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Distinguishability Condition and the Future Subsemigroup

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Dedicated to Karl H. Hofmann on the occasion of his 60$^{\text{th}}$ birthday

Abstract

The paper deals with two simply connected solvable four-dimensional Lie groups M_1 and M_2. The first group is a direct product of the nilpotent Heisenberg Lie group and the one-dimensional Lie group. The second one is a direct product of the two-dimensional non-abelian Lie group and the two-dimensional abelian Lie group. Applying Methods of [4, 6] we investigate the causal structure of left-invariant Lorentzian metrics on M_1 [8] and M_2 [7]. Here we focus our attention on one concrete metric on M_1 and on a certain one-parameter family g_q, $q > 0$ of metrics on M_2. We have proved in [7, 8] these Lorentzian spaces to be geodesically complete, satisfying the causality condition with a violation of uniform stable causality. In the present paper, we prove these spaces to be future distinguishing (that involves, because of their homogeneity, also the conditions of past distinguishing, strong causality, stable causality and continuity of causality). This result is of interest in causality theory since in accordance with [9], respectively, [5] the chronological (respectively, causal) structure of such spaces codes their conformal structure. It also characterizes the structure of the subsemigroup I^+, respectively, J^+ which defines the chronological, respectively, causal structure of the considered Lorentzian Lie group.

For all unfamiliar definitions, the reader is referred to [1, 6].

1. General method to prove future distinguishability of a Lorentzian Lie group

Assume M to be a solvable connected Lie group and fix a symmetric non-degenerated form of Lorentzian signature $+,\dots,+,-$ in the Lie algebra L of M. After the choice of future cone K^+ in L the group M becomes *a Lorentzian Lie group*, or LLG for short. If an ideal $[L,L]$ is *lightlike*, i.e., its intersection with K^+ is a single ray lying in ∂K^+, then such an LLG M satisfies the causality

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condition [6, Theorem 4.2]. If the intersection of K^+ and $[L,L]$ is $\{0\}$, then M is uniformly stably causal [6, Theorem 4.1], hence distinguishing. We may, therefore, restrict our attention to the case $K^+ \cap [L,L] = \ell$ where ℓ is a light ray in L.

Suppose that the hyperplane N contains ℓ and is a support hyperplane of K^+. Introduce in M a *canonical coordinate system* $(x_1, \dots, x_n)$ *of the second type associated with* N. Then the Lie subgroup corresponding to N is characterized by the equation $x_n = 0$.

A Lorentzian manifold M with a prescribed time orientation is said to be *future distinguishing* (see [1], p. 24) if for any $x, y \in M$ the assumption $I_x^+ = I_y^+$ implies $x = y$, where I_x^+, respectively, I_x^-, as usual, denotes the chronological future, respectively, past of x. If $x = \mathbf{1}$ then we shall simply write I^+ instead of $I_{\mathbf{1}}^+$ etc. Also the causal future, respectively, past, of x will be denoted by J_x^+, respectively, J_x^- (and J^+, respectively, J^- if $x = \mathbf{1}$.). We want to make use of a result due to R. Penrose: If M fails to be future distinguishing, then Condition (e) of his Theorem 4.31 from [10] is valid. The latter condition deals with a certain light geodesic γ. Suppose now that our LLG M fails to be future distinguishing. It follows from the proof of Theorem 4.2 of [6] that the γ above corresponds to ℓ.

We recall Condition (e) itself:

For any $u, v \in \gamma$ with $u \leq v$, if $u \ll x$ and $y \ll v$, then $y \ll x$.

We may assume without loss of generality that $\mathbf{1} \in \gamma$.

