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Polar and Ol'shanskiĭ Decompositions

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The Cartan decomposition in a semisimple Lie group is a generalization of the polar decomposition of matrices. In this paper we consider an even more general setting in which one obtains an analogous decomposition. In the semisimple case, this decomposition was worked out in a seminal paper of G. I. Ol'shanskiĭ [13]. In this paper we give general necessary and sufficient conditions for this decomposition to exist in arbitrary real finite dimensional Lie algebras and discuss various contexts and examples where this decomposition obtains, particularly examples related to contraction semigroups.

1. Symmetric Lie Algebras

In this section we review some elementary and standard properties of Lie algebras equipped with an involution.

Let $\mathfrak{g}$ be a Lie algebra equipped with a Lie algebra involution σ; we call the pair $(\mathfrak{g},\sigma)$ a *symmetric Lie algebra*, or more briefly a *symmetric algebra*. We set

$$\mathfrak{g}_\pm := \{X \in V : \sigma(X) = \pm X\},$$

the eigenspaces for ± 1. The pair $(\mathfrak{g}_+,\mathfrak{g}_-)$ is called the *canonical decomposition* of $\mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$ with projections from $\mathfrak{g}$ to $\mathfrak{g}_\pm$ given by $\pi_\pm(X) = \frac{1}{2}\big(X \pm \sigma(X)\big)$.

For a symmetric algebra $(\mathfrak{g},\sigma)$, set $\sigma^\sharp := -\sigma$, and denote $\sigma^\sharp(X)$ by $X^\sharp$. The next proposition summarizes straightforward equivalent formulations of a symmetric algebra in terms of the canonical decomposition and the involution $\sigma^\sharp$.

Proposition 1.1. *Let $(\mathfrak{g},\sigma)$ be a symmetric algebra.*

(i) *The canonical decomposition $\mathfrak{g} = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$ satisfies $[\mathfrak{g}_+,\mathfrak{g}_\pm] \subseteq \mathfrak{g}_\pm$ and $[\mathfrak{g}_-,\mathfrak{g}_\pm] \subseteq \mathfrak{g}_\mp$. Conversely if a decomposition $\mathfrak{g} = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$ satisfies these two conditions, then the corresponding involution $\sigma(x+y) = x-y$ for $x \in \mathfrak{g}_+, y \in \mathfrak{g}_-$ is the unique involutive automorphism of $\mathfrak{g}$ that yields the given decomposition.*

(ii) *The mapping $y \mapsto y^\sharp = \sigma^\sharp(y)$ is an involutive antiautomorphism. We have $y = y^\sharp \Leftrightarrow y \in \mathfrak{g}_-$ and $y^\sharp = -y \Leftrightarrow y \in \mathfrak{g}_+$. Conversely given an involutive antiautomorphism $y \mapsto y^\sharp : \mathfrak{g} \to \mathfrak{g}$, there exists a unique*

$\sigma\colon \mathfrak{g} \to \mathfrak{g}$, namely $\sigma(y) = -y^\sharp$, such that $(\mathfrak{g}, \sigma)$ is a symmetric space with associated involutive antiautomorphism $y \mapsto y^\sharp$. ■

We turn now to an analogous notion for groups.

Definition 1.2. An *involutive group* is a pair $(G, \widehat{\sigma})$, where G is a Lie group and $\widehat{\sigma}$ is an involutive automorphism on G. For $g \in G$, we define $g^* = \widehat{\sigma}(g)^{-1} = \widehat{\sigma}(g^{-1})$. We also set $G_+ := \{g \in G\colon \widehat{\sigma}(g) = g\}$ and $G_- := \{g \in G\colon g^* = g\}$.

We observe

Remark 1.3. For an involutive group $(G, \widehat{\sigma})$, the following observations are immediate.

(i) The mapping $g \mapsto g^*$ is an involutive antiautomorphism on G. The mappings $g \mapsto \widehat{\sigma}(g)$ and $g \mapsto g^*$ generate each other by composing with the inverse and hence one uniquely determines the other.

(ii) G_+ is a closed subgroup of G, and G_- is a closed subset of G which is closed under inversion. If $x, y \in G_-$ and commute with each other, then their product is again in G_-.

A involutive group always gives rise to a symmetric algebra, and the converse holds in the simply connected case.

