrak{g} = \mathfrak{L}(G)$ there exists a semigroup S in G such that $\mathfrak{L}(S) = \mathfrak{w}$. Such wedges $\mathfrak{w} \subseteq \mathfrak{g}$ we call *global with respect to* G and simply *global* if G is simply connected. First results are to be found in [4], and better insights are due to NEEB. As a first orientation we can formulate the following necessary and sufficient condition:

Proposition 1.8. *A* LIE *wedge* $\mathfrak{w} \subseteq \mathfrak{L}(G)$ *is global with respect to a connected Lie group* G *if and only if the analytic subgroup* H *with* $\mathfrak{L}(H) = \mathfrak{h}(\mathfrak{w})$ *is closed and there is a smooth function* $\tau: G \to \mathbb{R}$ *such that for all* $X \in \mathfrak{w} \setminus \mathfrak{h}(W)$ *and all* $g \in G$ *the number* $\langle d\tau(g), d\lambda_g(1)(X)\rangle$ *is positive.* ■

Such functions τ are also called *strictly positive.* Their geometric significance is as follows: One may say that the homogeneous space $M = G/H$ carries a *causal structure* defined by the transport via the left action of G on M of the pointed cone $\mathfrak{w}/\mathfrak{h}(W)$ to each tangent space $T(M)_{gH}$, yielding a unique pointed cone $\theta(gH)$ in this tangent space. The fact that $\mathfrak{w}$ is a LIE wedge is exactly adequate for this formalism to work. The prescription of strictly positive functions on G as specified above and that of strictly positive functions on M are one and the same thing; on M one could call such a function a global time since every "time like" curve $t \mapsto x(t): [t_0, T] \to M$ with $\dot{x}(t) \in \theta\big(x(t)\big)$ then can be assigned an eigentime $\tau\big(x(t)\big) = \tau\big(x(t_0)\big) + \int_{t_0}^{t} \langle d\tau\big(x(s)\big), \dot{x}(s)\rangle\, ds$ which allows a reparametrisation of the curve to an equivalent curve $\xi: [\tau\big(x(t_0)\big), \tau\big(x(T)\big)] \to M$ given by $\xi(r) = x\big((\tau \circ x)^{-1}(r)\big)$ with $\tau\big(\xi(r)\big) = r$.

Special LIE wedges are always global since half-space LIE wedges are global and the property of globality is compatible with the formation of intersections. Invariant wedges are not always global. NEEB's lecture will contain more on this. The issue of globality of invariant cones will settle the question of globality of LIE semialgebras.

An observation on LIE*'s Third Theorem*

Concerning the consideration of pairs $(\mathfrak{g}, \mathfrak{w})$ one should observe that the statement that $\mathfrak{w}$ generates $\mathfrak{g}$ sometimes imposes undesireable restrictions. Let us discuss this aspect briefly, illustrated in the context of split wedges $\mathfrak{w} = \mathfrak{c} + \mathfrak{h}$ with $\mathfrak{g} = \mathfrak{q} + \mathfrak{h}$ and $\mathfrak{c} = \mathfrak{w} \cap \mathfrak{q}$, $\mathfrak{h} = \mathfrak{h}(\mathfrak{w})$.

We form the semidirect sum $\mathfrak{g} \rtimes_{\mathrm{ad}} \mathfrak{h}$ with $\mathrm{ad}: \mathfrak{h} \to \mathrm{Der}(\mathfrak{g})$ given by $\mathrm{ad}(X)(Y) = [X, Y]$. In other words, on $\mathfrak{g} \times \mathfrak{h}$ we consider the componentwise vector spaces structure and the bracket given by

$$(1)\qquad [(X, Y), (X', Y')] = ([X, X'] + \mathrm{ad}(Y)(X') - \mathrm{ad}(Y')(X), [Y, Y']).$$

The we have a surjective morphism of LIE algebras $\sigma: \mathfrak{g} \rtimes_{\mathrm{ad}} \mathfrak{h} \to \mathfrak{g}$ with kernel

$$\ker(\sigma) = \{(-Y, Y) : Y \in \mathfrak{h}\} \cong \mathfrak{h}.$$

If $Y \in \mathfrak{h}$ then $\mathrm{ad}(0,Y)$ on $\mathfrak{g} \rtimes_{\mathrm{ad}} \mathfrak{h}$ is $(\mathrm{ad}\,Y, \mathrm{ad}\,Y)$ by (1) and thus $e^{\mathrm{ad}(0,Y)} = (e^{\mathrm{ad}\,Y}, e^{\mathrm{ad}\,Y})$. It follows that $\mathfrak{g}^{\#} \stackrel{\mathrm{def}}{=} \mathfrak{q} \times \mathfrak{h}$ is a subalgebra of $\mathfrak{g} \rtimes_{\mathfrak{h}}$ such that $\sigma|\mathfrak{g}^{\#}: \mathfrak{g}^{\#} \to \mathfrak{g}$ is an isomorphism on account of $\ker(\sigma) \cap (\mathfrak{q} \times \mathfrak{h}) = \{(0,0)\}$. Furthermore, the wedge $\mathfrak{w}^{\#} \stackrel{\mathrm{def}}{=} \mathfrak{c} \times \mathfrak{h} \subseteq \mathfrak{g} \rtimes_{\mathrm{ad}} \mathfrak{h}$ is a Lie wedge. The restriction and corestriction $\sigma|\mathfrak{w}^{\#}: \mathfrak{w}^{\#} \to \mathfrak{w}$ is an isomorphism, so that $(\mathfrak{g}^{\#}, \mathfrak{w}^{\#}) \cong (\mathfrak{g}, \mathfrak{w})$. However, WEISS' Theorem given us a topological semigroup T and a homomorphism $p: T \to G$ such that for suitable open identity neighborhoods $f(U)$ is a local subsemigroup of G with respect to V and that $\mathfrak{L}\big(p(U)\big) = \mathfrak{c}$. If WEISS' construction allows us to choose T in such a fashion that H acts continuously on T as a group of continuous automorphisms via $\alpha: H \to \mathrm{Aut}(T)$ such that $p\big(\alpha(h)(t)\big) = hp(t)h^{-1}$—which is certainly the case if $\mathfrak{h}$ is compactly embedded into $\mathfrak{g}$, i. e., if $\langle e^{\mathrm{ad}\,\mathfrak{h}} \rangle$ is relatively compact in $\mathrm{Aut}(\mathfrak{g})$—then $S = T \rtimes_{\alpha} H$ is a semigroup such that $P: S \to G \rtimes_I H$, $I(h)(g) = hgh^{-1}$, $P(t,h) = \big(p(t), h\big)$ is a homomorphism mapping $U \times H$ homeomorphically onto its image $p(U) \times H$. such that $P(U \times U')$ is a local subsemigroup of $G \rtimes_I H$ with respect to $V \times U'$ for a suitable identity neighborhood U' of H. Thus the Conjecture 1.6 holds for $\mathfrak{w}^{\#} \subseteq \mathfrak{g} \rtimes_{\mathrm{ad}} \mathfrak{h}$ whereas we have at present no way of asserting that it holds for $\mathfrak{w}^{\#} \subseteq \mathfrak{g}^{\#}$, that is, for $\mathfrak{w} \subseteq \mathfrak{g}$.

The main open problem in the domain of LIE's fundamental theorems for semigroups is a general proof or refutation of Conjecture 1.6.

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