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Convexity theorems in Harmonic Analysis

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I. Kostant's Convexity Theorem

Let G be a connected semisimple Lie group, $\mathfrak{g} = \mathfrak{L}(G)$ its Lie algebra, and $B(X,Y) = \mathrm{tr}(\mathrm{ad}\, X \,\mathrm{ad}\, Y)$ its Cartan Killing form. Then a *Cartan involution* θ is an involutive automorphism of $\mathfrak{g}$ such that the bilinear form $B_\theta(X,Y) := -B(X, \theta Y)$ is positive definite. It is well know that Cartan involutions always exist ([6, p.185]). Fix such an involution θ. Then we set

$$\mathfrak{k} := \{X \in \mathfrak{g} : \theta(X) = X\} \quad \text{and} \quad \mathfrak{p} := \{X \in \mathfrak{g} : \theta(X) = -X\}.$$

Then $\mathfrak{k}$ is a compact subalgebra of $\mathfrak{g}$ and all operators $\mathrm{ad}\, X$, $X \in \mathfrak{p}$ are semisimple with real spectrum because they are selfadjoint with respect to B_θ ([6, p.184]). We choose a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. Since all Cartan involutions on $\mathfrak{g}$ are conjugate under inner automorphisms ([6, p.183]) and all maximal abelian subspaces of $\mathfrak{p}$ are conjugate under $e^{\mathrm{ad}\,\mathfrak{k}}$ ([6, p.247]), we don't loose any generality by making this choice. For a linear functional $\alpha \in \mathfrak{a}^*$ we set $\mathfrak{g}^\alpha := \{Y \in \mathfrak{g} : (\forall X \in \mathfrak{a})[X,Y] = \alpha(X)Y\}$. Since the operators $\mathrm{ad}\, X$, $X \in \mathfrak{a}$ commute and are semisimple, we have a direct vector space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha,$$

where $\Delta := \{\alpha \in \mathfrak{a}^* : \mathfrak{g}^\alpha \neq \{0\}\}$ is called the *system of restricted roots with respect to* $\mathfrak{a}$. Let $X_0 \in \mathfrak{a}$ be a regular element, i.e. an element of $\mathfrak{a}$ such that $\alpha(X_0) \neq 0$ for all non-zero elements of Δ (such an element exists because we only have to avoid a finite union of hyperplanes). A *positive system* $\Delta^+ \subseteq \Delta$ is a subset which is given by

$$\Delta^+ = \{\alpha \in \Delta : \alpha(X_0) > 0\},$$

where X_0 is a regular element. We fix a positive system $\Delta^+ \subseteq \Delta$ and define

$$\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha.$$

This is a nilpotent subalgebra of $\mathfrak{g}$ and we have the direct vector space decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

called the *Iwasawa decomposition of* $\mathfrak{g}$.

On the group level we have an involution, which we also denote with θ, whose differential agrees with the corresponding involution on the Lie algebra and which has the following properties:

1) $K := \langle \exp \mathfrak{k} \rangle$ agrees with the fixed point set of θ.
2) The mapping $K \times \exp \mathfrak{p} \to G, (k, X) \mapsto k \exp(X)$ is a diffeomorphism ([6, p.253]).

Now we set $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$. These are closed simply connected subgroups of G and the mapping

$$K \times A \times N \to G, \quad (k, a, n) \mapsto kan$$

is a diffeomorphism ([6, p.270]). This is called an *Iwasawa decomposition* of the group. These decompositions are very useful in harmonic analysis on the *Riemannian symmetric space* G/K. They provide well behaved parametrizations of the group which can be used to compute integrals or at least to obtain estimations for certain integrals (see Faraut's article in this volume). Clearly $G = KAN$ is not a direct product decomposition of a group and therefore one has to analyse the way the pieces are built together. One theorem which gives some information in this direction is Kostant's Convexity Theorem.