Lemma . *γ is entirely contained in $\overline{I^+} \cap \overline{I^-}$.*

Proof. In Condition (e) we take $u = \mathbf{1}$, $v \in \gamma \cap J^+$. Let $\mathbf{1} \ll x$, i.e., $x \in I^+$ and $y \ll v$, i.e., $y \in I_v^-$. Let y tend to v in I_v^- and let x tend to $\mathbf{1}$ in I^+. This choice is possible, since $v \in I_v^- \subset \overline{I_v^-}$, $\mathbf{1} \in J^+ \subset \overline{I^+}$. Taking into consideration the continuous dependence of $I_y^\pm$ on the point y itself, we deduce $v \in \overline{I^-}$. ∎

We return to M and the canonical coordinates $(x_1, \dots, x_n)$. Let $x = x(t)$ denote a future timelike curve λ issuing from $\mathbf{1} = (0, \dots, 0)$. The subsemigroup I^+, the chronological future of $\mathbf{1}$, consists of the points on all such λ. Observe that the component z_n of the product $z = x \cdot y$ equals $x_n + y_n$. Thus the coordinate x_n of the point $x(t)$ is increasing while we move along λ from $\mathbf{1}$ to the future. But the above lemma states that it must somehow reach the vicinity of $\gamma^- = J^- \cap \gamma$.

Let exp denote the exponential map (in the geometrical sense) defined on some neighborhood U of $\mathbf{1}$. When $t > 0$ is sufficiently small, the points $x(t)$ "concentrate near" exp K^+. They can return to γ^- only above such a "level" x_n, at or below which there are conjugate points to $\mathbf{1}$ along null geodesics issuing from $\mathbf{1}$. Therefore the we have following result.

Theorem 1. *If, under the assumptions above, in a certain slice $\mathcal{U} \overset{\text{def}}{=} \{x : 0 < x_n < \varepsilon\}$ there are no points conjugate to $\mathbf{1}$ along future null geodesics issuing from $\mathbf{1}$, and if the set of all points on all lightlike future geodesics from $\mathbf{1}$ in $\mathcal{U}$ divides $\mathcal{U}$ into two components, then the Lorentzian Lie group M is future distinguishing.* ∎

We note that similar arguments have been used in [5] in the course of proving the future distinguishability of a certain class of Lorentzian symmetric spaces.

Note added in 1992. In [2] the authors introduced the notion of strict causality. For homogeneous Lorentzian spaces this concept agrees with that of distinguishability (see e.g. [4, 6]). In particular, the Lorentzian in the present article as well as the symmetric spaces in [5] are strictly causal.

We also avail ourselves the opportunity of pointing out that, in the English translation [5] of our article "Prescribing the conformal geometry..." the formula labelled (3) was inadvertantly omitted. It should read

$$ds^2 = \sum_{i=1}^{n-2} dx_i^2 - 2dx_{n-1}dx_n - \left(\sum_{i=1}^{n-2} \lambda_i x_i^2\right) dx_n^2. \tag{3}$$

2. Future distinguishability of the space M_1

The Lie algebra $L_1 = L(M_1)$ can be defined by the commutation rule

$$[e_4, e_2] = e_1. \tag{1}$$

We can realize M_1 as $\mathbb{R}^4$ with multiplication $z = x \cdot y$ given by

$$z = (x_1 + y_1 - x_2y_4,\ x_2 + y_2,\ x_3 + y_3,\ x_4 + y_4)\ . \tag{2}$$

We fix a Lorentzian form $\widetilde{g} = (\widetilde{g}_{ij})$ with $\widetilde{g}_{13} = \widetilde{g}_{31} = \widetilde{g}_{22} = \widetilde{g}_{44} = 1$ in L_1 and extend it to M_1 via left translation to get a homogeneous Lorentzian manifold. We also fix the future cone $K^+ = \{a \in L_1 : a^2 < 0,\ a_3 > 0\}$ and use for this LLG the same notation M_1 as for the Lie group itself.

It was proved in [8] that M_1 fails to be uniformly stably causal whence it is causal with $I^+ \subset \{x : x_3 > 0\}$.

To apply Theorem 1 we find the points conjugate to $\mathbf{1}$ along future light geodesics issuing from $\mathbf{1}$.

In the system (2) the equations for a geodesic $x = x(t)$ passing through $\mathbf{1}$ with the initial tangent vector $a \in L_1$ are as follows:

$$\begin{aligned}
\dot{x}_1 &= a_1 + a_3x_2^2 - a_4x_2,\\
\dot{x}_2 &= a_2 + a_3x_4,\\
\dot{x}_3 &= a_3,\\
\dot{x}_4 &= -a_3x_2 + a_4,
\end{aligned} \tag{3}$$

with the initial conditions $x(0) = \mathbf{1}$. The geodesic (3) is lightlike iff its initial tangent vector a is lightlike, i.e. satisfies $a^2 = 0$.

