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Toric varieties and algebraic monoids

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Introduction

Let $\mathbb{K}$ be a field and $\mathbb{K}^* = G_m$ the multiplictive group of $\mathbb{K}$ viewed as an algebraic group over $\mathbb{K}$. A $\mathbb{K}$-torus is an algebraic $\mathbb{K}$-group isomorphic to a direct product of such groups. In the theory of toric varieties or toroidal embeddings (cf. [5] and also [4], [8]) one studies algebraic varieties which occur as closures of torus orbits for algebraic actions of a torus on an algebraic variety. These varieties are called *toric varieties* and they can be obtained by affine toric varieties via a certain glueing process (cf. [8, Ch. 1]).

In this note we show how the structure of affine toric varieties M, and those which occur by linear actions on projective spaces, can be analysed using the concept of affine algebraic monoids (cf. [7]). The basic observation is that an affine toric variety carries a natural structure of an algebraic monoid. One of the main results states that the lattice of orbit closures of the torus T in M coincides with the finite lattice $E(M)$ of idempotents of the monoid M.

To compute such a lattice in concrete situations one also needs a method to read it off the structural data given by the torus action. Such a method is provided by the identification of the set $E(M)$ with a lattice of faces of a polyhedral cone in a real vector space which can be computed directly from the data defining the torus action. In the case of a linear action on a projective space the lattice of orbit closures can be identified with the lattice of faces of a convex polytope.

Throughout the whole paper, if not stated otherwise, $\mathbb{K}$ denotes an *arbitrary* field, in Section IV $\mathbb{K}$ is an infinite field, and in Section V a local field. For algebraically closed fields the main results of Sections I - IV are in one or the other form contained in [5] and [7]. To be more precise, the fact that a toric monoid M over an algebraically closed field $\mathbb{K}$ is the product of the lattice of idempotents and its group of units can be found in [7, p.23] and the correspondence between orbit closures and faces of a certain cone is Theorem 2 in [5, p.9].

We think that the advantage of our approach is that the use of the monoid structure on the orbit closures permits relative elementary proofs for the relevant results. We also tried to keep the exposition as self-contained as possible. In Section I we collect some material on polyhedral cones in real vector spaces and in Section II we consider finitely generated subsemigroups of $\mathbb{Z}^n$ and integral polyhedral cones. The main difficulty arising in this context is that such a finitely generated semigroup need not be saturated, i.e., the intersection of a cone in $\mathbb{R}^n$ with $\mathbb{Z}^n$. Section III contains some generalities on diagonalizable monoids over an arbitrary field $\mathbb{K}$. In Section IV we prove the Structure Theorem for Toric Monoids and we show how the lattice of idempotents of such a monoids can be viewed as the lattice of faces of a polyhedral cone. These results are applied to the analysis of the lattice of orbit closures of a torus action on an affine variety and a linear action on projective space. Motivated by the observation that the structure of a toric monoid M is very similar to the structure of a compact abelian semigroup, we show in Section V how toric varieties over local fields can also be described via compact semigroups.

I. Polyhedral cones in real vector spaces

In this section V always denotes a finite dimensional real vector space. We collect some material on polyhedral cones in real vector space which will be complemented by similar material on polyhedral cones in free abelian groups in the next section.

Definition I.1. A closed convex cone W in a finite dimensional vector space V is called a *wedge*. The vector space $H(W) := W \cap (-W)$ is called *the edge of* W. We say that W is *pointed* if $H(W) = \{0\}$ and that W is *generating* if $W - W = V$, i.e., if $\mathrm{int}(W) \neq \emptyset$. A *polyhedral wedge* is defined to be a finite intersection of closed half spaces. The *dual wedge* is the set

$$W^\star := \{\alpha \in V^* : (\forall w \in W)\alpha(w) \geq 0\}.$$

A *face* F of a wedge W is a subsemigroup such that $W \setminus F$ is a semigroup ideal in W. The complete lattice of faces of a wedge W is denoted $\mathrm{Fa}(W)$. ∎

Proposition I.2. *Let V be a finite dimensional real vector space and $W \subseteq V$. Then* (1)-(3) *are equivalent and imply* (4),(5).

(1) *There exists a finite subset $E \subseteq V$ such that $W = W_E := \sum_{e\in E} \mathbb{R}^+ e$.*

(2) *W is a polyhedral wedge.*

(3) *The dual wedge $W^\star \subseteq V^*$ is polyhedral.*

(4) *For every face $F \in \mathrm{Fa}(W)$ there exists a finite subset $D \subseteq E$ such that the following assertions hold:*

  (a) $F = W_D$.