In the following we consider $\mathfrak{a}$ as an euclidean vector space, where the scalar product comes from the restriction of the positive definite form B_θ to $\mathfrak{a} \times \mathfrak{a}$. Since $\mathfrak{a} \subseteq \mathfrak{p}$, this is the same as the restriction of the Cartan Killing form to $\mathfrak{a}$. For a root $\alpha \in \Delta$ we write s_α for the orthogonal reflection on the hyperplane $\ker \alpha$ in $\mathfrak{a}$. Then the *Weyl group* $\mathcal{W} = \mathcal{W}(\Delta)$ is the group generated by the reflections s_α, $\alpha \in \Delta$. It may be identified with the quotient

$$N_K(A)/Z_K(A),$$

where

$$N_K(A) = \{k \in K : kAk^{-1} \subseteq A\} \quad \text{and} \quad Z_K(A) = \{k \in K : (\forall a \in A) kak^{-1} = a\}$$

([6, p.289]). Let us write $L : KAN \to \mathfrak{a}$ for the analytic mapping determined by

$$g \in K \exp L(g) N.$$

Now we have all definitions to state Kostant's theorem:

Theorem 1. (Kostant's Convexity Theorem) *Let* $a \in A$. *Then the set* $L(aK)$ *equals the convex hull of the Weyl group orbit* $\mathcal{W} \log a$. ∎

In the following we describe how this theorem can be generalized from Riemannian symmetric spaces to symmetric spaces of regular type, and which methods are used to prove this theorem.

II. Symmetric spaces of regular type

Let G be a connected semisimple Lie group, and τ an involutive endomorphism of G. We write H for the $\mathbf{1}$-component of the group G^τ of fixed points of τ in G. Note that this implies that $M := G/H$ is a symmetric space. Let us use the same letter for the corresponding involution of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$. We choose a Cartan involution θ such that τ and θ commute ([9, p. 153]). We set

$$\mathfrak{h} := \{X \in \mathfrak{g} : \tau(X) = X\} \quad \text{and} \quad \mathfrak{q} := \{X \in \mathfrak{g} : \tau(X) = -X\}.$$

Then the compatibility of τ and θ show that we have a direct vector space decomposition

$$\mathfrak{g} = \mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{p} + \mathfrak{q}_\mathfrak{k} + \mathfrak{q}_\mathfrak{p},$$

where $\mathfrak{h}_\mathfrak{k} := \mathfrak{h} \cap \mathfrak{k}, \mathfrak{h}_\mathfrak{p} := \mathfrak{h} \cap \mathfrak{p}$ etc., and all summands are invariant under τ and θ. It is convenient to assume that $\mathfrak{h}$ contains no non-trivial ideals of $\mathfrak{g}$, i.e. that $Z_\mathfrak{h}(\mathfrak{q}) = \{0\}$, and that G/H is an irreducible symmetric space. For all our main results formulated below an easy reduction shows that they remain true without this additional hypothesis ([10, I]).

Definition 2. Write $\mathfrak{h}^a := \mathfrak{h}_\mathfrak{k} + \mathfrak{q}_\mathfrak{p}$ for the subalgebra of $\tau\theta$-fixed points in $\mathfrak{g}$. Then it is shown in [8, Chapert 3] that

$$\begin{aligned} Z(\mathfrak{h}^a) \cap \mathfrak{q} = \mathfrak{c} &:= \{X \in \mathfrak{q}_\mathfrak{p} : [X, \mathfrak{q}_\mathfrak{p}] = \{0\}\} \\ &= \{X \in \mathfrak{q}_\mathfrak{p} : [X, \mathfrak{h}_\mathfrak{k}] = \{0\}\}. \end{aligned}$$

Let us write $Z_\mathfrak{q}(\mathfrak{c})$ for the centralizer of $\mathfrak{c}$ in $\mathfrak{q}$. Then we say that the symmetric space G/H or equivalently the symmetric Lie algebra $(\mathfrak{g}, \tau)$ is of *regular type* if the condition

$$Z_\mathfrak{q}(\mathfrak{c}) = \mathfrak{q}_\mathfrak{p}$$

is satisfied. ∎

It is explained in [8, Chapter 4] that this condition is related to the existence of certain H-invariant cones in $\mathfrak{q}$. As it stands it is not very instructive. But let us remark that it is satisfied by two essential classes of symmetric spaces.