We need $ds^2 = g_{ij}dx^idx^j$ in the coordinate basis. The matrix $g = (g_{ij})$ is equal to $B^T\widetilde{g}B$, where $B = A^{-1}$ and $A = \frac{\partial z_i}{\partial y_k}$ is the derivative of the left translation L_x at $\mathbf{1}$. We now compute the Christoffel symbols

$$\Gamma^i_{jk} = \frac{g^{im}}{2}\left(\frac{\partial g_{mj}}{\partial x_k} + \frac{\partial g_{mk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_m}\right)\ , \tag{4}$$

where $(g^{im}) = g^{-1}$. The non-zero components are:

$$\begin{aligned} \Gamma^1_{23} = \Gamma^1_{32} = -\frac{x_2}{2},\ \Gamma^1_{24} = \Gamma^1_{42} = \frac{1}{2}, \\ \Gamma^2_{34} = \Gamma^2_{43} = -\frac{1}{2},\ \Gamma^4_{23} = \Gamma^4_{32} = \frac{1}{2}. \end{aligned} \tag{5}$$

The non-zero connection one-forms ${\Gamma^i}_j = \Gamma^i_{jk}\ dx^k$ are:

$$\begin{aligned} {\Gamma^1}_2 = -\frac{x_2}{2}dx^3 + \frac{1}{2}dx^4,\ {\Gamma^1}_3 = -\frac{x_2}{2}dx^2,\ {\Gamma^1}_4 = \frac{1}{2}dx^2\ , \\ {\Gamma^2}_3 = -\frac{1}{2}dx^4,\ {\Gamma^2}_4 = -\frac{1}{2}dx^3,\ {\Gamma^4}_2 = \frac{1}{2}dx^3,\ {\Gamma^4}_3 = \frac{1}{2}dx^2\ . \end{aligned}$$

The components R^i_{jkm} of the curvature tensor $\mathcal{R}$ may be found from the equality ${\theta^i}_j = \frac{R^i_{jmk}}{2} dx^m \wedge dx^k$ where curvature two-forms ${\theta^i}_j = d{\Gamma^i}_j + {\Gamma^i}_s \wedge {\Gamma^s}_j$. Non-zero of them are:

$$\begin{aligned} {\theta^1}_2 = -\frac{dx^2 \wedge dx^3}{4},\ {\theta^1}_3 = \frac{x_2 dx^3 \wedge dx^4}{4},\ {\theta^1}_4 = \frac{dx^3 \wedge dx^4}{4}, \\ {\theta^2}_3 = \frac{dx^2 \wedge dx^3}{4},\ {\theta^4}_3 = -\frac{dx^3 \wedge dx^4}{4}\ . \end{aligned} \tag{6}$$

The Jacobi field $y = y(t)$ along the geodesic γ given by $x = x(t)$ is found as the solution of the system

$$\frac{\mathcal{D}^2 y}{dt^2} + \mathcal{R}(y, \dot{x})\dot{x} = 0\ , \tag{7}$$

where $\mathcal{D}/dt$ is the covariant derivative along γ.

From the 3rd equation of (7) we extract $y_3 \equiv 0$ since we are searching for only those Jacobi fields which become zero at $\mathbf{1}$ (i.e. when $t = 0$), and at least at one additional point of γ with $t > 0$. Let us also consider the three other equations. The second and third of them form the subsystem

$$\begin{aligned} \ddot{y}_2 - a_3\dot{y}_4 = 0\ , \\ \ddot{y}_4 + a_3\dot{y}_2 = 0\ , \end{aligned} \tag{8}$$

which can be easily integrated to yield $y_2 = \frac{\lambda_2(1-C)-\lambda_1 S}{a_3}$, $y_4 = \frac{\lambda_1(1-C)-\lambda_2 S}{a_3}$, where we write S, C for $\sin a_3 t$ and $\cos a_3 t$, respectively, and where λ_1, λ_2 are integration constants. These solutions also fulfil the initial conditions $y_2(0) = y_4(0) = 0$.