  (b) $W - F = W_{E \cup -D}$ *is a wedge.*

  (c) $H(W - F) = F - F = W_{D \cup -D}$.

  (d) $(F - F) \cap W = F$.

(5) *The mapping*

$$\mathrm{op} : \mathrm{Fa}(W) \to \mathrm{Fa}(W^\star), \quad F \mapsto F^\perp \cap W^\star$$

defines an antiisomorphism of finite lattices. Moreover,

$$\mathrm{op}(F) - \mathrm{op}(F) = F^\perp \qquad \forall F \in \mathrm{Fa}(W).$$

(6) $\dim F + \dim \mathrm{op}(F) = \dim V$ *for all* $F \in \mathrm{Fa}(W)$.

Proof. (1) $\Rightarrow$ (2): We prove by induction over the cardinality of E that W_E is closed. This is clear for $|E| = 0, 1, 2$. Assume that $2 < |E|$ and that all cones which are sum of less than $|E|$ rays are closed. Pick $e \in E$ and set $E' := E \setminus \{e\}$. Then $W_E = W_{E'} + \mathbb{R}^+ e$ and $W_{E'}$ is closed. Factoring the edge of $W_{E'}$ we even may assume that $W_{E'}$ is pointed. If $-\mathbb{R}^+ e \cap W_{E'} = \{0\}$, then the convex set

$$K := \mathrm{conv}(E) = \mathrm{conv}(E' \cup \{e\})$$

does not contain 0. Hence this is a compact convex set which is separated from 0, so $W_E = \mathbb{R}^+ K$ is closed.

If $-e \in W_{E'}$, then $W_E = W_{E \cup \{-e\}}$. Let $q\colon V \to V/\mathbb{R}e$ denote the quotient mapping. Then

$$q(W) = q(W_{E\setminus\{e\}}) = q(W_{E'})$$

is closed by induction hypothesis. Hence W is closed since $\mathbb{R}e \subseteq H(W)$.

Since the dual wedge

$$W^\star = \{\omega \in V^* : (\forall e \in E)\omega(e) \geq 0\}$$

is polyhedral, it follows from [2, I.4.4] that W_E is polyhedral.
(2) $\Rightarrow$ (3): This is a consequence of [2, I.4.4].
(3) $\Rightarrow$ (1): [2, I.4.2].
(4), (a) It is a direct consequence of the definition of $\mathrm{Fa}(W)$ that $\sum_{e\in E} \lambda_e e \in F$ implies that $e \in F$ whenever $\lambda_e > 0$. Hence $F = W_D$ for $D := F \cap E$.
(b) $W - F = W_{E \cup -D}$ follows from (a) and the closedness from (1).
(c) The inclusion $F - F \subseteq H(W - F)$ is clear. The converse follows from the fact that $w - f = f' - w' \in W - F$, where $w, w' \in W, f, f' \in F$, implies that $w + w' = f + f' \in F$ and hence that $w \in F$.
(d) Let $f - f' = w \in W$. Then $f = f' + w$ and therefore $w \in F$.
(5) It follows from (4) that every face F of W is *exposed*, i.e., that

$$F = H(\overline{W - F}) \cap W$$

([2, I.2.2]). So the first assertion follows from [2, I.2.4, I.4.3].

The second assertion is a consequence of

$$\mathrm{op}(F) = F^\perp \cap W^\star = (W - F)^\star$$

which implies that

$$\mathrm{op}(F) - \mathrm{op}(F) = (W - F)^\star - (W - F)^\star = H(W - F)^\perp = (F - F)^\perp = F^\perp.$$

(6) This equality follows from

$$\dim F + \dim \mathrm{op}(F) = \dim F + \dim\big(\mathrm{op}(F) - \mathrm{op}(F)\big) = \dim F + \dim F^\perp = \dim V.$$

■

II. Subsemigroups of free abelian groups

In this section X denotes a free abelian group with $\mathrm{rank}(X) = n$, i.e., $X \cong \mathbb{Z}^n$. We write

$$\check{X} := \mathrm{Hom}(X, \mathbb{Z})$$

for the group of all homomorphisms into $\mathbb{Z}$. Note that $\check{X} \cong \mathbb{Z}^n$ and that we have a pairing

$$X \times \check{X} \to \mathbb{Z}, \quad (x, \omega) \mapsto \omega(x).$$

Example II.1. $X := \mathbb{Z}(1,1) \oplus \mathbb{Z}(-1,1) \subseteq \mathbb{R}^2$ is a free abelian group of rank 2. Its dual group may be realized as

$$\check{X} := \mathbb{Z}(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}(-\frac{1}{2}, \frac{1}{2}).$$

Note that $X \subseteq \check{X}$ is a subgroup of index 4. ■

Definition II.2. A subsemigroup $C \subseteq X$ is called an *integral polyhedral cone* if there exists a finite subset $D \subseteq \check{X}$ such that

$$C = D^\star := \{x \in X : (\forall \chi \in D)\chi(x) \geq 0\}.$$

In general we define $D^\star \subseteq X$ for a subset $D \subseteq \check{X}$ as above, and similary $E^\star \subseteq \check{X}$ for a subset $E \subseteq X$.

