Seminar Sophus Lie
1 (1991) 47–54

A short course on the Lie theory of semigroups III
Globality of invariant wedges

Karl-Hermann Neeb

Let G be a connected Lie group, $\mathbf{g} = \mathbf{L}(G)$ its Lie algebra and $\exp : \mathbf{g} \to G$ the corresponding exponential function. In Part I of this sequence we have defined the *tangent wedge* of a subsemigroup $S \subseteq G$ to be the set of *subtangent vectors* $L_{\mathbf{1}}(S)$ in the unit element $\mathbf{1}$. Since this set is the same for the closed semigroup $\overline{S}$ we will restrict our attention in the following to closed subsemigroups of G. For a closed subsemigroup the *tangent wedge* $L_{\mathbf{1}}(S)$ agrees with the wedge

$$\mathbf{L}(S) = \{X \in \mathbf{g} : \exp(\mathbb{R}^+ X) \subseteq S\}$$

of *infinitesimal generators* of S. We say that a closed subsemigroup S of G is a *Lie semigroup* if it is reconstructable from its tangent wedge $\mathbf{L}(S)$, i. e. if

$$S = \overline{\langle \exp \mathbf{L}(S) \rangle}.$$

We have also seen that $\mathbf{L}(S)$ is a *Lie wedge* in the Lie algebra $\mathbf{g}$ and that, in general, not every Lie wedge $W \subseteq \mathbf{g}$ is the tangent wedge of a subsemigroup S of G. Those who are will be called *global in G*.

Note that the subalgebras $\mathbf{a}$ of $\mathbf{g}$ which are global in G are exactly those which are tangent to closed subgroups of G. Nevertheless there exists for every subalgebra $\mathbf{a}$ of $\mathbf{g}$ a Lie group A such that $\mathbf{a} \cong \mathbf{L}(A)$ (Lie's Third Theorem).

The situation in the semigroup case is more complicated. As we will see in the following there exist Lie semialgebras W, even invariant wedges, which are not global in any Lie group G with $W \subseteq \mathbf{L}(G)$, where the inclusion $W \to \mathbf{L}(G)$ is compatible with the local semigroup structure on a neighborhood of 0 in W (cf. Part II of this sequence).

In this third part we will consider the globality problem for *invariant Lie semigroups*, i. e. Lie semigroups which are invariant under all inner automorphisms of G. It follows directly from the definitions that the pairs (S, G), S an invariant Lie semigroup in G, are in one-to-one correspondence with the pairs $(W, \mathbf{g})$, W an invariant wedge in $\mathbf{g}$ which is global in G. So the problem is to characterize those invariant wedges $W \subseteq \mathbf{g}$ which are global in G. We will mainly concentrate on simply connected Lie groups G because all the information about the simply connected group associated with a given Lie algebra $\mathbf{g}$ is contained in the algebraic structure of $\mathbf{g}$, and we may reasonably hope that in this case one can find algebraic and geometric characterizations of the global wedges $W \subseteq \mathbf{g}$.

If W is a subalgebra then it is an ideal of $\mathbf{g}$ and the answer is simple.

Lemma 1. *Let G be simply connected Lie group then every ideal $\mathbf{a} \subseteq \mathbf{g}$ is global in G.*

Proof. This follows from the fact that normal analytic subgroups of simply connected Lie groups are closed ([3, p.135]). ■

This lemma shows how we can reduce the problem.

Proposition 2. *Let G be a simply connected Lie group, $W \subseteq \mathbf{g}$ an invariant wedge, $\mathbf{a} := W - W$, and $H(W) = W \cap (-W)$ the edge of W. Then the following assertions hold:*

a) *$\mathbf{a}$ is an ideal in $\mathbf{g}$, W is an invariant wedge in $\mathbf{a}$, the subgroup $A := \langle \exp \mathbf{a} \rangle$ is closed and simply connected, and W is global in G if and only if it is global in A.*

b) *$H(W)$ is an ideal in $\mathbf{g}$, $W/H(W)$ is an invariant wedge in the Lie algebra $\mathbf{g}/H(W)$, the normal subgroup $H := \langle \exp H(W) \rangle$ is closed, the quotient group G/H is simply connected, and W is global in G if and only if $W/H(W)$ is global in G/H.*

Proof. [3, p.135], [6, Prop. III.9]. ■

The Infinitesimal Part: Invariant Cones in Lie Algebras

According to the preceeding proposition we may restrict our attention to the case where the invariant wedge $W \subseteq \mathbf{g}$ satisfies the following two additional conditions:

1) W is *generating*, i.e. $W - W = \mathbf{g}$. This is equivalent to say that the interior $\operatorname{int} W$ of W is non-empty.