Such a Jacobi field is orthogonal to the tangent vector of the geodesic γ [1, p. 294]. The component $y_1(t)$, therefore, may be found from the equation

$$0 = \langle y, \dot{x} \rangle = y_1 a_3 + \dot{x}_2 y_2 + 2x_2 y_4 a_3 + \dot{x}_4 y_4\ .$$

Note that there is only one null-geodesic through $\mathbf{1}$ with $a_3 = 0$. It coincides with a one-parameter subgroup, is contained in $T = \{x \colon x_3 = 0\}$, and has no points conjugate to $\mathbf{1}$ along itself. That is why our Jacobi field becomes zero iff $x_3 = \pi k$ with $k \in \mathbb{Z}$. Therefore, the slice $0 < x_3 < \pi$ has no points conjugate to $\mathbf{1}$ along null geodesics. Thus the first hypothesis of Theorem 1 is satisfied. In order to verify the second we note that the equations (3) are readily integrated and have the following solutions for $a_3 \neq 0$:

We set $C(t) = \cos a_3 t$, $S(t) = \sin a_3 t$, $S_2(t) = \sin(2a_3 t)$ and $C_2(t) = \cos(2a_3 t)$. Then we have

$$x(t; a_1, \ldots, a_4) \begin{cases} x_1(t; a_1, \ldots, a_4) = \frac{a_2 a_4}{2a_3^2} + \frac{a_2 a_4}{2a_3^2}(C_2(t) - 2C(t)) + \\ \qquad\qquad (a_1 + \frac{a_2^2 + a_4^2}{2a_3})t + \frac{a_4^2 - a_2^2}{4a_3^2} S_2(t) - \frac{a_4^2}{a_3^2} S(t), \\ x_2(t; a_1, \ldots, a_4) = \frac{a_4}{a_3}(1 - C(t)) + \frac{a_2}{a_3} S(t), \\ x_3(t; a_1, \ldots, a_4) = a_3 t, \\ x_4(t; a_1, \ldots, a_4) = \frac{a_2}{a_3}(C(t) - 1) + \frac{a_4}{a_3} S(t). \end{cases}$$

The function $f\colon L_1 \to M_1$, $f(a_1, a_2, a_3, a_4) = x(a_3; a_1, a_2, 1, a_4)$ maps a suitable open subset $\mathcal{V}$ in L_1 diffeomorphically onto the slice $\mathcal{U}$ in M_1. Let γ^+ denote the future geodesic ray from the identity in the direction of e_1. The f maps the portion $\partial K^+ \setminus \gamma^+$ of the boundary of the light cone onto the set $\partial I^+ \cap \mathcal{U}$ of all points on all light like geodesics in $\mathcal{U}$. Since $\partial K^+ \setminus \gamma^+$ divides $\mathcal{V}$ into two components, then $\partial I^+ \cap \mathcal{U}$ separates $\mathcal{U}$ into two components. Thus the hypotheses of Theorem 1 are satisfied and we obtain the following result:

Theorem 2. *A Lorentzian Lie group M_1 is future distinguishing (hence also strongly causal, stably causal, causally continuous in view of homogeneity).* ∎

3. Future distinguishability of the spaces $M_2 = M_2(q)$.

The Lie algebra $L_2 = L(M_2)$ is defined via

$$[e_4, e_1] = e_1 \; . \tag{9}$$

The Lie group M_2 itself is $\mathbf{R}^4$ with $z = x \cdot y$ given by

$$z = (x_1 + y_1 e^{x_4},\ x_2 + y_2,\ x_3 + y_3,\ x_4 + y_4) \; . \tag{10}$$

The commutation rule for its Lie algebra in this coordinate system is exactly (9). We fix a Lorentzian form

$$\widetilde{g} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

at $\mathbf{1} = (0, 0, 0, 0)$ and extend it to M_2 via left translations. We thus get for every $q > 0$ a homogeneous Lorentzian space, also denoted by M_2. It is proved in [7] that M_2 is not uniformly stably causal. If a future cone in L_2 is fixed by

$K^+ = \{a \in L_2 : a^2 < 0,\ a_2 > 0\}$, then I^+ is entirely contained in the halfspace $x_2 > 0$ and M_2 possesses no causal cycles.