A *face* F of a semigroup S is a subsemigroup satisfying

$$(S \setminus F) + S \subseteq S \setminus F.$$

The set of all faces of S is denoted $\mathrm{Fa}(S)$.

A subsemigroup S of a group G is said to be *saturated* if $g \in G$, $n \in \mathbb{N}$ and $g^n \in S$ entails that $g \in S$. ■

Remark II.3. Let $X_{\mathbb{R}} := \mathbb{R} \otimes X$ and identify $\check{X}$ with a subset of the dual vector space $X_{\mathbb{R}}^*$. For a finitely generated subsemigroup $C \subseteq X$ we write $C_{\mathbb{R}}$ for the polyhedral cone in $X_{\mathbb{R}}$ spanned by the generators of C, i.e., $C_{\mathbb{R}}$ is the smallest wedge in $X_{\mathbb{R}}$ containing C.

Then it is clear that every integral polyhedral cone in $C = D^\star$ is the intersection of the polyhedral cone

$$C_{\mathbb{R}} = \{x \in X_{\mathbb{R}} : (\forall \chi \in D)\chi(x) \geq 0\}$$

with the lattice X. This extension of C often permits to use results from real convex geometry to obtain results on integral cones. ■

Lemma II.4. *Let $C \subseteq X$ be a finitely generated subsemigroup and $C_{\mathbb{R}} \subseteq X_{\mathbb{R}}$ the corresponding polyhedral cone. Then $X \cap C_{\mathbb{R}}$ is finitely generated.*

Proof. Suppose that C is generated by the set E. Then

$$C_{\mathbb{R}} \subseteq C + \sum_{e \in E} [0,1]e.$$

The set $K := \sum_{e \in E} [0,1]e$ is compact so that it contains at most finitely many elements of the discrete group X. It follows that $(C_{\mathbb{R}} \cap X) \cap K$ is finite. Thus $C_{\mathbb{R}} \cap X = \langle E \cup (C_{\mathbb{R}} \cap X \cap K) \rangle$ is finitely generated. ∎

Theorem II.5. (Gordan's Lemma) *A polyhedral integral cone C is a finitely generated saturated subsemigroup of X.*

Proof. (cf. [1, p.200]) If $x \in X$ and $n \in \mathbb{N}$ with $nx \in C$, then $\chi(nx) = n\chi(x) \geq 0$ for all $\chi \in D$. Whence $\chi(x) \geq 0$ and thus $x \in C$. This shows that C is saturated.

Let C be a polyhedral integral cone in X. We consider the extension $X_{\mathbb{R}} = \mathbb{R} \otimes X$ and set $W := \sum_{\chi \in D} \mathbb{R}^+ \chi$. This is a polyhedral cone in $X_{\mathbb{R}}^*$ and $C_{\mathbb{R}} = W^\star$ is a polyhedral cone in $X_{\mathbb{R}}$ satisfying

$$C_{\mathbb{R}} \cap X = C.$$

Thus it is generated by finitely many extremal rays, i.e., there exists a finite set $E' \subseteq C_{\mathbb{R}}$ with

$$C_{\mathbb{R}} = \sum_{e' \in E'} \mathbb{R}^+ e'$$

such that $\mathbb{R}^+ e'$ is a one-dimensional face of $C_{\mathbb{R}}$ for every $e' \in E'$. Let $n = \dim X_{\mathbb{R}}$. Then Proposition I.2(6) entails that $\mathrm{op}(\mathbb{R}^+ e')$ has dimension $n-1$. In view of Proposition I.2(4), we find $n-1$ linearly independent elements $\chi_1, \ldots, \chi_{n-1}$ in $D \subseteq X_{\mathbb{R}}^*$ such that

$$\mathbb{R}^+ e' = C_{\mathbb{R}} \cap \bigcap_{i=1}^{n-1} \ker \chi_i.$$

Since the functionals χ_i are integral on X, the Rank-Theorem for finitely generated abelian groups shows that