Proposition 1.4. *If $(G, \widehat{\sigma})$ is an involutive group, then $(\mathfrak{g}, \sigma)$ is a symmetric algebra, where $\sigma = d\widehat{\sigma}(e)$. We have the following commutative diagrams:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \ \sigma\ \ } & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\ \ \widehat{\sigma}\ \ ]{} & G \end{array} \qquad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ x \mapsto x^\sharp\ } & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\ g \mapsto g^*\ ]{} & G \end{array}$$

Thus $\widehat{\sigma}(\exp(Y)) = \exp(Y)$ if $\sigma(Y) = Y$ and $\exp(Y)^* = \exp(Y)$ if $Y = Y^\sharp$. Also

$$\mathfrak{L}(G_\pm) := \{Y \in \mathfrak{L}(G)\colon \exp(tY) \in G_\pm \text{ for all } t \geq 0\} = \mathfrak{g}_\pm.$$

The subgroup generated by $\exp(\mathfrak{g}_+)$ is the identity component of G_+, hence closed.

Conversely let $(\mathfrak{g}, \sigma)$ be a symmetric algebra, and let $\exp\colon \mathfrak{g} \to G$, where G is a simply connected Lie group. Then there exists an unique $\widehat{\sigma}\colon G \to G$ such that $(G, \widehat{\sigma})$ is an involutive group and $d\widehat{\sigma} = \sigma$. ■

2. Examples and Constructions

Proposition 2.1. *Let A be an associative unital Banach algebra, and let σ (resp. $\sigma^\sharp$) be an involutive algebra automorphism (resp. antiautomorphism) on A. Let $G(A)$ denote the group of invertible elements of A, and $\exp : A \to G(A)$ the exponential mapping. Then $\big(G(A), \sigma|G(A)\big)$ (resp. $\big(G(A), \sigma^\sharp|G(A)\big)$) is the involutive group with corresponding symmetric algebra (A, σ) (resp. $(A, \sigma^\sharp)$) (where A is the Lie algebra equipped with the commutator product).* ■

We consider now some examples.

Example 2.2. Let $\mathfrak{g} = M_n(\mathbb{F})$ denote the Lie algebra of $n \times n$ matrices over $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The matrix antiautomorphism $A \mapsto A^*$ (the transpose for $\mathbb{R}$ and conjugate transpose for $\mathbb{C}$ and $\mathbb{H}$) is an involutive algebra antiautomorphism and gives rise (via Proposition 1.1) to a symmetric algebra structure on $\mathfrak{g}$. For $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{C}$) the space $\mathfrak{g}_-$ is the space of symmetric (resp. Hermitian) matrices and $\mathfrak{g}_+$ is the space of skew-symmetric (resp. skew-Hermitian) matrices.

Let $G = \mathrm{GL}_n(\mathbb{F})$ be the group of invertible $n \times n$ matrices. By Proposition 2.1 the matrix antiautomorphism $A \mapsto A^*$ restricted to G is the mapping $g \mapsto g^*$ on G. The corresponding involutive automorphism $\widehat{\sigma}$ is then the composition with inversion, $A \mapsto (A^*)^{-1}$. For $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{C}$) the group G_+ is then the set of orthogonal (resp. unitary) matrices and G_- is the set of invertible symmetric (resp. hermitian) matrices. ■

Example 2.3. Let $(V, \langle,\rangle)$ denote a vector space equipped with $\langle,\rangle$, a nondegenerate indefinite symmetric (resp. sesquilinear) form. Let $A^\sharp$ denote the adjoint of A with respect to the form. Then $A \mapsto A^\sharp$ is again an involutive antiautomorphism for the algebra $\mathrm{gl}(V)$. Again applying Propositions 1.1 and 2.1, we conclude $A \mapsto A^\sharp$ is an involutive antiautomorphism for a symmetric structure on $\mathrm{gl}(V)$. The canonical decomposition is given by the spaces of skew-symmetric ($\mathrm{gl}(V)_+$) and self-adjoint ($\mathrm{gl}(V)_-$) operators, the mapping $A \mapsto A^\sharp$ cut down to $\mathrm{GL}(V)$ is the corresponding antiautomorphism at the group level, the group G_+ is the set of unitary (with respect to the given form) linear operators, and G_- is the set of invertible self-adjoint linear operators. ■

We consider now an important general construction.