Remark 3. If G/H is a Riemannian symmetric space, i.e. if τ is a Cartan involution, then $\mathfrak{h} = \mathfrak{k}$ and $\mathfrak{q} = \mathfrak{p}$. Therefore $\mathfrak{c} = \{0\}$ because $[X, \mathfrak{p}] = \{0\}$ implies that

$$B(X, X) \in B(X, \mathfrak{p}) = B(X, [\mathfrak{p}, \mathfrak{k}]) = B([X, \mathfrak{p}], \mathfrak{k}) = \{0\}.$$

So $Z_\mathfrak{q}(\mathfrak{c}) = \mathfrak{p} = \mathfrak{q}_\mathfrak{p}$. ∎

Remark 4. Let $\mathfrak{g}$ be a simple hermitean Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. We define $\tau(X+iY) := X - iY$. Then $(\mathfrak{g}_{\mathbb{C}}, \tau)$ is of regular type. To see this, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = (\mathfrak{k} + i\mathfrak{p}) + (i\mathfrak{k} + \mathfrak{p})$ is a Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$ and therefore $\mathfrak{q}_{\mathfrak{p}} = i\mathfrak{k}$. So $\mathfrak{c} = iZ(\mathfrak{k})$ and the claim follows from the fact that $\mathfrak{k}$ is the centralizer of its center ([6, p.382]). ■

Lemma 5. *Let $\mathfrak{a} \subseteq \mathfrak{q}_{\mathfrak{p}}$ be a maximal abelian subspace. Then $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$ and in $\mathfrak{p}$. Moreover $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{q}_{\mathfrak{p}}$.*

Proof. Since $\mathfrak{a}$ contains $Z(\mathfrak{q}_{\mathfrak{p}})$, the regularity condition implies that $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$.

Write $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ for the dual Lie algebra. Then $\mathfrak{k}^c := \mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{p}}$ contains $i\mathfrak{c}$ in its center and we have to show that $Z_{\mathfrak{g}^c}(i\mathfrak{c}) = \mathfrak{k}^c$. This is proved in [11, pp.130,131]. Note that the condition that $\mathfrak{h}$ contains no non-trivial ideal of $\mathfrak{g}$ is used in the proof. ■

Definition 6. As explained in Section I we get a root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We write

$$\Delta_p := \{\alpha \in \Delta : \alpha(\mathfrak{c}) \neq \{0\}\} \quad \text{and} \quad \Delta_k := \{\alpha \in \Delta : \alpha(\mathfrak{c}) = \{0\}\}.$$

The elements of Δ_k are called the *compact roots* and the elements of Δ_p the *non-compact roots*. Note that a root α is compact if and only if $\mathfrak{g}^\alpha \subseteq Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{h}^a = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{q}_{\mathfrak{p}}$ (Lemma 5). So Δ_k is the system of restricted roots with respect to the subalgebra $\mathfrak{h}^a$.

We choose a positive system corresponding to a regular element X_0 with $\alpha(X_0) < \beta(X_0)$ for all $\beta \in \Delta_p^+$ and $\alpha \in \Delta_k^+$ (One has to choose X_0 near to a non-zero element of $\mathfrak{c}$). We write $\mathcal{W} := \mathcal{W}(\Delta_k^+)$ for the Weyl group generated by the reflections s_α with $\alpha \in \Delta_k^+$ and set

$$C_{\max} := \{X \in \mathfrak{a} : (\forall \alpha \in \Delta_p^+)\alpha(X) \geq 0\}, \qquad C_{\min} = C_{\max}^\star,$$

and

$$C_k := \{X \in \mathfrak{a} : (\forall \alpha \in \Delta_k^+)\alpha(X) \geq 0\}.$$

Then $C_k \cap C_{\max}$ is the positive *Weyl chamber* of the root system and C_k is the Weyl chamber with respect to Δ_k^+. The subalgebra $\mathfrak{n}$, and the groups A and N are defined as in Section I. ■

Theorem 7. (Generalized Iwasawa decomposition)

1) $\mathfrak{g} = \mathfrak{h} + \mathfrak{a} + \mathfrak{n}$ is a direct vetor space sum.

2) The mapping

$$G^\tau \times A \times N \to G, \quad (h, a, n) \mapsto han$$

is a diffeomorphism onto the open subset $G^\tau AN$ of G.