$$\bigcap_{i=1}^{n-1} \ker \chi_i \cap X = \mathbb{Z}e$$

for an element $e \in \mathbb{R}^+ e'$. This proves that

$$C_{\mathbb{R}} = \sum_{e \in E} \mathbb{R}^+ e,$$

where $E \subseteq C$ is a finite subset with the property that

$$\mathbb{R}e \cap X = \mathbb{Z}e \qquad \forall e \in E.$$

Now Lemma II.4 shows that $C = C_{\mathbb{R}} \cap X$ is finitely generated. ∎

Remark II.6. Let $E \subseteq C$ be as in the proof of Theorem II.5 and suppose that D generates $\check{X}$, and that $n = \operatorname{rank} X$. Then $D^\perp = H(C) = \{0\}$ and for every $e \in E$ there exists $D_e := \{\chi_1, \ldots, \chi_{n-1}\} \subseteq D$ with

$$D_e^\perp = \mathbb{Z}e \quad \text{and} \quad D_e^\perp \cap C = \mathbb{N}_0 e.$$

It follows in particular that

$$F_e := \mathbb{N}_0 e \in \operatorname{Fa}(C)$$

because $-\mathbb{N}e \cap C = \emptyset$.

In contrast to the real situation, it is no longer true that C is generated by such elements. A simple example is given by

$$X = \mathbb{Z}^2, \quad D = \{(1,1), (1,-1)\}$$

and

$$C = D^\star = \{(n,m) \in X : n \geq |m|\}.$$

Let $E = D$. Then the cyclic faces of C are generated by the elements of E and

$$C \setminus \langle E \rangle = 2\mathbb{N} \times \{0\}.$$

It follows that a minimal system of generators for C contains at least three elements, f.i. $\{(1,1), (1,-1), (1,0)\}$ is such a system. ■

Lemma II.7. *Let $C \subseteq X$ be a finitely generated semigroup. Then the smallest integral polyhedral cone containing C is given by*

$$(C^\star)^\star = C_s := \{x \in X : (\exists n \in \mathbb{N}) nx \in C\}.$$

Moreover, the mapping

$$\operatorname{Fa}(C) \to \operatorname{Fa}(C_s), \quad F \mapsto F_s := \{x \in X : (\exists n \in \mathbb{N}) nx \in F\}$$

is an isomorphism of finite lattices.

Proof. First we note that C_s is saturated.

Let E be a finite set generating C, and $C_\mathbb{R} \subseteq X_\mathbb{R}$ the corresponding real polyhedral cone generated by the finite set E. Then

$$C^\star \subseteq \check{X}$$

may be obtained as

$$C^\star = (C_\mathbb{R})^\star \cap \check{X} = E^\star \cap \check{X}.$$

Using Theorem II.5, we see that $C^\star$ is a finitely generated subsemigroup of $\check{X}$ and that $C_\mathbb{R}^\star$ is the polyhedral cone generated by $C^\star$. Suppose that D is a finite generating subset for $C^\star$. Then

$$C_\mathbb{R} = \big((C_\mathbb{R})^\star\big)^\star = D^\star$$

and it remains to show that
$$C_{\mathbb{R}} \cap X = C_s.$$

The inclusion $C_s \subseteq C_{\mathbb{R}} \cap X$ is trivial, so let $c \in C_{\mathbb{R}} \cap X$. Since $C_{\mathbb{R}}$ is generated by E, Caratheodory's Theorem shows that there exists a linearly independent generating subset $E' \subseteq E$ with $c \in \sum_{e' \in E'} \mathbb{R}^+ e'$. Now the group $X_{E'} := \langle E' - E' \rangle$ has finite index in the group $X_{E'} + \mathbb{Z}c$ so that there exist $n \in \mathbb{N}$ and $n_{e'} \in \mathbb{N}_0$ with

$$nc = \sum_{e' \in E'} n_{e'} e' \in X_{E'}.$$

Next $nc \in \sum_{e' \in E'} \mathbb{R}^+ e'$ shows that $n_{e'} \in \mathbb{N}_0$ for all $e' \in E'$. Now $nc \in C$ and $c \in C_s$ follows. We have thus proved that $C_s = C_{\mathbb{R}} \cap X$.

Let $F \in \mathrm{Fa}(C)$ and $x, y \in C_s$ with $x + y \in F_s$. Pick $n, m, k \in \mathbb{N}$ with $nx, my \in C$ and $k(x+y) \in F$. Then

$$kmnx + kmny \in F \quad \text{and} \quad kmnx, kmny \in C.$$

Whence $kmnx, kmny \in F$ and therefore $x, y \in F_s$. Thus $F_s \in \mathrm{Fa}(C_s)$. That $F_s \cap C = F$ follows immediately from the definition of a face.