Example 2.4. Let (V, τ) be a vector space equipped with a vector space involution τ. Define $\sigma\colon \mathrm{gl}(V) \to \mathrm{gl}(V)$ by $\sigma(A) = \tau \circ A \circ \tau$. Then

$$\sigma(AB) = \tau AB\tau = \tau A\tau\tau B\tau = \sigma(A)\sigma(B).$$

Thus σ is an algebra and hence Lie algebra homomorphism. Also $\sigma^2(A) = \tau^2 A\tau^2 = A$, so σ is an involution. Thus $(\mathrm{gl}(V), \sigma)$ is a symmetric algebra. Then $A^\sharp = -\sigma A = -\tau A\tau$.

Suppose $\sigma(A) = A$. Then $\tau A\tau = A$, and applying τ to both sides we obtain $\tau A = A\tau$. Thus A is an endomorphism of the involutive space (V, τ). This is equivalent to $A(V_\pm) \subseteq V_\pm$. Since $A^\sharp = -\sigma(A)$, this is also equivalent to $A^\sharp = -A$. Similarly $A^\sharp = A$ if and only if $A(V_\pm) \subseteq V_\mp$.

By Proposition 2.1 $\mathrm{GL}(V)$ has corresponding involutive automorphism the restriction of σ to $\mathrm{GL}(V)$. We then have for A invertible, $A^* = \sigma(A)^{-1} = \tau A^{-1}\tau$.

An important special case of the preceding arises when the involutive vector space is itself a symmetric Lie algebra $(\mathfrak{g}, \tau)$. We then have the Lie algebra homomorphism $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$. Then for $x, y \in \mathfrak{g}$,

$$\sigma(\mathrm{ad}x)(y) = \tau\big(\mathrm{ad}(x)\tau(y)\big) = \tau([x, \tau y]) = [\tau x, y] = \mathrm{ad}(\tau x)(y).$$

Hence ad: $(\mathfrak{g}, \tau) \to (\mathrm{gl}(\mathfrak{g}), \sigma)$ is a morphism of symmetric algebras. Of course this holds if the codomain is cut down to any subalgebra containing the image of ad and invariant under σ.

If $\exp\colon \mathfrak{g} \to G$, and $g \in G$, we then have $\mathrm{Ad}(g)^* = \tau \mathrm{Ad}(g)^{-1}\tau = \tau \mathrm{Ad}(g^{-1})\tau$. ■

Example 2.5. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ denote the complexification of $\mathfrak{g}$. Then the operation of taking complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ is a Lie algebra involution. This involution lifts to the simply connected group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and by the construction of Example 2.4 it also lifts to its adjoint group $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. ■

3. The Decomposition Theorem

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\exp : \mathfrak{g} \to G$ the exponential mapping. Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. A *wedge* in $\mathfrak{g}$ is a closed convex set which is also closed under multiplication by non-negative scalars.

Lemma 3.1. *If $X \in \mathfrak{g}$ and* $\mathrm{ad}\,X$ *has real spectrum, then* exp *is regular at X.* ■

Lemma 3.2. *Let $\mathbf{W}$ be wedge in $\mathfrak{g}$ such that* $\mathrm{ad}\,X$ *has real spectrum for each $X \in \mathbf{W}$. The following are equivalent:*

(1) *The mapping* exp *restricted to $\mathbf{W}$ is injective.*

(2) *If $Z \in \mathfrak{z} \cap (\mathbf{W} - \mathbf{W})$ and* $\exp Z = e$, *then $Z = 0$.* ■

Remark. A function α defined on a manifold M is said to be a diffeomorphism when restricted to some subset A if α restricted to A is a homeomorphism onto $\alpha(A)$ and if its restriction to some open subset containing A is a local diffeomorphism onto some open subset of the codomain. If α is differentiable and regular at each point of A and its restriction to A is a homeomorphism, then it follows from the inverse function theorem that α restricted to A is a diffeomorphism.