2) W is *pointed*, i.e. $H(W) = \{0\}$.

Therefore the first step to the globaliy problems is a detailed analysis of the infinitesimal situation. What can be said about the pairs $(W, \mathbf{g})$, where W is a pointed generating invariant cone in the Lie algebra $\mathbf{g}$?

Indeed, the existence of a pointed generating invariant cone W has strong structural consequences for the Lie algebra $\mathbf{g}$ as is shown by the following theorem.

Theorem 3. *Let W be a pointed generating invariant cone in a finite dimensional real Lie algebra $\mathbf{g}$. Then the following assertions hold:*

a) *$\mathbf{g}$ contains a compactly embedded Cartan algebra $\mathbf{h}$.*

b) *For every compactly embedded Cartan algebra $\mathbf{h}$ we have that*

$$\operatorname{int} W = \langle e^{\operatorname{ad} \mathbf{g}} \rangle \operatorname{int}_{\mathbf{h}}(W \cap \mathbf{h}).$$

Proof. a) [2, Theo. III.2.14], b) [2, Theo. III.2.15]. ■

As the example of the Lie algebra of the group of euclidean motions of the plane shows, there exist Lie algebras with compactly embedded Cartan algebras which do not contain any pointed generating invariant cone. So one has to look for additional properties of these Lie algebras. The main method to do this rests on a real root decomposition of $\mathbf{g}$ with respect to a compactly embedded Cartan algebra. Before we can state the Characterization Theorem we have to describe briefly this root decomposition to give the necessary definitions.

Let us consider the adjoint action of a fixed compactly embedded Cartan algebra $\mathbf{h}$ of a Lie algebra $\mathbf{g}$. Then, since $\mathbf{h}$ is compactly embedded, the Lie algebra decomposes into a direct sum of vector subspaces

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus \{\mathbf{g}^{|\omega|} : \omega \in \mathbf{h}^*\},$$

where $\mathbf{g}^{|\omega|}$ are the isotypical components of the $\mathbf{h}$-action and

$$\mathbf{g}^{|\omega|} = \{X \in \mathbf{g} : \big[H, [H, X]\big] = -\omega(H)^2 X \ \forall H \in \mathbf{h}\}.$$

We write Ω for the set of all non-zero linear functionals ω on $\mathbf{h}$ for which $\mathbf{g}^{|\omega|} \neq \{0\}$. Then $\Omega = -\Omega$ and we say that Ω^+ is a *positive system of roots* if there exists an open half space E in $\mathbf{h}^*$ whose boundary does not meet Ω and for which $\Omega^+ = \Omega \cap E$. To every choice of a positive system corresponds a complex structure I on the root spaces such that

$$[H, X] = \omega(H) I X \qquad \forall X \in \mathbf{g}^{|\omega|}, H \in \mathbf{h}, \omega \in \Omega^+.$$

Let us denote the radical of the Lie algebra $\mathbf{g}$ with $\mathbf{r}$. Then one finds that $[IX, X]$ is contained in the center $Z(\mathbf{r})$ of $\mathbf{r}$ whenever $X \in \mathbf{g}^\omega \cap \mathbf{r}$ ([2, Cor. III.6.24]).

Definition 4. We say that a Lie algebra $\mathbf{g}$ with a compactly embedded Cartan algebra $\mathbf{h}$ has *strong cone potential* if there exists a linear functional ν on $Z(\mathbf{r})$ and a positive system Ω^+ of roots such that

$$\langle \nu, [IX, X] \rangle > 0 \qquad \forall X \in \mathbf{g}^\omega \cap \mathbf{r} \setminus \{0\}, \omega \in \Omega^+.$$

■

Theorem 5. (Characterization Theorem for Lie algebras with invariant cones) *A finite dimensional Lie algebra $\mathbf{g}$ contains a pointed generating invariant cone W if and only if the following conditions are satisfied:*

i) $\mathbf{g}$ *ist not compact semisimple.*

ii) *The semisimple Lie algebra $\mathbf{g}/\mathbf{r}$ contains only compact or hermitean simple ideals.*

iii) $\mathbf{g}$ *has strong cone potential.*

Proof. [9, Theorem III.36]. ■

In the case of simple Lie algebras this theorem is already contained in [16]. In [12] and [13] one finds explicit descriptions of the invariant cones in hermitean simple Lie algebras and in [14] of the invariant cones in solvable Lie algebras. Invariant cones in general Lie algebras have been studied in [2], [15], [17], and [9].