If $F' \in \mathrm{Fa}(C_s)$, then $F' \cap C \in \mathrm{Fa}(C)$ and for every $f' \in F'$ there exists $n \in \mathbb{N}$ with $nf' \in C \cap F'$. Hence $F' \subseteq (F' \cap C)_s$ and the inclusion $(F' \cap C)_s \subseteq F'$ follows from the fact that $F' \in \mathrm{Fa}(C_s)$. Whence we have proved that the mappings
$$\mathrm{Fa}(C) \to \mathrm{Fa}(C_s), \quad F \mapsto F_s$$
and
$$\mathrm{Fa}(C_s) \to \mathrm{Fa}(C), \quad F' \mapsto F' \cap C$$
are inverses of each other. ■

The following theorem gives an algebraic characterization of integral polyhedral cones without refering to the order structure of $\mathbb{Z}$.

Theorem II.8. *Let X be a finitely generated free abelian group and $C \subseteq V$ a subsemigroup. Then the following are equivalent:*

(1) *C is finitely generated and saturated.*

(2) *C is an integral polyhedral cone.*

Proof. (1) $\Rightarrow$ (2): This follows from Lemma II.7.
(2) $\Rightarrow$ (1): This is Theorem II.5. ■

Proposition II.9. *Let X be a finitely generated free abelian group and $C \subseteq X$ a subsemigroup generated by the finite set E. Then the following assertions hold:*

(i) *The dual cone $C^\star \subseteq \check{X}$ is integral polyhedral.*

(ii) *For every face $F \in \mathrm{Fa}(C)$ there exists a subset $E' \subseteq E$ such that the following assertions hold:*

  (a) *$F = C_{E'} := \langle E' \rangle$.*

(b) $C - F = C_{E \cup -E'}$.

(c) $H(C-F) = F - F = C_{E' \cup -E'}$.

(d) $(F - F) \cap C = F$.

(e) *If C is saturated, then the same holds for F.*

(iii) *The mapping*

$$\mathrm{op} : \mathrm{Fa}(C) \to \mathrm{Fa}(C^\star), \quad F \mapsto F^\perp \cap C^\star$$

defines an antiisomorphism of finite lattices. Moreover,

$$\mathrm{op}(F) - \mathrm{op}(F) = F^\perp \qquad \forall F \in \mathrm{Fa}(C).$$

(iv) $\mathrm{rank}(F - F) + \mathrm{rank}\big(\mathrm{op}(F) - \mathrm{op}(F)\big) = \mathrm{rank}\, X$ *for all* $F \in \mathrm{Fa}(C)$.

Proof. (i) If C is generated by the finite set E, then $C^\star = E^\star$ is integral polyhedral by definition.
(ii), (a) It is a direct consequence of the definition of $\mathrm{Fa}(C)$ that $\sum_{e\in E} \lambda_e e \in F$ implies that $e \in F$ whenever $\lambda_e > 0$. Hence $F = C_{E'}$ for $E' := F \cap E$.
(b) $C - F = C_{E\cup -E'}$ follows from (a).
(c) The inclusion $F - F \subseteq H(C - F)$ is clear. The converse follows from the fact that $c - f = f' - c' \in C - F$, where $c, c' \in C, f, f' \in F$, implies that $c + c' = f + f' \in F$ and hence that $c \in F$.
(d) Let $f - f' = c \in C$. Then $f = f' + c$ and therefore $c \in F$.
(e) If $x \in X$ with $nx \in F$, then $x \in C$ since C is saturated. Thus $x \in F$ because F is a face.
(iii) In view of Proposition I.2, it suffices to show that the mapping

$$\mathrm{Fa}(C_{\mathbb{R}}) \to \mathrm{Fa}(C), \quad F \mapsto F \cap C$$

is bijective and inverted by

$$\mathrm{Fa}(C) \to \mathrm{Fa}(C_{\mathbb{R}}), \quad F \mapsto F_{\mathbb{R}}.$$

First we note that $F_{\mathbb{R}}$ is a polyhedral cone since F is finitely generated by (ii)(a). If $F \in \mathrm{Fa}(C_{\mathbb{R}})$, then, by Proposition I.2(4)(a), $F = C_{\mathbb{R},F\cap E}$. Then (ii)(a) implies that

$$F \cap C = C_{F\cap C\cap E} = C_{F\cap E}$$

holds for the face $F \cap C$ of C. Thus $(F \cap C)_{\mathbb{R}} = C_{\mathbb{R},F\cap E} = F$.