Lemma 3.3. *Let $\mathbf{W}$ be a wedge in $\mathfrak{g}$ such that* exp *restricted to $\mathbf{W}$ is injective and* $\mathrm{ad}\,X$ *has real spectrum for each $X \in \mathbf{W}$. The following are equivalent:*

(1) *The mapping* exp *from $\mathbf{W}$ to* $\exp(\mathbf{W})$ *is a homeomorphism (diffeomorphism) and* $\exp(\mathbf{W})$ *is closed in G.*

(2) *For each non-zero $X \in \mathbf{W}$, the closure of* $\exp(\mathbb{R}X)$ *is not compact.*

(3) *For each non-zero $X \in \mathbf{W} \cap \mathfrak{z}$, the closure of* $\exp(\mathbb{R}X)$ *is not compact.* ■

One can define by the analytic functional calculus an analytic log function inverse to the exponential function from the linear operators on $\mathfrak{g}$ with positive real spectrum to the linear operators with real spectrum. Indeed in this case it can be defined directly by the formula

$$\log(T) = \log\bigl(\mathrm{Tr}(T)\bigr)I - \sum_{n=1}^{\infty} \frac{1}{n}(I - \mathrm{Tr}(T)^{-1}T)^n \tag{3.1}$$

where Tr stands for the trace and I for the identity (see [10], p. 172). This log function plays a crucial role in the proof of the preceding lemma.

Let M be a manifold, let A be a closed subset, and let X be a vector field on M. The Bony-Brezis Theorem states that if a trajectory for X begins in A, then it remains in A, provided $X(x)$ is a subtangent vector of A for every $x \in A$. A nice application of this theorem is given in [9, V.4.57,58] to derive the following result. (The result is stated there for pointed cones and connected H, but exactly the same proof holds for wedges and closed H.)

Theorem 3.4. *Let $(G, \widehat{\sigma})$ be a involutive Lie group, and let $H \subseteq G_+$ be a closed subgroup containing the identity component of G_+. Let $\mathbf{W}$ be a wedge in $\mathfrak{g}_-$ which is invariant under the adjoint action of H. Suppose further that*

(i) $\mathrm{ad}\, X$ *has real spectrum for* $X \in \mathbf{W}$*;*

(ii) *If* $Z \in \mathfrak{z} \cap (\mathbf{W} - \mathbf{W})$ *satisfies* $\exp Z = e$*, then* $Z = 0$*.*

Then if $S := (\exp W)H$ is closed, it is a semigroup with subtangent wedge $\mathfrak{L}(S) = \mathbf{W} \bigoplus \mathfrak{g}_+$ at the identity, and the mapping $(X, h) \mapsto (\exp X)h \colon \mathbf{W} \times H \to S$ is a diffeomorphism. In the case that H is connected, S is strictly infinitesimally generated. ∎

The earlier lemmas of this section allow a sharpening of this result.

Theorem 3.5. *Let $(G, \widehat{\sigma})$ be an involutive Lie group, and let $H \subseteq G_+$ be a closed subgroup containing the identity component of G_+. Let $\mathbf{W}$ be a wedge in $\mathfrak{g}_-$ which is invariant under the adjoint action of H and for which $\mathrm{ad}\, X$ has real spectrum for each $X \in \mathbf{W}$. Then the following conditions are equivalent:*

(1) $(X, h) \mapsto (\exp X)h \colon W \times H \to (\exp \mathbf{W})H$ *is a diffeomorphism onto a closed subset of* G.

(2) *The mapping* $\mathrm{Exp} \colon \mathfrak{g}_- \to G/H$ *defined by* $\mathrm{Exp}(X) = (\exp X)H$ *restricted to* $\mathbf{W}$ *is a diffeomorphism onto a closed subset of* G/H.

(3) *The mapping* $\exp$ *restricted to* $\mathbf{W}$ *is a diffeomorphism onto a closed subset of* G.

(4) (i) *If* $Z \in \mathfrak{z} \cap (\mathbf{W} - \mathbf{W})$ *satisfies* $\exp Z = e$*, then* $Z = 0$*.*

(ii) *For each non-zero* $X \in \mathbf{W} \cap \mathfrak{z}$*, the closure of* $\exp(\mathbb{R}\mathbf{W})$ *is not compact.*

If these conditions hold, then $S := (\exp \mathbf{W})H$ is a closed (and strictly infinitesimally generated in the case H is connected) semigroup with subtangent wedge $\mathfrak{L}(S) = \mathbf{W} \bigoplus \mathfrak{g}_+$ at the identity. ∎

Semigroups S arising as in Theorem 3.5 are called *Ol'shanskiĭ semigroups* and the factorization $S = (\exp \mathbf{W})H$ is called the *Ol'shanskiĭ decomposition.*

In [6] N. Dörr has shown that the conclusions of Theorem 3.4 hold in the case that the group $G_{\mathbb{C}}$ is the simply connected group corresponding to complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ (where the involution in $\mathfrak{g}_{\mathbb{C}}$ is conjugation, see Example 2.5). However, it now follows from Theorem 3.5 that one always obtains the appropriate factorization in the simply connected case.