Proof. 1) Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and $\mathfrak{q}$, it follows that $\mathfrak{g}^0 \subseteq \mathfrak{a} + \mathfrak{h}_{\mathfrak{k}}$. If $X \in \mathfrak{g}^{-\alpha}$, where $\alpha \in \Delta^+$, then $X = \big(X + \tau(X)\big) - \tau(X) \in \mathfrak{h} + \mathfrak{n}$. So the root decomposition shows that $\mathfrak{g} = \mathfrak{h} + \mathfrak{a} + \mathfrak{n}$. To see that the decomposition is direct, let $X = Y + Z \in \mathfrak{h} \cap (\mathfrak{a} + \mathfrak{n})$ with $Y \in \mathfrak{a}$ and $Z \in \mathfrak{n}$. Then

$\tau(X) = X = Y + Z = -Y + \tau(Z)$, i.e. $2Y + Z - \tau(Z) = 0$. Now the directness of the root decomposition implies that $Z = 0$ and $Y = 0$, because $\tau(Z) \in \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^{-\alpha}$.
2) In view of 1) and the fact that G^τ and AN are subgroups of G, one only has to show that $G^\tau \cap AN = \{\mathbf{1}\}$. But this follows as in 1) from $\tau(N) \cap AN = \{\mathbf{1}\}$ (Bruhat decomposition) ([6, p.406]). ■

The main deficiency of the generalized Iwasawa decomposition is that $G^\tau AN$ is in general different from the whole group (see Hilgert's article for an example). Since G^τ is in general not connected, the same holds for the set $G^\tau AN$ and so we restrict our considerations to the connected component HAN. As in Section I we consider the analytic mapping:

$$L : HAN \to \mathfrak{a}, \qquad g \mapsto L(g)$$

which is determined by $g \in H \exp L(g) N$.

The set $L(aH)$ is only well defined if $aH \subseteq HAN$. The following Lemma tells us when this is true.

Lemma 8. *The condition $aH \subseteq HAN$ is equivalent to $a \in \exp(C_{\max})$.*

Proof. The proof can be found in [10]. It bases on the fact that

$$\overline{H \exp(C_{\max}) H} = \{g \in G : gHAN \subseteq HAN\}$$

which, more or less in this formulation, is proved in [8]. See also the discussion in Section 2 of [3] ■

The preceding lemma describes those candidates in A for which the set $L(aH)$ is well defined. This explains the restriction in the assumptions of the following theorem.

Theorem 9. (The Convexity Theorem) *Let $a \in A$ with $0 \neq \log a \in C_{\max}$. Then*

$$L(aH) = \mathrm{co}(\log a) + C_{\min},$$

where $\mathrm{co}(\log a)$ *denotes the closed convex hull of the $\mathcal{W}$-orbit of* $\log a$. ■

Remark 10. If $\tau = \theta$, then $\Delta_k = \Delta$ and $H = K$. Therefore $C_{\min} = \{0\}$ and Theorem 9 reduces to Theorem 1. ■

In the remainder we explain the methods which are used in the proof of the convexity theorem. The details can be found in [10]

One starts with the invariance properties of the set $L(aH)$.

Proposition 11. *$\mathrm{co}(Y) \subseteq L(aH)$ for every $Y \in L(aH)$.*

Proof. (Sketch) The essential idea is to use rank-1-reduction. In view of Lemma 10.4 in [7] one only has to show that the line segments $\{s_\alpha(Y), Y\}$ are contained in $L(aH)$ for every element $Y \in L(aH)$ and every reflection s_α, $\alpha \in \Delta_k^+$. If this holds for the reflections associated to the elements α of a basis of the root system, then we conclude that $L(aH)$ is invariant under the Weyl group, and the general assertion follows from the fact that every root is conjugate to a root in the basis. If α is a root in the basis, then N decomposes as a semidirect product $N \cong N^\alpha N'$, where $N^\alpha = \exp \mathfrak{g}^\alpha$ is contained in the subgroup $H^a := \langle \exp \mathfrak{h}^a \rangle$. This paves the way to rank-1-reduction. For the details we refer to [10] or [7, pp.476,477]. ∎

The next step is a description of the sets $\mathrm{co}(Y) + C_{\min}$ in a more convenient form.