If, conversely, $F \in \mathrm{Fa}(C)$, then $F = C_{F\cap E}$ and $F_{\mathbb{R}} = C_{\mathbb{R},F\cap E}$. We claim that $F_{\mathbb{R}} \in \mathrm{Fa}(C)$. We write F' for the face generated by $F_{\mathbb{R}}$. Then the algebraic interior of F' contains an element f of $F_{\mathbb{R}}$. Using Caratheodory's Theorem, we find a linearly independent subset $E' \subseteq F \cap E$ which generates $F_{\mathbb{R}} - F_{\mathbb{R}}$ such that $f \in C_{\mathbb{R},E'}$. Let $B \subseteq (F' - F')$ be a convex neighborhood of 0 such that $f + B \subseteq F'$ and pick $n \in \mathbb{N}$ with $E' \subseteq nB$. Then $n(f+B) = nf + nB \subseteq \mathrm{algint}\, F'$ contains an element of F. Thus F' is even generated by F and we find that

$$F' = F_{\mathbb{R}} \cap (F - C_{\mathbb{R}}).$$

Thus, in view of Lemma II.7,

$$\begin{aligned} F' \cap C &\subseteq C \cap (F - C_{\mathbb{R}}) = C \cap (F - C_{\mathbb{R}} \cap X) \\ &= C \cap (F - C_s) = C \cap \big(C_s \cap (F - C_s)\big) \\ &\subseteq C \cap \big(C_s \cap (F_s - C_s)\big) = C \cap F_s = F. \end{aligned}$$

We conclude that

$$F' = C_{\mathbb{R}, F' \cap E} = C_{\mathbb{R}, F \cap E} = F_{\mathbb{R}}$$

and that

$$F' \cap C = F.$$

(iv) This follows from Proposition I.2(6), (iii) above, and

$$\dim F_{\mathbb{R}} = \operatorname{rank}(F - F) \qquad \forall F \in \operatorname{Fa}(C).$$

■

Remark II.10. If $C \subseteq X$ is a saturated subsemigroup such that $C^\star$ is integral polyhedral, then C need not be polyhedral (finitely generated). A simple example is $C := \langle \mathbb{N} \times \{1\} \rangle \subseteq \mathbb{Z}^2$. This semigroup is not finitely generated but $C^\star = \langle (1,0), (0,1) \rangle$ is integral polyhedral. ■

We conclude this section with the statement of the main tool that we will use in the following sections to analyze the structure of affine toric varieties.

Corollary II.11. *Let $C \subseteq X$ be generated by the finite set E. Then the following assertions hold:*

(i) $\operatorname{Fa}(C)$ *is a finite lattice with $H(C)$ as minimal and C as maximal element. This lattice is isomorphic to* $\operatorname{Fa}(C_{\mathbb{R}})$.

(ii) *We say that a subset $E' \subseteq E$ is extremal if there exists a face $F \in \operatorname{Fa}(C)$ with $E' = E \cap F$. Then the lattice* $\operatorname{Fa}(C)$ *is isomorphic to the set of extremal subsets of E.*

Proof. The second assertion follows from Proposition II.9(ii) and the first assertion is a consequence of the second and the proof of Proposition II.9. ■

Note that a subset $E' \subseteq E$ is extremal if and only if there exists a subset $D' \subseteq D$ with $E' = E \cap D'^{\perp}$.

III. Diagonalizable monoids

In this section we follow Hochschild's terminology for affine varieties (cf. [3]), i.e., an affine variety over a field $\mathbb{K}$ is a pair $\big(V, P(V)\big)$ of a set V and a finitely generated $\mathbb{K}$-algebra of functions on V such that

(V1) every $\mathbb{K}$-algebra homomorphism $P(V) \to \mathbb{K}$ is a point evaluation

$$v^* : f \mapsto f(v),$$

and

(V2) $P(V)$ separates the points of V.

Then each morphism $\alpha: M \to N$ of algebraic varieties induces a $\mathbb{K}$-algebra morphism

$$\alpha^* : P(N) \to P(M), \qquad f \mapsto f \circ \alpha$$

and, conversely, every $\mathbb{K}$-algebra morphism

$$\beta : P(N) \to P(M)$$

induces a morphism $\beta_*: M \to N$ which is uniquely determined by the condition ${\beta_*}^* = \beta$.

An *algebraic monoid M over a field* $\mathbb{K}$, or a $\mathbb{K}$*-monoid*, is an affine $\mathbb{K}$-variety $\big(M, P(M)\big)$ such that M is a monoid and the multiplication $m_M: M \times M \to M$ is a morphism of $\mathbb{K}$-varieties. We write

$$\lambda_m : M \to M, \quad x \mapsto mx$$

and

$$\rho_m : M \to M, \quad x \mapsto xm$$

for the left and right multiplication on M.