Corollary 3.6. *Let $(G, \widehat{\sigma})$ be a simply connected involutive Lie group, and let $H \subseteq G_+$ be a closed subgroup containing the identity component of G_+.*

Let $\mathbf{W}$ *be a wedge in* $\mathfrak{g}_-$ *which is invariant under the adjoint action of* H *and for which* $\operatorname{ad} X$ *has real spectrum for each* $X \in \mathbf{W}$. *Then the conditions of Theorem 3.5 hold. In particular,* $(X, h) \mapsto (\exp X)h \colon \mathbf{W} \times H \to (\exp \mathbf{W})H$ *is a diffeomorphism onto a closed subsemigroup of* G. ∎

Example 3.7. Ol'shanskiĭ [13] gives the following fundamental example. Let $\mathfrak{g}$ be a simple hermitian Lie algebra over $\mathbb{R}$, and let $\mathfrak{g}^{\mathbb{C}}$ be the complexification with involution conjugation (see Example 2.5). Let $G_{\mathbb{C}}$ be a corresponding Lie group to which the involution lifts, and let G be the connected Lie subgroup corresponding to the subalgebra $\mathfrak{g}$ (G is the identity component of the fixed points of the involution). Now $\mathfrak{g}$ has a maximal invariant pointed cone $\mathbf{W}$, and $S := \exp(i\mathbf{W})G$ is an Ol'shanskiĭ semigroup in $G_{\mathbb{C}}$.

Now $X_0 := G/K$ (where K is the maximal compact subgroup of G) is a hermitian symmetric space which embeds in its compact dual symmetric space X_c as a bounded symmetric domain (Borel embedding theorem). The group $G_{\mathbb{C}}$ acts naturally on X_c; the corresponding transformations are precisely the elements of the connected component of the identity in the group of all holomorphic transformations of X_c. Ol'shanskiĭ shows that the semigroup S consists precisely of those transformations in $G_{\mathbb{C}}$ which carry X_0 into itself. ∎

We close with an observation on the method by which the factorization of Theorem 3.5 is obtained. This method arose in the proof of that theorem and is useful to have on record.

Proposition 3.8. *Assume the setting and hypotheses of Theorem 3.5, including the four equivalent conditions. The factorization of* $s \in S$ *is given by* $s = \exp(\frac{1}{2}X)h$, *where* X *is the unique member of* $\mathbf{W}$ *such that* $\exp X = ss^*$ *and* $h = \exp(-\frac{1}{2}X)s$. ∎

4. Expansion and Contraction Semigroups

Subsemigroups of Lie groups arise in a variety of contexts and have varied descriptions or definitions. In certain circumstances one would like to determine that the semigroup under consideration is an Ol'shanskiĭsemigroup, i.e., has a description along the lines of Theorems 3.4 and 3.5. In the case of involutive groups, a positive determination can sometimes be made by constructing a polar-like decomposition for members of the semigroup, which then turns out to be the Ol'shanskiĭ decomposition of the previous section. In this section we carry out this determination for general types of expansion and contraction semigroups.

Lemma 4.1. *Suppose that a Lie algebra* $\mathfrak{g}$ *has a faithful representation in* $gl(V)$, *where* V *is a finite dimensional real or complex vector space. If* $X \in \mathfrak{g}$ *has a real spectrum as an operator on* V, *then* $\operatorname{ad} X$ *has a real spectrum.* ∎

Lemma 4.2. *Let* V *be a finite dimensional real or complex vector space. If* $T \in \mathrm{GL}(V)$ *has positive real spectrum, then there exists a unique* $S \in \mathrm{gl}(V)$ *with real spectrum such that* $\exp(S) = T$, *namely* $S = \log(T)$, *where* $\log(T)$ *is given by formula (3.1).* ∎

Example 4.3. (*Expansion Semigroups*). Let V be a complex finite dimensional vector space equipped with a nondegenerate sesquilinear (i.e. Hermitian) form $\langle , \rangle$. We consider the semigroup

$$S^{\geq} := \{T \in \mathrm{GL}(V) \colon \langle Tx, Tx \rangle \geq \langle x, x \rangle \text{ for all } x \in V\},$$

the semigroup of length increasing or expansive operators. We list standard properties about this semigroup.