Proposition 12. *Let E be a euclidean vector space, $\mathcal{H}$ a finite system of hyperplanes in E, $\mathcal{W}$ the group generated by the reflections on the elements of $\mathcal{H}$, and suppose that $\mathcal{W}$ is finite and leaves $\mathcal{H}$ invariant. If C is a chamber of $\mathcal{H}$, $Y \in \overline{C}$, and C' a $\mathcal{W}$-invariant closed convex cone in E, then*

$$\mathrm{co}(Y) + C' = \mathrm{co}(Y + C') = \bigcap_{s \in \mathcal{W}} s(X + C' - C^\star).$$

Proof. ([10, I.12]) The proof is rather elementary, one only has to deal with basic properties of finite groups generated by reflections ([1, Ch. V]) and the finite dimensional version of the Hahn-Banach Separation Theorem. The only difficulties come from the non-compactness of the set C', but they can be overcome with the crucial observation that $C' - C^\star$ is always closed in this situation. ∎

Let us keep the notation from Proposition 12 for a moment.

Corollary 13. *If $X_0 \in \overline{C}$ and $C' := \sum_{s\in\mathcal{W}} \mathbb{R}^+ s(X_0)$, then*

$$\mathrm{co}(Y) + C' = \mathrm{co}(Y + \mathbb{R}^+ X_0) = \bigcap_{s \in \mathcal{W}} s(X + \mathbb{R}^+ X_0 - C^\star).$$

Proof. One only has to show that the cones $C' - C^\star$ and $\mathbb{R}^+ X_0 - C^\star$ are equal. But this follows from the fact that $X_0 - s_\alpha(X_0) \in C^\star$ ([10, I.3]). ∎

Now we can already proof one inclusion of the convexity theorem. Let $X := \log a$ and assume that $X \neq 0$ (otherwise there is nothing to prove). For a compact root α we choose an element X_α orthogonal to $\ker \alpha$ such that $\alpha(X_\alpha) = 1$. If Y_α is a non-zero vector in $\mathfrak{g}^\alpha$, then $\mathrm{span}\{Y_\alpha, \theta(Y_\alpha), X_\alpha\}$ is a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. After conjugation with $\mathcal{W}$ we may assume that $X_\alpha \in \overline{C_k}$ which implies that $\alpha(X) > 0$. An explicit calculation in $\mathrm{Sl}(2,\mathbb{R})$ (see Hilgert's article) now shows that $L(aH)$ contains the ray $X + \mathbb{R}^+ X_\alpha$. Therefore

$$\mathrm{co}(X + \mathbb{R}^+ X_\alpha) = \mathrm{co}(X) + C_{\min} \subseteq L(aH)$$

because $C_{\min} = \sum_{s\in\mathcal{W}} \mathbb{R}^+ s(X_\alpha)$ ([10]).

The other inclusion of the theorem is harder to prove. It bases on Harish Chandra's construction of the holomorphic discrete series for a linear hermitean

Lie group. The first observation is that, in view of Propositions 11 and 12, it suffices to show that

$$L(aH) \subseteq \log a + C_{\min} - C_k^\star.$$

But $-C_{\min} + C_k^\star = (-C_{\max} \cap C_k)^\star$ because it is closed, as we have already remarked above. So it would be sufficient to prove that

$$\lambda\big(L(ah) - \log a\big) \leq 0$$

for all functionals in a set $\mathcal{P} \subseteq (-C_{\max}) \cap C_k$ which has the property that $\mathbb{R}^+\mathcal{P}$ is dense. It is shown in [10, II.1] that the following set satisfies this condition. Let $E \subseteq \mathfrak{a}$ denote the abelian group generated by the vectors corresponding to the roots under the identification $\mathfrak{a} \cong \mathfrak{a}^*$. As $\mathcal{P}$ we take the subset of all those elements λ in E which correspond to dominant integral weights of the subalgebra $\mathfrak{h}_\mathfrak{k} + \mathfrak{q}_\mathfrak{p}$ and which satisfy the additional condition that

$$\langle \lambda + \rho, \beta \rangle < 0 \qquad \forall \beta \in \Delta_p^+,$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$.