We also write $\mathbb{M}(n, \mathbb{K})$ for the algebraic monoid obtained by the set $\mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)$ together with the algebra of polynomial functions on it. As an algebraic $\mathbb{K}$-variety this set is isomorphic to $\mathbb{K}^{n^2}$. If V is a finite dimensional $\mathbb{K}$-vector space, we simply write $\mathbb{M}(V)$ for the algebraic monoid structure on $\mathrm{End}_{\mathbb{K}}(V)$ obtained by a choice of a basis and transport of the structure from $\mathbb{M}(n, \mathbb{K})$.

We start with a collection of some general results on abelian algebraic monoids which prepare the analysis of toric monoids in Section IV.

Let $\mathbb{K}$ be a field and S an abelian finitely generated monoid. The $\mathbb{K}$-semigroup algebra of S is denoted $\mathbb{K}[S]$. The diagonal mapping

$$\Delta_S : S \to S \times S, \quad s \mapsto (s, s)$$

and the zero-mapping

$$\omega : S \to \{0\}, \quad s \mapsto 0$$

are monoid homomorphisms. Thus they induce $\mathbb{K}$-algebra homomorphisms

$$\delta : \mathbb{K}[S] \to \mathbb{K}[S \times S] \cong \mathbb{K}[S] \otimes \mathbb{K}[S]$$

and

$$\varepsilon : \mathbb{K}[S] \to \mathbb{K}[\{0\}] \cong \mathbb{K}, \quad \sum_{s \in S} \lambda_s s \mapsto \sum_{s \in S} \lambda_s.$$

It follows that

$$(\mathbb{K}[S], \mathbf{1}, m_S, \varepsilon, \delta)$$

is a finitely generated cocommutative $\mathbb{K}$-bialgebra. Thus we obtain an algebraic $\mathbb{K}$-monoid by the prescription

$$\widehat{S} := \mathrm{Hom}_{\mathbb{K}-Alg}(\mathbb{K}[S], \mathbb{K}).$$

Let $\mathbb{K}_m$ denote the multiplicative monoid $(\mathbb{K}, \cdot)$. Then every monoid homomorphism $S \to \mathbb{K}_m$ extends to a $\mathbb{K}$-Algebra morphism $\mathbb{K}[S] \to \mathbb{K}$ and conversely every such algebra morphism restricts to a monoid morphism on S. It follows that

$$\widehat{S} \cong \mathrm{Hom}_{monoid}(S, \mathbb{K}_m)$$

may be identified with the set of monoid morphism $S \to \mathbb{K}_m$. Such monoid morphisms are called *characters* and therefore the algebraic monoid $\widehat{S}$ is called the *character monoid* of S.

From the comultiplication δ on $\mathbb{K}[S]$ we get the multiplication on $\widehat{S}$ by

$$(\chi_1\chi_2)(s) := (\chi_1 \otimes \chi_2) \circ \delta(s) = \chi_1(s)\chi_2(s) \qquad \forall \chi_1, \chi_2 \in \widehat{S}.$$

It follows that the multiplication on $\widehat{S}$ is the pointwise multiplication of characters.

Note that the algebra $P(\widehat{S})$ of polynomial functions is isomorphic to $\mathbb{K}[S]$ if and only if the characters of $\mathbb{K}[S]$ separate the points. To see that the converse also holds, we first note that every element of S defines a polynomial character $\eta(s)$ of $\widehat{S}$ by $\eta(s)(\chi) := \chi(s)$.

Definition III.1. Let M be an algebraic $\mathbb{K}$-monoid. We write

$$X(M) := \mathrm{Hom}_{\mathbb{K}-monoid}(M, \mathbb{K}_m)$$

for the set of all polynomial characters of M. Let $x = \mathrm{id}_{\mathbb{K}}$. Then $P(\mathbb{K}_m) \cong \mathbb{K}[x]$ is the algebra of functions on $\mathbb{K}$ generated by x. If $\mathbb{K}$ is infinite, then $P(\mathbb{K}_m) \cong \mathbb{K}[X]$ is the polynomial algebra in one variable. The comultiplication on $P(\mathbb{K}_m)$ is given by $\delta_{\mathbb{K}_m}(x) = x \otimes x$.

Thus, if $\chi \in X(M)$ and $\chi^* : \mathbb{K}[x] \to P(M)$ the induced morphism of $\mathbb{K}$-Bialgebras, then $f := \chi^*(x)$ satisfies

$$\delta(f) = f \otimes f. \tag{3.1}$$

By elementary duality theory, it follows that $X(M)$ may be identified with the set of all elements in $P(M)$ satisfying (3.1).

