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Non-Commutative Covariant Differential Calculi on Quantum Spaces and Quantum Groups

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1. Introduction

There are two alternative fundamental approaches to the field of quantum groups. The first one is based on a deformation of the universal enveloping algebra, while the second one works with deformations of function algebras on the group. Here we shall adopt the second view point and we think of a “quantum group” as a q-deformation of some classical matrix group. (Fundamental papers for this approach are [4] and [10, 11, 12]. A very elementary introduction was given by the author in [7].) Then it is quite natural to ask for a generalization of the classical differential calculus, say on matrix groups, to quantum groups. A general framework for a bicovariant differential calculus on quantum groups was developed by S. L. Woronowicz [12] following general ideas of A. Connes [2]. Concrete examples of such calculi were constructed in: [10] (the 3D-calculus on $\mathrm{SU}_q(2)$ which is left-covariant, but not bicovariant), [12] (the bicovariant $4D_\pm$-calculus on $\mathrm{SU}_q(2)$), [9] and [6] (left-covariant calculus on $\mathbb{C}_q^n$), [8] (left-covariant calculus on $\mathrm{GL}_q(2)$), and [3] (bicovariant calculi on $\mathrm{SU}_q(n)$ and $\mathrm{SO}_q(n)$).

The proofs for the results mentioned in Sections 3 and 4 will be given in a forthcoming paper of the author.

2. Covariant Differential Calculi on Quantum Spaces

Throughout this paper we assume that all algebras are over the complex field $\mathbb{C}$ and admit a unit element denoted by 1. Algebra homomorphisms are always meant to be unit preserving. Let X be an algebra.

Definition 2.1. A *first order differential calculus* (or briefly, a *differential calculus*) over X is a pair (Γ, d), where Γ is a bimodule for X and $d\colon X \to \Gamma$ is a linear mapping such that $d(xy) = dx\cdot y + x\cdot dy$ for $x, y \in X$, and where Γ is the linear span of elements $x\cdot dy$ whith $x, y \in X$. ■

Definition 2.2. Two differential calculi (Γ_1, d_1) and (Γ_2, d_2) over X are said to be *isomorphic* if there exists a bimodule isomorphism $\psi\colon \Gamma_1 \to \Gamma_2$ such that $\psi(d_1 x) = d_2 x$ for all $x \in X$. ■

The above Definition 2.1 is adopted from the following classical picture: Suppose M is a compact smooth manifold. Let Γ be the space $C^\infty\big(T^*(M)\big)$ of sections of the cotangent bundle and let d be the exterior derivative. Then (Γ, d) is a differential calculus over the algebra $X = C^\infty(M)$. Sometimes it is more convenient to work with a smaller algebra X which is invariant under d. For instance, if M is a matrix group, one could take the coordinate algebra for X. Having this classical example in mind, we consider the bimodule Γ in Definition 2.1. as a non-commutative variant of the space of first order differential forms.

Now let A be a Hopf algebra. The comultiplication, the counit and the antipode of A are denoted by Δ, ε, and κ, respectively.

Definition 2.3. A *left quantum space* (or briefly, a *quantum space*) for A is an algebra X together with an algebra homomorphism $\varphi\colon X \to A \otimes X$ such that $(id \otimes \varphi)\varphi = (\Delta \otimes \mathrm{id})\varphi$ and $(\varepsilon \otimes id)\varphi = \mathrm{id}$. Also, φ is called the *action* of A on X.

In other words, φ has to be both an algebra homomorphism and a coalgebra homomorphism or, equivalently, φ is an algebra homomorphism, and X is a left comodule for A. We may think of a "quantum space" X as a "homogenous space" where the "quantum group" A acts.

Definition 2.4. Let X be a quantum space for A with action φ. A differential calculus (Γ, d) over X is said to be *left-covariant* with respect to A if there exists a linear mapping $\varphi\colon \Gamma \to A \otimes \Gamma$ such that:

(i) $(id \otimes \varphi)\varphi = (\Delta \otimes \mathrm{id})\varphi$ and $(\varepsilon \otimes \mathrm{id})\varphi = \mathrm{id}$.

(ii) $\varphi(x\omega) = \varphi(x)\varphi(\omega)$ and $\varphi(\omega x) = \varphi(\omega)\varphi(x)$ for $x \in X$, $\omega \in \Gamma$.

(iii) $\varphi \cdot d = (\mathrm{id} \otimes d)\varphi$.