(1) Each operator T on V has an adjoint operator $T^\sharp$ satisfying

$$\langle Tx, y \rangle = \langle x, T^\sharp y \rangle \quad \text{for all } x, y \in V.$$

The mapping $T \mapsto T^\sharp$ is an involutive algebra antiautomorphism on $\mathrm{gl}(V)$ (see Example 2.3).

(2) The Lie algebra involution $A \mapsto -A^\sharp$ on $\mathfrak{g} := \mathrm{gl}(V)$ has for $\mathfrak{g}_+$ the set of skew-symmetric matrices and for $\mathfrak{g}_-$ the set of self-adjoint matrices (both with respect to $\langle , \rangle$) (see Propositions 1.1 and 2.1 and Example 2.3).

(3) The closed subgroup H of $\mathrm{GL}(V)$ consisting of those matrices which are unitary with respect to $\langle , \rangle$ are precisely the fixed points of the involution $T \mapsto (T^\sharp)^{-1}$ on $\mathrm{GL}(V)$ (see Proposition 2.1 and Example 2.3).

(4) The group H is the (unique) maximal subgroup of $S^{\geq}$.

(5) Recall that the subtangent set of the semigroup $S^{\geq}$ is defined by

$$\mathfrak{L}(S^{\geq}) := \{X \in \mathrm{gl}(V) : \exp(tX) \in S^{\geq} \text{ for all } t \geq 0\}.$$

Then

$$\mathfrak{L}(S^{\geq}) = \{X \in \mathrm{gl}(V) : \langle (X + X^\sharp)v, v \rangle \geq 0 \text{ for all } v \in V\}.$$

(6) Since H acting by inner automorphisms on $\mathrm{GL}(V)$ leaves invariant both $S^{\geq}$ and $\mathrm{GL}(V)_-$, the adjoint action of H on $\mathrm{gl}(V)$ leaves invariant their (sub)tangent sets $\mathfrak{L}(S^{\geq})$ and $\mathfrak{g}_-$ (see Proposition 1.4).

(7) Section V.1 of [9] summarizes basic facts about the subtangent set $\mathfrak{L}(S)$ of a semigroup S; in particular, $\mathfrak{L}(S)$ is always a wedge. We set $\mathbf{W} := \mathfrak{L}(S^{\geq}) \cap \mathfrak{g}_-$; then $\mathbf{W}$ is also invariant under the adjoint action of H. Since H is the maximal subgroup of $S^{\geq}$ and $\mathfrak{L}(H) = \mathfrak{g}_+$, it follows that $\mathbf{W}$ is a pointed cone. Since members of $\mathfrak{g}_-$ are self-adjoint, it follows from (5) that

$$\mathbf{W} = \{X \colon X = X^\sharp, \langle Xv, v \rangle \geq 0 \text{ for all } v \in V\},$$

i.e., $\mathbf{W}$ consists of all self-adjoint positive semidefinite operators. Note that $\mathfrak{L}(S^{\geq}) = \mathbf{W} + \mathfrak{g}_+$, a direct sum.

(8) If $A = A^\sharp$ and A is positive semidefinite, then A has real spectrum (see p. 35, p. 147 of [1]). Hence by Lemma 4.1 and (7), if $A \in \mathbf{W}$, then $\mathrm{ad}(A)$ has real spectrum.

(9) If $0 \neq Z \in \mathfrak{z}$, the center of $\mathrm{gl}(V)$, then Z is a scalar operator. If also $Z = Z^\sharp$, then the eigenvalues of Z come in conjugate pairs (Proposition 2.4 of [7]); thus the scalar must be real and non-zero. It then follows directly that condition (4) of Theorem 3.5 is satisfied.

(10) From the preceding results, the hypotheses of Theorem 3.5 are satisfied, and $S := \exp(\mathbf{W})H$ is an Ol'shanskiĭ semigroup. It follows from parts (3), (5), and (7) that $S \subseteq S^{\geq}$.

(11) We recall the principal results from Section 3 of [4], which are derived via the spectral decomposition of length increasing (or positive semidefinite) self-adjoint transformations with respect to a given non-degenerate sesquilinear form. Let $T \in S^{\geq}$. Then $TT^{\sharp}$ is self-adjoint, has positive spectrum, and has a unique self-adjoint square root T_0 with positive spectrum. There exists an unique unitary $U \in H$ and an unique self-adjoint operator in $S^{\geq}$, namely T_0, such that T has a (polar) decomposition of the form $T = T_0 U$. Furthermore, there exists an unique positive semidefinite self-adjoint operator A such that $\exp A = T_0$. By step (7), we have $A \in \mathbf{W}$. Thus $T = (\exp A)U$ is the Ol'shanskiĭ decomposition for T and $T \in \exp(\mathbf{W})H = S$, the Ol'shanskiĭ semigroup. Hence the Ol'shanskiĭ semigroup S is equal to $S^{\geq}$.