The Global Part: Invariant Lie Semigroups

As we have already mentioned above we will focus our attention mainly on simply connected Lie groups. Therefore, in the whole section, G denotes a simply connected Lie group.

In view of Proposition 2, we know that the question whether an invariant wedge W with $H(W) \neq \{0\}$ is global or not may be reduced to the corresponding problem for a pointed cone of lower dimension. So one first looks for a converse of this in the sense that one tries to obtain globality information for an invariant cone W from globality information for a cone V of lower dimension or, equivalently, with higher dimensional edges. The main outgrowth of this idea is given in the following theorem:

Theorem 6. (The Reduction Theorem) *Let G be a simply connected Lie group and $W_1 \subseteq W_2$ invariant wedges. Then in each of the two following cases the globality of W_2 in G implies the globality of W_1 in G:*

i) $W_1 \cap H(W_2) \subseteq H(W_1)$.

ii) $H(W_2)$ *is a nilpotent ideal.*

Proof. The first part of the theorem is a special case Proposition IV.25 in [9] (cf. [2, Cor. VI.5.2], [6, Prop. III.1]). It rests on the method of positive functions which, based on ideas in [2, Chapter VI], is developed in [6].

The second part is Theorem VIII.7 in [9]. The main ingredients in its proof are the method of positive functions, an extension of the Baker-Campbell-Hausdorff multiplication to a uniform neighborhood of the nilpotent normal subgroup $\exp H(W_2)$ (cf. [4]), and the fact that W_1 is a Lie semialgebra. ∎

Since the quotient of a Lie algebra $\mathbf{g}$ modulo its nilradical $\mathbf{n}$ is a reductive Lie algebra, the second part of the Reduction Theorem may be used in some cases to reduce the globality problem to the reductive case.

Lemma 7. *Let W be a pointed generating invariant cone in the Lie algebra $\mathbf{g}$, G the associated simply connected Lie group, $\mathbf{n}$ the nilradical of $\mathbf{g}$ and $\mathbf{k}$ the sum of all compact ideals of $\mathbf{g}$. Then $\mathbf{a} := \mathbf{n} + \mathbf{k}$ is an ideal of $\mathbf{g}$, the wedge $V := \overline{W + \mathbf{a}}/\mathbf{a}$ is a pointed generating invariant cone in $\mathbf{g}/\mathbf{a}$, and W is global in G if V is global in $G/\langle \exp \mathbf{a} \rangle$.*

Proof. ([8, Lemma VIII.8]) The main part of the proof consists in applications of the geometric theory of invariant cones in Lie algebras and modules of compact groups ([9, Sections I, III]). Then one uses the second part of the Reduction Theorem. ∎

After this partial reduction to reductive algebras we consider their building blocks, the simple ideals.

Definition 8. We say that a hermitean simple Lie algebra $\mathbf{g}$ is of *tubular type* if for a Cartan decomposition $\mathbf{g} = \mathbf{k} + \mathbf{p}$ the symmetric domains $G/\exp \mathbf{k}$

are of tubular type (cf. [5]). Among the hermitean simple Lie algebras these are the following classical algebras

$$\mathrm{su}(n,n),\ \mathrm{so}(n+2,2),\ \mathrm{sp}(n,\mathbb{R}),\ \mathrm{so}^*(4n+4) \quad \text{for} \quad n \in \mathbb{N}$$

and the exceptional algebra $e_{7(-25)}$ of type E_7. Note that

$$\mathrm{sl}(2,\mathbb{R}) \cong \mathrm{su}(1,1) \cong \mathrm{so}(1,2), \quad \text{and} \quad \mathrm{so}(2,2) \cong \mathrm{sl}(2,\mathbb{R})^2.$$

■

The globality question in simple Lie groups was already satisfatorily answered by Ol'shanskiĭ and Vinberg ([11], [Vi80]).