Let us briefly discuss the conditions (i)-(iii) in Definition 2.4. Condition (i) means only that Γ is a left comodule for the Hopf algebra A, that is, the "quantum group A acts on the space Γ of differential forms". (ii) says that this action φ of A on Γ is compatible with the action φ of A on X. A bimodule Γ of the quantum space X together with a linear mapping $\varphi\colon \Gamma \to A \otimes \Gamma$ satisfying (i) and (ii) is called a *left-covariant bimodule.* Finally, (iii) means that the action φ is compactible with the differentiation d. Condition (iii) is the requirement that the following diagram is commutative:

$$\begin{array}{ccc} x & \xrightarrow{\quad d \quad} & \Gamma \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ A \otimes X & \xrightarrow[\mathrm{id}\otimes d]{} & A \otimes \Gamma. \end{array}$$

We check that the mapping φ occuring in Definition 2.4 is uniquely determined by φ and d. For an arbitrary form $\omega = \sum_n x_n \cdot dy_n \in \Gamma$ we have

$$\varphi(\omega) = \varphi(\sum_n x_n \cdot dy_n) = \sum_n \varphi(x_n)(\varphi \cdot d)(y_n) = \sum \varphi(x_n)(\mathrm{id} \otimes d)\varphi(y_n) \tag{1}$$

and the right-hand side of (1) depends only on d and φ.

A characterization of the left-covariance which avoids use of the action φ is given by

Proposition 2.5. *A differential calculus* (Γ, d) *over a quantum space* X *for* A *is left-covariant with respect to* A *if and only if* $\sum_n x_n \cdot dy_n = 0$ *always implies that* $\sum_n \varphi(x_n)(id \otimes d)\varphi(y_n) = 0$.

The necessity of this condition is clear by (1). The proof of the sufficiency (in the case $X = A$, $\varphi = \Delta$) is given in [12], Section 2. Note that in [12] the condition in Proposition 2.5 is used to define the left-covariance. ■

The definitions of a right quantum space, a right-covariant bimodule and a right-invariant differential calculus are similar. In the latter case (iii) has to be replaced by $\varphi \cdot d = (d \otimes id)\varphi$.

Clearly, the quantum group A itself is both a left quantum space and a right quantum space for A with the action $\varphi = \Delta$. (The corresponding conditions in Definition 2.3 are valid by the Hopf algebra axioms for A.) With this action, we can speak about left- or right-covariant differential calculi on A. A differential calculus on A is called *bicovariant* if it is left-covariant and right-covariant. Similarly, a bicovariant bimodule of A is a bimodule which is both left- and right-covariant.

3. Two Examples: The Quantum Hyperboloid and the Quantum Plane

Let δ be a fixed complex number. Let $X_{q,\delta}$ denote the universal algebra with unit 1 and two generators satisfying the relation $xy - qyx = \delta \cdot 1$. That is, $X_{q,\delta}$ is the quotient $\mathbb{C}\langle x,y\rangle/(xy - qyx - \delta \cdot 1)$ of the free algebra with unit $\mathbb{C}\langle x,y\rangle$ generated by x and y modulo the two-sided ideal $(xy - qyx - \delta \cdot 1)$ of this algebra which is generated by the element $xy - qyx - \delta \cdot 1$. In the case that $\delta = 0$, the algebra $X_{q,\delta}$ is nothing but the *quantum plane* $\mathbb{C}_q^2$ discussed already in [7]. If $\delta \neq 0$, we shall call $X_{q,\delta}$ the *quantum hyperboloid.*

For arbitrary $\delta \in \mathbb{C}$, the algebra $X_{q,\delta}$ is a (left) quantum space for the quantum group $SL_q(2)$, where its action φ is given by the "matrix multiplication", i.e., $\varphi(x) = a \otimes x + b \otimes y$ and $\varphi(y) = c \otimes x + d \otimes y$. More precisely, φ has to be extended to an algebra homomorphism of $X_{q,\delta}$ into $\mathrm{SL}_q(2) \otimes X_{q,\delta}$. It is not difficult to check that φ is well-defined, i.e., $\varphi(x)\varphi(y) - q\varphi(y)\varphi(x) - \delta(1 \otimes 1) = 0$ if $xy - qyx - \delta \cdot 1 = 0$.

It should be noted that the cases $\delta \neq 0$ and $\delta = 0$ are different in several respects. We mention one of them. If we set $q = 1$, then $\mathbb{C}_q^2 = X_{q,0}$ is just the polynomial algebra $\mathbb{C}[x,y]$ in two commuting variables x and y. That is, for $q = 1$ and $\delta = 0$ we get the coordinate algebra of a "classical (commutative) space". However, if $\delta \neq 0$, then $q = 1$ does not give a "classical space". For instance, for $\delta = 1$ and $q = 1$, the algebra $X_{q,\delta}$ is the Weyl algebra.