(12) Let $T \in S^{\geq}$. Then $T = \exp(\frac{1}{2}X)U \in \exp(\mathbf{W})H$, where X can be computed by $X = \log(TT^*)$, where the logarithm is computed by formula (3.1). Thus one has a specific algorithm for computing the Ol'shanskiĭ decomposition.■

Analogous results hold for non-degenerate bilinear forms on real vector spaces. By passing to inverses, length increasing transformations become length decreasing transformations (contractions), and this time we state our results in the dual (via inversion) context.

Example 4.4. Let V be a finite-dimensional real vector space and $\langle , \rangle$ be a non-degenerate symmetric bilinear form on V. Then $\mathrm{gl}(V)$ is a symmetric Lie algebra (see Example 2.3). The contraction semigroup

$$S^{\leq} := \{T \in \mathrm{GL}(V) \colon \langle Tx, Tx \rangle \leq \langle x, x \rangle \text{ for all } x \in V\},$$

is an Ol'shanskiĭ semigroup, and each $T \in S^{\leq}$ has a unique factorization in the form $T = (\exp A)U$, where A is in the cone $\mathbf{W}$ of self-adjoint negative semidefinite operators and U is a member of the group of orthogonal operators preserving the bilinear form. ■

The next example appears in Section 1.4 of [2]. (see also [12] and the section on contraction semigroups, pp. 434ff, in [9], particularly Example V.4.55). It is a specific case of Example 4.4, but the derivation is much more elementary and straightforward.

Example 4.5. Let S be the subsemigroup of expansive matrices with positive determinant of $\mathrm{GL}(n, \mathbb{R})$ consisting of all matrices P such that $(Px, Px) \geq (x, x)$ for all $x \in \mathbb{R}^n$, where (x, y) denotes the usual Euclidean inner product on $\mathbb{R}^n$. Then the tangent wedge is given by $\mathfrak{L}(S) = \{A \colon A + A^T \text{ is non-negative definite}\}$. The algebra anti-involution $A \mapsto A^T$ on $\mathrm{gl}(n, \mathbb{R})$ leaves fixed the symmetric matrices; thus the skew-symmetric matrices are the fixed points of the involution $A \mapsto -A^T$, and these matrices form the maximal subspace of $\mathfrak{L}(S)$. Note that if $\mathbf{W}$ denotes the intersection of $\mathfrak{L}(S)$ and the set of symmetric matrices, then $\mathbf{W}$ consists of all non-negative semidefinite symmetric matrices. One verifies that $\mathbf{W}$ is a cone which is invariant under the adjoint action of the orthogonal group. Since each member of W is symmetric, it has a real spectrum as an operator on $\mathbb{R}^n$, and hence has a real spectrum under the adjoint representation *ad.* Thus $\exp \mathbf{W} \cdot \mathrm{SO}(n)$ is an Ol'shanskiĭ semigroup, where SO(n) denotes the special orthogonal group of orthogonal matrices of determinant 1. Finally by considering the polar decomposition $P = R\Theta$ of any expansive matrix in S, where R is symmetric and Θ in orthogonal, we conclude that $R = P\Theta^{-1}$ is expansive, and hence in the exponential image of W (the argument involves diagonalizing R by an orthogonal matrix).

5. Restricted Decompositions

We saw in Example 4.4 that the type of polar decompositions that we have been considering for contractions and expansions, namely the Ol'shanskiĭ decompositions, carry over when one restricts from the complex to the real case. These decompositions maintain under much more general types of restrictions.