We collect the essential facts in the following theorem:

Theorem 14. *Let $\mathcal{P}$ be as defined above, suppose that G is contained in a complex Lie group $G_\mathbb{C}$ with $\mathfrak{L}(G_\mathbb{C}) = \mathfrak{g}_\mathbb{C}$, and set $G^c := \langle \exp(\mathfrak{h} + i\mathfrak{q}) \rangle$. Then the following assertions hold:*

1) *For every $\lambda \in \mathcal{P}$ there exists an irreducible unitary representation $\pi_\lambda : G^c \to \mathcal{U}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ which has the following properties:*
   a) *Let $K^c := \langle \exp(\mathfrak{h}_\mathfrak{k} + i\mathfrak{q}_\mathfrak{p}) \rangle$. Then the subspace $\mathcal{H}^K$ of K^c-finite vectors consists of analytic vectors and therefore carries the structure of a $\mathfrak{g}_\mathbb{C}$-module. As $\mathfrak{g}_\mathbb{C}$-module it is a module with highest weight λ.*
   b) *For $\alpha \in \mathfrak{a}^*$ we set $\mathcal{H}^\alpha := \{v \in \mathcal{H} : (\forall X \in \mathfrak{a}) d\pi_\lambda(X)v = \alpha(X)v\}$ and call α a weight if $\mathcal{H}^\alpha \neq \{0\}$. Then all weights are contained in the set*
   $$\mathcal{W}(\lambda - C_{\min}).$$
2) *Let $S_{\min}$ denote the subsemigroup of G generated by H and $\exp(C_{\min})$. Then $S_{\min} = \overline{H \exp(C_{\min})H}$ and the restriction $\pi_\lambda|_H$ permits a continuation to a representation*
   $$\widetilde{\pi}_\lambda : S_{\min} \to \mathcal{C}(\mathcal{H}),$$
   where $\mathcal{C}(\mathcal{H})$ denotes the semigroup of contractions on $\mathcal{H}$. This representation is continuous with respect to the weak operator topology on $\mathcal{C}(\mathcal{H})$ and analytic on the interior of $S_{\min}$.
3) *The restriction $\widetilde{\pi}_\lambda|_{\exp C_{\min}}$ permits a continuation to a representation $\widehat{\pi}_\lambda$ of $\exp(C_{\max})$ by bounded operators on H such that*
   $$||\widehat{\pi}_\lambda(\exp X)|| = e^{\lambda(X)} \qquad \forall X \in C_{\max}.$$

4) Let $v \in \mathcal{H}^K$ be a unit vector of highest weight λ, $h \in H$, and $a \in \exp C_{\max}$ such that $ah \in h_1 \exp L(ah)N$. Then

$$\pi_\lambda(h_1)^{-1}\widehat{\pi}_\lambda(a)\pi_\lambda(h) = e^{\lambda\left(L(ah)\right)}v.$$

Proof. For a more precise formulation of the results we refer to [10]. 1) is essentially due to Harish Chandra ([4], [5]), and 2) is a result of Ol'shanskiĭ who uses a theorem on analytic continuation of Lüscher and Mack ([12]). The rest is proved in [10]. Assertion 3) bases on the fact, which follows from 1)b), that the spectrum of the essentially selfadjoint operators $d\pi_\lambda(X)$, $X \in C_{\max}$ is bounded from above. To prove 4) one has to make use of the analyticity of the representation $\widetilde{\pi}_\lambda$ on $\mathrm{int}(S_{\min})$ for analytic continuation arguments because neither $\widetilde{\pi}_\lambda$ nor $\widehat{\pi}_\lambda$ is defined on the whole semigroup $\exp(C_{\min})N \subseteq G$. ∎

With this theorem the proof of the convexity theorem is almost complete. So let $\lambda \in \mathcal{P}$, $a \in \exp(C_{\max} \cap C_k)$, $h \in H$, $ah \in h_1 \exp L(ah)N$, and v a unit vector of highest weight λ. Then

$$\begin{aligned} e^{\lambda\left(L(ah)\right)} &= ||e^{\lambda\left(L(ah)\right)}v|| = ||\pi_\lambda(h_1)^{-1}\widehat{\pi}_\lambda(a)\pi_\lambda(h)v|| \\ &\leq ||\pi_\lambda(h_1)^{-1}||\,||\widehat{\pi}_\lambda(a)||\,||\pi_\lambda(h)v|| = e^{\lambda(\log a)}. \end{aligned}$$

This proves that $\lambda\big(L(ah)\big) \leq \lambda(\log a)$ and the theorem follows. ∎

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