Motivated by the classical situation we shall restrict ourselves to differential calculi (Γ, d) for which the following is true:

$$(*) \qquad \{dx, dy\} \text{ is a basis for the left module } \Gamma.$$

We discuss the two cases $\delta \neq 0$ and $\delta = 0$ separately.

Example 1. *The quantum hyperboloid $X_{q,\delta}$ with $\delta \neq 0$.*
First suppose that if $q \neq 1$, then $q^m \neq 1$ for $m = 2, 3$. Then there exist precisely two non-isomorphic differential calculi (Γ, d) on $X_{q,\delta}$ which are left-covariant with respect to $\mathrm{SL}_q(2)$ and satisfy the assumption $(*)$. The corresponding formulae describing Γ are:

$$\begin{aligned}
x{\cdot}dx &= p\,dx{\cdot}x + (1-p^{-1})\delta^{-1}\omega_{\mathrm{inv}}x^2,\\
y{\cdot}dy &= p\,dy{\cdot}y + (1-p^{-1})\delta^{-1}\omega_{\mathrm{inv}}y^2,\\
x{\cdot}dy &= (p-p^{-1}q^{-1})\,dx{\cdot}y + p^{-1}\,dy{\cdot}x + (1-p^{-1})\delta^{-1}\omega_{\mathrm{inv}}xy,\\
y{\cdot}dx &= p^{-1}\,dx{\cdot}y + (p-qp^{-1})\,dy{\cdot}x + (1-p^{-1})\delta^{-1}\omega_{\mathrm{inv}}yx,
\end{aligned}$$

where

$$\omega_{\mathrm{inv}} := dx{\cdot}y - q dy{\cdot}x \quad \text{and} \quad p := \pm(q^{1/2} + q^{-1/2}) - 1.$$

In the proof of this statement one has to decompose tensor product representations of $\mathrm{SL}_q(2)$. Here one has similar results as for the classical $\mathrm{SL}(2,\mathbb{C})$ if q is not a root of unity. This is the point where the assumption concerning q is used. It is interesting that there are more left-covariant differential calculi if $q = \frac{1}{2}(-1 \pm \sqrt{3}i)$. Then for arbitrary non-zero $\alpha \in \mathbb{C}$ there exists a left-covariant differential calculus over $X_{q,\delta}$ which satisfics $(*)$. The corresponding formulae are:

$$\begin{aligned}
x{\cdot}dx &= \alpha\,dx{\cdot}x + \widetilde{\alpha}\omega_{\mathrm{inv}}x^2,\\
y{\cdot}dy &= \alpha\,dy{\cdot}y + \widetilde{\alpha}\omega_{\mathrm{inv}}y^2,\\
x{\cdot}dy &= (\alpha - \alpha^{-1}q^{-1})\,dx{\cdot}y + \alpha^{-1}\,dy{\cdot}x + \widetilde{\alpha}\omega_{\mathrm{inv}}xy,\\
y{\cdot}dx &= \alpha^{-1}\,dx{\cdot}y + (\alpha - q\alpha^{-1})\,dy{\cdot}x + \widetilde{\alpha}\omega_{\mathrm{inv}}yx,
\end{aligned}$$

where

$$\widetilde{\alpha} := -(\alpha + \alpha^{-1} + 1)\delta^{-1}.$$

Example 2. *The quantum plane $\mathbb{C}_q^2$.*
Suppose that $q^m \neq 1$ for all $m \in \mathbb{N}$ if $q \neq 1$. Then for each $\alpha \in \mathbb{C}$ there is a left-covariant differential calculus over $\mathbb{C}_q^2$ with respect to $\mathrm{SL}_q(2)$ which statisfies assumption $(*)$:

$$\begin{aligned}
x{\cdot}dx &= q^2\,dx{\cdot}x + \alpha\omega_{\mathrm{inv}}x^2,\\
y{\cdot}dy &= q^2\,dy{\cdot}y + \alpha\omega_{\mathrm{inv}}y^2,\\
x{\cdot}dy &= q\,dy{\cdot}x + (q^2-1)\,dx{\cdot}y + \alpha\omega_{\mathrm{inv}}xy,\\
y{\cdot}dx &= q\,dx{\cdot}y + \alpha\omega_{\mathrm{inv}}yx.
\end{aligned}$$

Further, other left-covariant differential calculi over $\mathbb{C}_q^2$ satisfying $(*)$ are obtained if we replace in the above formulas q by q^{-1}, x by y and y by x. These are all differential calculi over $\mathbb{C}_q^2$ with the described properties.