We recall the following definition from Section XV, Chapter IV of [5] (see also [8, p. 449]). Let Q be a subset of the general linear group $\mathrm{GL}(n,\mathbb{C})$. Let g_{ij} $(1 \le i,j \le n)$ denote the matrix elements of $g \in \mathrm{GL}(n,\mathbb{C})$, and let $x_{ij}(g)$ and $y_{ij}(g)$ be the real and imaginary parts of g_{ij}. The subset Q is called a *pseudoalgebraic* subset of $\mathrm{GL}(n,\mathbb{C})$ if there exists a set of polynomials P_β in $2n^2$ arguments such that $g \in Q$ if and only if $g \in \mathrm{GL}(n,\mathbb{C})$ and $P_\beta(\ldots x_{ij}(g), y_{ij}(g), \ldots) = 0$ for all P_β. The following lemma is a variant of part of Lemma 2.3, Chapter X of [8].

Lemma 5.1. *Let Δ be a pseudoalgebraic subset of $\mathrm{GL}(n,\mathbb{C})$ satisfying $g \in \Delta$ implies $g^n \in \Delta$ for all $n \ge 1$. If a matrix A has all real eigenvalues and $\exp A \in \Delta$, then $\exp tA \in \Delta$ for all $t \in \mathbb{R}$.* ■

In the following proposition we adopt the setting and notation of Theorem 3.5.

Theorem 5.2. *Let $(G,\widehat{\sigma})$ be an involutive Lie group which is a subgroup of $\mathrm{GL}(n,\mathbb{C})$, and let $H \subseteq G_+$ be a closed subgroup containing the identity component of G_+. Let $\mathbf{W}$ be a wedge in $\mathfrak{g}_-$ which is invariant under the adjoint action of H and for which each matrix $X \in \mathbf{W}$ has real spectrum. Suppose further that*

(i) *If $Z \in \mathfrak{z} \cap (\mathbf{W} - \mathbf{W})$ satisfies $\exp Z = e$, then $Z = 0$.*

(ii) *For each non-zero $X \in \mathbf{W} \cap \mathfrak{z}$, the closure of $\exp(\mathbb{R}\mathbf{W})$ is not compact.*
Then $S := (\exp \mathbf{W})H$ is a closed (and strictly infinitesimally generated in the case H is connected) semigroup and the mapping $(X,h) \mapsto (\exp X)h \colon W \times H \to S$ is a diffeomorphism onto S.

Let Γ be a subgroup of G which is closed under the involution and is a pseudoalgebraic subset. Then $S \cap \Gamma = \big(\exp(\mathbf{W} \cap \mathfrak{L}(\Gamma)\big)(\Gamma \cap H)$ is also an Ol'shanskiĭ semigroup and the factorization on the right is the Ol'shanskiĭ factorization (where $\mathfrak{L}(\Gamma)$ denotes the Lie algebra of Γ). ■

Example 5.3. This example appears in [4] and arose originally in problems of extending representations on real sympletic groups to complex sympletic groups. Let E denote a complex even-dimensional vector space equipped with a non-degenerate Hermitian form $\langle,\rangle$ and a non-degenerate symplectic form $\{,\}$ which are related by having a common orthonormal basis $\{f_i : 1 \le i \le 2m\}$, that is, a basis satisfying

$$\langle f_i, f_i\rangle = 1, \qquad \langle f_{i+m}, f_{i+m}\rangle = -1, \qquad 1 \le i \le m,$$

and

$$\{f_i, f_{i+m}\} = 1 = -\{f_{i+m}, f_i\}, \qquad 1 \le i \le m,$$

are the only non-zero terms.

We consider the group $G := \mathrm{GL}(E)$ equipped with the adjoint anti-involution $A \mapsto A^*$ for the Hermitian form. From Example 4.3 we know that the semigroup $S^{\geq}$ is an Ol'shanskiĭ semigroup of the form $(\exp \mathbf{W})U(m,m)$, where $\mathbf{W}$ consists of all positive semidefinite transformations and $U(m,m)$ is the group of $\langle,\rangle$-isometries. Let $\mathrm{Sp}(2m,\mathbb{C})$ denote the group of matrices preserving the sympletic form $\{,\}$. A direct calculation establishes that $\mathrm{Sp}(2m,\mathbb{C})$ is closed under the given anti-involution $A \mapsto A^*$ on G. It is clear that the matrices preserving the sympletic form are defined by a set of equations, hence pseudo-algebraic. Thus by Theorem 5.2 the semigroup $S^{\geq} \cap \mathrm{Sp}(2m,\mathbb{C})$ is an Ol'shanskiĭ semigroup with factorization $\exp(\mathrm{sp}(2m,\mathbb{C}) \cap \mathbf{W})\big(\mathrm{sp}(2m,\mathbb{C}) \cap U(n,n)\big)$. ■

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