Theorem 9. (Ol'shanskiĭ) *Let G be a simply connected simple Lie group and $\mathbf{g}$ its Lie algebra. Then one of the following two cases occurs:*

i) *If $\mathbf{g}$ is of tubular type then all invariant wedges $W \subseteq \mathbf{g}$ are global in G.*

ii) *If $\mathbf{g}$ is not of tubular type then there exist invariant cones in $\mathbf{g}$ which are not global in G. Moreover, there exists an invariant cone W_0 such that an invariant cone W is global in G if and only if*

$$W \subseteq W_0 \quad or \quad -W \subseteq W_0.$$

■

Now we apply the reduction idea.

Definition 10. We say that a Lie algebra $\mathbf{g}$ is of *globality type* if the semisimple Lie algebra $\mathbf{g}/\mathbf{r}$ contains only compact ideals or ideals of tubular type. ■

Theorem 11. (The First Globality Theorem) *Let G be a simply connected Lie group and suppose that $\mathbf{g}$ is of globality type. Then every generating invariant wedge $W \subseteq \mathbf{g}$ is global in G.*

Proof. The proof given in [9, Theorem VIII.12] rests on the second part of the reduction theorem and on a generalization of Ol'shanskiĭ's result to the semisimple case ([9, Theorem VIII.10]). ■

Corollary 12. *Let G be a simply connected solvable Lie group and $W \subseteq \mathbf{g}$ an invariant wedge. Then W is global in G.* ■

This result was also obtained by Gichev ([1]) who also gave very explicit descriptions of the invariant semigroups which are generated by the invariant cones.

The greatest disadvantage of the First Globality Theorem is that it says nothing about Lie algebras which are not of globality type. It generalizes only the first half of Ol'shanskiĭ's theorem. To get a generalization of the second part one has to use techniques which are much more involved, namely the structure theorem of Lie algebras with strong cone potential ([9, Theorem II.38]) which is developed with the aid of Spindler's universal construction of Lie algebras with cone potential ([15]).

This leads to another reduction theorem:

Theorem 13. (The Second Globality Theorem) *Let W be a pointed generating invariant cone in $\mathbf{g}$ and $\mathbf{s}_0$ the maximal semisimple ideal of $\mathbf{g}$. Then $\mathbf{s}_0$ is a direct summand which is complemented by its centralizer. Let $\pi : \mathbf{g} \to \mathbf{s}_0$ the projection homomorphism onto $\mathbf{s}_0$. Then $V := \overline{\pi(W)}$ is a pointed generating invariant cone in $\mathbf{s}_0$ and W is global whenever V is global.* ∎

Corollary 14. *Let W be a pointed generating invariant cone in the Lie algebra $\mathbf{g}$ which contains no simple ideals. Then W is global in the associated simply connected Lie group G.* ∎

Note that the condition that $\mathbf{g}$ contains no simple ideals means that no ideal of a Levi subalgebra acts trivially on the radical. Therefore the corollary applies only in the non-reductive case. These are often the most complicated Lie algebras.

Together with Ol'shanskiĭ's theorem, the Second Globality Theorem can be used for a derivation of the following generalization of the second part of Theorem 9.

Theorem 15. (The Third Globality Theorem) *Let G be a simply connected Lie group and W a pointed generating invariant cone in $\mathbf{g}$. Then there exists a pointed generating invariant wedge $V \subseteq W$ which is global in G.* ∎

Continuous Orders on Lie Groups

The Third Globality Theorem permits a remarkable application to the problem of the characterization of Lie groups with continuous group orders.

Definition 16. A partial order $\leq$ on a Lie group G is said to be a *group order* if $a \leq b$ implies that

$$ga \leq gb \quad \text{and} \quad ag \leq bg \qquad \forall g \in G.$$

It is said to be *continuous* if the semigroup

$$S := \{g \in G : \mathbf{1} \leq g\}$$

of positive elements is closed and for every neighborhood U of $\mathbf{1}$ in G we have that

$$S = \overline{\langle S \cap U \rangle},$$

i.e. S is *locally generated*. We say that $\leq$ is *non-degenerate* if $\mathbf{1}$ is a cluster point of the interior of S. ∎

The connection to Lie semigroups is given by the following theorem.