Of particular interest are the two calculi which are obtained if we set $\alpha = 0$. These calculi were introduced independently in [9] and in [6]. Another approach to the differential calculus on $\mathbb{C}_q^2$ (or more generally, on $\mathbb{C}_q^n$) was given by [5].

4. Bicovariant Bimodules over a Quantum Group

If (Γ, d) is a bicovariant differential calculus over a quantum group A, then Γ is, in particular, a bicovariant bimodule. The structure of such modules has been completely characterized by WORONOWICZ [12], cf. Theorem 2.4. We state this result in the following theorem. Its formulation uses convolutions of elements x of the Hopf algebra A and of linear functionals f on A which are defined by $f * x \overset{\text{def}}{=} (\mathrm{id} \otimes f)\Delta(x)$ and $x * f \overset{\text{def}}{=} (f \otimes \mathrm{id})\Delta(x)$, respectively.

Theorem 4.1. *Suppose Γ is a bicovariant bimodule for a Hopf algebra A. Let φ_L and φ_R denote the left resp.right actions of A on Γ and let $(\omega_i)_{i \in I}$ be a basis of the vector space $\Gamma_{\mathrm{inv}} := \{\omega \in \Gamma : \varphi_L(\omega) = 1 \otimes \omega\}$ of left-invariant elements in Γ. Then there exist linear functionals f_{ij}, $i, j \in I$, on A and elements R_{ij}, $i, j \in I$, of A such that for all $x, y \in A$ and $i, j \in I$:*

(2) $\omega_i x = \sum_j (f_{ij} * x)\omega_j$,
(3) $\varphi_R(\omega_i) = \sum_j \omega_j \otimes R_{ji}$,
(4) $f_{ij}(xy) = \sum_k f_{ik}(x) f_{kj}(y), f_{ij}(1) = \delta_{ij}$,
(5) $\Delta(R_{ij}) = \sum_k R_{ik} \otimes R_{kj}, \varepsilon(R_{ij}) = \delta_{ij}$,
(6) $\sum_k R_{ki}(x * f_{kj}) = \sum_k (f_{ik} * x) R_{jk}$.

Further, $(\omega_i)_{i \in I}$ is a left module basis of Γ.

Conversely, if functionals f_{ij} on A and elements R_{ij} of A are given such that (4)-(6) *is satisfied and if $(\omega_i)_{i \in I}$ is a basis of a certain vector space $\widetilde{\Gamma}$, then there exists a unique bicovariant bimodule Γ such that $\widetilde{\Gamma} = \Gamma_{\mathrm{inv}}$,* (2) *and* (3) *hold.* ■

We add a few comments. (4) and (5) have nice algebraic interpretations: (4) says that the mapping $x \mapsto \big(f_{ij}(x)\big)$ is a representation of the algebra A on the vector space Γ_{inv} and (5) means that (R_{ij}) is the matrix for a representation of the coalgebra A. Conditions (6) expresses a compatibility requirement for the right and the left actions.

5. Bicovariant Differential Calculi on Quantum Groups of Type B_n, C_n, and D_n

In this section we let A denote the Hopf algebra for one of the quantum groups B_n, C_n, or D_n as defined in [4], Subsection 1.4. These quantum groups are q-deformations of the classical matrix groups $\mathrm{SO}(n+1)$, $\mathrm{SO}(2n)$ and $\mathrm{Sp}(n)$. As an algebra, A is the quotient $\mathbb{C}\langle u^i_j : i, j = 1, \ldots, N\rangle / J$ of the free algebra generated by N^2 elements u^i_j, $i, j = 1, \ldots, N$ by the two-sided ideal J generated by the matrix elements of $\widehat{R}(u \otimes u) - (u \otimes u)\widehat{R}$, $u(cu^t c^{-1}) - I$, and $cu^t c^{-1} u - I$. Here R is the so-called R-matrix and $\widehat{R}$ arises from R through the "flip operator" for the tensor product. Further, u denotes the matrix (u^i_j)

and c is another invertible $N \times N$ matrix. Its concrete form is not needed here. Let $b = (b^i_j)$ denote the inverse of c.

We consider a bicovariant differential calculus (Γ, d) over A which satisfies the following "natural" assumption:

$(*)$ $\qquad \{du^i_j; i, j = 1, ..., N\}$ is a basis of the left module Γ.