Here are the equations for a geodesic $\gamma : x = x(t)$ passing through $\mathbf{1}$ with initial tangent vector a :

$$\begin{aligned} \dot{x}_1 &= a_1 e^{x_4} + a_2(e^{2x_4} - e^{x_4}), \quad \dot{x}_2 = a_2 e^{x_4} \ , \\ \dot{x}_3 &= a_3, \quad \dot{x}_4 = -\frac{a_2 x_1}{q} + a_4 \ . \end{aligned} \tag{11}$$

They are more complicated than (3), but we shall solve our problems without integrating them.

We find $ds^2 = g_{ij} dx^i dx^j$ in the coordinates (10) as in Section 2:

$$ds^2 = 2e^{-x_4} dx_1 dx_2 - dx_2^2 + dx_3^2 + q dx_4^2 \ .$$

The non-zero connection one-forms are

$$\begin{aligned} &\Gamma^1{}_1 = \frac{dx^4}{2}, \ \Gamma^1{}_2 = -\frac{e^{x_4} dx^4}{2}, \ \Gamma^1{}_4 = -\frac{dx^1}{2} - \frac{e^{x_4} dx^2}{2}, \\ &\Gamma^2{}_2 = -\frac{dx^4}{2}, \ \Gamma^2{}_4 = -\frac{dx^2}{2}, \ \Gamma^4{}_1 = \frac{e^{-x_4} dx^2}{2q}, \ \Gamma^4{}_2 = \frac{e^{-x_4} dx^1}{2q}. \end{aligned}$$

Similarly to (6), the expressions for the curvature two-forms are

$$\begin{aligned} &\theta^1{}_1 = -\frac{e^{-x_4} dx^1 \wedge dx^2}{4q}, \ \theta^1{}_2 = \frac{dx^1 \wedge dx^2}{4q}, \ \theta^1{}_4 = -\frac{dx^1 \wedge dx^4}{2}, \\ &\theta^2{}_2 = \frac{e^{-x_4} dx^1 \wedge dx^2}{4q}, \ \theta^2{}_4 = -\frac{dx^2 \wedge dx^4}{4}, \ \theta^4{}_1 = \frac{e^{-x_4} dx^2 \wedge dx^4}{4q}, \\ &\theta^4{}_2 = \frac{e^{-x_4} dx^1 \wedge dx^4}{4q} - \frac{dx^2 \wedge dx^4}{4q}. \end{aligned}$$

As in Section 2, we are searching for the solution $y(t)$ of the system (7) with $y(0) = y(t) = 0$ for some $t > 0$. That is why $y_3 \equiv 0$. The other equations form a system for y_1, y_2, y_4 as follows:

$$\begin{aligned} \ddot{y}_1 - \dot{y}_1 \dot{x}_4 - \dot{y}_2 \dot{x}_2 e^{x_4} - \dot{y}_4(\dot{x}_1 + \dot{x}_2) - y_4 \dot{x}_2 \dot{x}_4 = 0, \\ \ddot{y}_2 - \dot{y}_2 \dot{x}_4 - \dot{y}_4 \dot{x}_2 = 0 \ , \\ \ddot{y}_4 + \frac{a_2 \dot{y}_1 + \dot{x}_1 e^{-x_4} \dot{y}_2 - a_2 \dot{x}_1 y_4}{q} = 0. \end{aligned} \tag{12}$$

We find y_1 from $0 = \langle y, \dot{x} \rangle = g_{ij} y^i \dot{x}^j$ and get a linear first-order system for $\dot{y}_2$, $\dot{y}_4$:

$$\begin{aligned} &\ddot{y}_2 - \dot{x}_4 \dot{y}_2 - \dot{x}_2 \dot{y}_4 = 0 \ , \\ &\ddot{y}_4 + \frac{\dot{x}_2 \dot{y}_2}{q} - \dot{x}_4 \dot{y}_4 = 0 \ . \end{aligned} \tag{13}$$