Theorem 17. *For every continuous group order $\leq$ on G the semigroup of positive elements is an invariant Lie semigroup with trivial group of units, and, conversely, if $S \subseteq G$ is an invariant Lie semigroup with $H(S) = \{\mathbf{1}\}$ then the prescription $x \leq y$ if $x^{-1}y \in S$ defines a continuous group order $\leq_S$ on G such that S is the semigroup of positive elements. The order $\leq_S$ is non-degenerate if and only if the wedge $\mathbf{L}(S)$ is generating in $\mathbf{L}(G)$.*

Proof. Theorem II.12 and Corollary III.15 are in [8]. ■

Combining Theorem 17 with Theorem 15 we obtain:

Theorem 18. *A simply connected Lie group G admits a non-degenerate continuous group order if and only if there exists a pointed generating invariant cone in $\mathbf{L}(G)$.* ■

This result confirms the philosophy that every property of a simply connected Lie group G is reflected in a property of the corresponding Lie algebra. In view of Theorem 5 it yields a characterization of the simply connected Lie groups with non-degenerate continuous group orders in terms of algebraic and geometric properties of the Lie algebra.

If we drop the assumption that G is simply connected it is much more difficult to obtain results of this type and the methods leading to them are more technical. Nevertheless we have a general characterization of those solvable or reductive Lie groups which admit non-degenerate continuous group orders ([10]). In the solvable case our result extends these of Gichev ([1]) and in the reductive case we have generalized the result of Vinberg that a simple Lie group G admits a continuous non-trivial group order iff $\mathbf{g}$ is hermitean and the fundamental group $\pi_1(G)$ is finite.

References

[1] Gichev, V. M., *Invariant Orderings in Solvable Lie Groups*, Sib. Mat. Zhurnal **30**(1989), 57–69.

[2] Hilgert, J., K. H. Hofmann, and J. D. Lawson, "Lie Groups, Convex Cones, and Semigroups", Oxford University Press, 1989.

[3] Hochschild, G., "The Structure of Lie Groups", Holden Day, San Franzisco, 1965.

[4] Hofmann, K. H., *A memo on the singularities of the exponential function*, Preprint TH Darmstadt, 1990.

[5] Korányi, A., and J. A. Wolf, *Realization of Hermitean Symmetric Spaces as Generalized Half Planes*, Ann. Math. **82**(1965), 332–350.

[6] Neeb, K.–H., *The Duality Between Subsemigroups of Lie Groups and Monotone Functions* , Transactions of the Amer. Math. Soc., to appear.

[7] —, *Globality in Semisimple Lie Groups*, Annales de l'Institut Fourier, to appear.

[8] —, *On the Foundations of Lie Semigroups*, submitted.

[9] —, *Invariant Subsemigroups of Lie groups*, submitted.

[10] —, *Invariant Orders on Lie Groups and Coverings of Ordered Homogeneous Spaces*, submitted.

[11] Ol'shanskiĭ, G. I., *Invariant orderings in simple Lie groups. The solution to* E. B. VINBERG *'s problem*, Funct. Anal. and Appl. **16**(1982), 311–313.

[12] Paneitz, S., *Invariant convex cones and causality in semisimple Lie algebras and groups*, J. Funct. Anal. **43**(1981), 313–359.

[13] —, *Determination of invariant convex cones in simple Lie algebras*, Arkiv för Mat. **21**(1984), 217–228.

[14] Poguntke, D., *Invariant Cones in Solvable Lie Algebras*, Preprint 1990.

[15] Spindler, K., "Invariante Kegel in Liealgebren", Mitt. aus dem mathematischen Sem. Gießen, Heft 188, 1988.

[16] Vinberg, E. B., *Invariant cones and orderings in Lie groups*, Funct. Anal. and Appl. **14**(1980), 1–13.

[17] Zimmermann, U., *Invariante Kegel in Cartan Algebren*, Diplomarbeit, Technische Hochschule Darmstadt, 1990.

Département de Mathématiques
Analyse Complexe et Géometrie
Université Paris VI
4, place Jussieu, 75252 Paris Cedex 05

Received February 12, 1991