Set $\omega_{(ij)} = \sum_{r,s} b^i_r \kappa(u^r_s) du^s_j$ for $i,\ j \in \{1, \ldots, N\}$. Small calculations show that $\varphi_L(\omega_{(ij)}) = 1 \otimes \omega_{(ij)}$ and $\varphi_R(\omega_{(ij)}) = \sum_{kl} \omega_{kl} \otimes u^k_i u^l_j$. The first equality means that $\omega_{(ij)}$ is left-invariant, i.e., $\omega_{(ij)} \in \Gamma_{\text{inv}}$. From $(*)$ it follows easily that the N^2 elements $\omega_{(ij)}$ form a basis of Γ_{inv}. The second equality shows that the elements $R_{(kl),(ij)}$ in Section 4 are $u^k_i u^l_j$, i.e., $R_{(kl)(ij)}$ is the matrix element of the tensor product representation $u \otimes u$. (Note that our indices are now pairs of numbers such as $(ij), (kl)$.) Let $f_{(ij),(kl)}$ be the linear functionals on A from Theorem 4.1 and put

(7) $T^{rkl}_{ijs} := f_{(ij),(kl)}(u^r_s)$, $i,\ j,\ k,\ l,\ r,\ s = 1, \ldots, N$.

Then $T \equiv (T^{rkl}_{ijs})$ belongs to $\text{End}(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$.

Theorem 5.1. *If* (Γ, d) *is a bicovariant differential calculus over* A *satisfying* $(*)$*, then we have:*

(8) $T(u \otimes u \otimes u) = (u \otimes u \otimes u)T$,

(9) $\widehat{R}_{12}\, T_{234}\, T_{123} = T_{234}\, T_{123}\, \widehat{R}_{34}$,

(10) $T \cdot (c_3^{-1} P_{132} T c_3)^{t_3} = (c_3^{-1} P_{132} T c_3)^{t_3} \cdot T = I$.

Conversely, if a mapping $T \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$ *fulfills the three conditions* (8)–(10)*, then there exists a unique bicovariant differential calculus over* A *such that* $(*)$ *and* (7) *are satisfied.* ∎

In other words, this theorem characterizes the bicovariant differential calculi over A satisfying $(*)$ in terms of the linear mapping T. The crucial condition therein is, of course, (8). It is derived from condition (6) in Theorem 4.1, Condition (8) says that the mapping T belongs to the intertwining space $\mathcal{M} := \text{Mor}(u \otimes u \otimes u, u \otimes u \otimes u)$. This space is a finite dimensional algebra whose structure is known. It is a quotient of the so-called *Birman-Wenzl-Murakami algebra* (see, for instance, [1]), a q-deformed version of the classical Brauer algebra.

Let us look at the simplest case $A = D_1$. It is known (cf. [4]) that the Hopf algebra of D_1 for q is isomorph to the Hopf algebra of $\text{SL}_{q^2}(2)$ and so of $\text{SU}_{q^2}(2)$ if q is real and $q^2 < 1$. Thus it suffices to consider the case of $\text{SL}_q(2)$. Suppose that q is not a non-trivial root of unity. Then the tensor product $u \otimes u \otimes u$ of the fundamental representation $u = d_{\frac{1}{2}}$ decomposes as $d_{1\frac{1}{2}} \oplus 2d_{\frac{1}{2}}$ similarly as in the classical case $q = 1$. Hence $\dim \mathcal{M} = 5$. More precisely, the algebra is a quotient of the Hecke algebra $H_3(q)$ by a one-dimensional ideal.

Let us come back to the construction and classification of bicovariant differential calculi over A. It is not difficult to check that the four endomorphism $T := \widehat{R}_{12}\widehat{R}_{23}^{-1}$, $\widehat{R}_{12}^{-1}\widehat{R}_{23}$, $\widehat{R}_{12}\widehat{R}_{23}$, $\widehat{R}_{12}^{-1}\widehat{R}_{23}^{-1}$ satisfy the conditions (8)-(10).

Therefore, by Theorem 5.1, each of the mappings yields a bicovariant differential calculus over A. It turns out that the first and the second endhomorphism give isomorphic differential calculi. But if $q^2 \neq 1$ then $\widehat{R}_{12}^{-1}\widehat{R}_{23}$, $\widehat{R}_{12}\widehat{R}_{23}$ and $\widehat{R}_{12}^{-1}\widehat{R}_{23}^{-1}$ define non-isomorphic differential calculi. In case $q^2 = 1$ we have $\widehat{R} = \widehat{R}^{-1}$, so that four mappings coincide and define the same differential calculus.

Here is a natural question: Are the four mappings T defined above all solutions of (8)-(10)?

I conjecture that this is true. In proving this the concrete structure of the intertwining space $\mathrm{Mor}(u \otimes u \otimes u, u \otimes u \otimes u)$ will be certainly helpful.

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