It can be writen as $\dot{z} = A(t)z$ with $z = (\dot{y}_2, \dot{y}_4)$ and

$$A = \begin{pmatrix} \dot{x}_4 & \dot{x}_2 \\ -\dot{x}_2/q & \dot{x}_4 \end{pmatrix}.$$

The matrix A of this system commutes with its integral

$$\begin{pmatrix} x_4 & x_2 \\ -x_2/q & x_4 \end{pmatrix}.$$

The latter makes it possible to find the fundamental solutions

$$\dot{y}_2 = e^{x_4} C(x_2),\ \dot{y}_4 = -\frac{e^{x_4} S(x_2)}{\sqrt{q}},$$

and

$$\dot{y}_2 = \sqrt{q} e^{x_4} S(x_2),\ \dot{y}_4 = e^{x_4} C(x_2),$$

where $C(x_2) = \cos \frac{x_2}{\sqrt{q}}$, $S(x_2) = \sin \frac{x_2}{\sqrt{q}}$.

In order to integrate these equations, we will use (11):

$$y_2(\tau) = \int_0^\tau e^{x_4(t)} C(x_2(t))\, dt = \int_0^\tau \frac{\dot{x}_2(t)}{a_2} C(x_2(t))\, dt = \frac{\sqrt{q}}{a_2} \int_0^{x_2(\tau)} dS = \frac{\sqrt{q} S(x_2(\tau))}{a_2}.$$

Therefore,

$$y_2 = \frac{\lambda_1 \sqrt{q} S(x_2) + \lambda_2 q(1 - C(x_2))}{a_2},\ y_4 = \frac{\lambda_1 (C(x_2) - 1) + \lambda_2 \sqrt{q} S(x_2)}{a_2} \tag{14}$$

is the general solution of (13). It is not difficult now to find $y_1(t)$, but for our purposes this will not be necessary.

There is only one null geodesic with $a_2 = 0$. It coincides with a one-parameter subgroup, is contained in $T = \{x : x_2 = 0\}$ and has no points conjugate to $\mathbf{1}$ along itself. We deduce, as in Section 2, that the points cojugate to $\mathbf{1}$ along null future geodesics closest with respect to x_2 lie in the hypersurface $x_2 = \pi\sqrt{q}$. So the first hypothesis of Theorem 1 is satisfied.

In order to verify the second, we integrate the equations (11): We set $b = 1 + a_3^2 + q a_4^2$, $\Delta = 4a_3^2 - b^2$. Note that $b > 0$ and thus $\Delta < 0$. Further set $S(t) = \sin \frac{a_3(t+\lambda)}{\sqrt{q}}$ and $C(t) = \cos \frac{a_3(t+\lambda)}{\sqrt{q}}$ with λ solving the equation

$$\sqrt{-\Delta} \sin \frac{a_3 \lambda}{\sqrt{q}} = b^2 - 2a_3^2.$$

Now we obtain

$$x(t; a_1, \ldots, a_4) = \begin{cases} x_1(t; a_1, \ldots, a_4) = q\left(a_4 + \frac{(a_3/\sqrt{q})\sqrt{-\Delta} C(t)}{\sqrt{-\Delta} S(t) - b}\right), \\ x_2(t; a_1, \ldots, a_4) = h + 2\sqrt{q} \arctan \frac{b \tan \frac{a_3(t+\lambda)}{2\sqrt{q}} - \sqrt{-\Delta}}{2a_3}, \\ x_3(t; a_1, \ldots, a_4) = a_3 t, \\ x_4(t; a_1, \ldots, a_4) = \log(2a_3^2) - \log(b - \sqrt{-\Delta} S(t)), \end{cases} \tag{15}$$

where h is a constant chosen so that $x_2(0) = 0$.

In the equation for x_2 in (15) it is understood that the values of the function are defined for all $t \in \mathbb{R}$ by continuous extension. By an argument similar to that used in the proof of Theorem 2 we now conclude that also the second hypothesis of Theorem 1 is satisfied. Therefore we obtain:

Theorem 3. *The Lorentzian Lie group $M_2(q)$ is future distinguishing for all $q > 0$. Hence is also strongly causal, stably causal and causally continuous in view of its homogeneity.*

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