An algebraic monoid M is said to be *diagonalizable* if $X(M)$ generates the algebra $P(M)$. Note that, by definition, for every finitely generated monoid S, the algebraic $\mathbb{K}$-monoid $\widehat{S}$ is diagonalizable. ∎

The following lemma is a useful generalisation of the corresponding result in the group case (cf. [7, p.81] for the case where M is irreducible).

Lemma III.2. *Let M be any monoid. Then the set $\widehat{M}$ of characters is linearly independent in $\mathbb{K}^M$.*

Proof. Suppose that this is false. Then we find $\lambda_1, \ldots, \lambda_n$ in $\mathbb{K}^*$ und $\chi_1, ..., \chi_n \in \widehat{M}$ such that

$$\lambda_1\chi_1 + ... + \lambda_n\chi_n = 0. \tag{3.2}$$

We may assume that n is minimal. Since $\chi_1 \neq \chi_2$, there exists $x \in M$ with $\chi_1(x) \neq \chi_2(x)$. Since either $\chi_1(x)$ or $\chi_2(x)$ is non-zero, we may assume that $\chi_1(x) \in \mathbb{K}^*$. Whence

$$(\lambda_1\chi_1 + \ldots + \lambda_n\chi_n)(xm) = \big(\lambda_1\chi_1(x)\big)\chi_1(m) + \ldots + \big(\lambda_n\chi_n(x)\big)\chi_n(m) = 0 \tag{3.3}$$

for all $m \in M$. Multiplying (3.2) with $\chi_1(x)$ and subtracting from (3.3), we find that

$$\lambda_2\big(\chi_2(x) - \chi_1(x)\big)\chi_2 + \ldots + \lambda_n\big(\chi_n(x) - \chi_1(x)\big)\chi_n = 0.$$

This contradicts the minimality of n. ∎

This has an interesting consequence for the character monoids of finitely generated monoids:

Proposition III.3. *Let S be a finitely generated monoid. Then*

$$P(\widehat{S}) \cong \mathbb{K}[S]$$

if and only if $\widehat{S}$ separates the points of S.

Proof. That $\widehat{S}$ separates the points if $P(\widehat{S}) \cong \mathbb{K}[S]$ has already been mentioned above. Suppose, conversely, that $\widehat{S}$ separates the points of S. This means that the monoid homomorphism

$$S \to P(\widehat{S})$$

is injective. According to Lemma III.2, the image is linearly indepent. Thus the extension

$$\mathbb{K}[S] \to P(\widehat{S})$$

is also injective. Since it is surjective by definition, the assertion follows. ∎

Proposition III.4. *Let M be an algebraic $\mathbb{K}$-monoid. Then the following are equivalent:*

(1) *M is diagonalizable.*

(2) *$X(M)$ is a finitely generated monoid,*

$$P(M) \cong \mathbb{K}[X(M)], \quad \textit{and} \quad M \cong \widehat{X(M)}.$$

(3) *If $\pi\colon M \to \mathrm{End}(V)$ is a finite dimensional polynomial representation of M, then there exist characters $\chi_1, \ldots, \chi_n$, such that*

$$V = V_{\chi_1} \oplus \ldots \oplus V_{\chi_n},$$

where

$$V_\chi = \{v \in V\colon (\forall m \in M) m.v = \chi(m)v\}.$$

Proof. (1) $\Rightarrow$ (2): As already mentioned above, we identify $X(M)$ with a submonoid of the algebra $P(M)$. According to Lemma III.2, this set is linearly independent. Since $X(M)$ is a submonoid generating the algebra $P(M)$, the set $X(M)$ is a vector space basis of $P(M)$,

$$P(M) \cong \mathbb{K}[X(M)], \quad \text{and} \quad M \cong \widehat{X(M)}.$$

Let $f_1, \ldots, f_n$ be a finite set generating $P(M)$. Then each f_i is a finite sum of characters. Hence there exists a finite subset of $X(M)$ generating $P(M)$. Since $X(M)$ is linearly independent, $X(M)$ is a finitely generated monoid because a linear combination of characters is a character if and only if it consists of only one summand which is a character.
(2) $\Rightarrow$ (3): Since $\dim V < \infty$, we may assume that $V \subseteq P(M)^n$, where M acts on $P(M)^n$ by the right-regular representation. Since $P(M)$ is a sum of one-dimensional M-submodules, $P(M)$ is a semisimple M-module and the same holds for $P(M)^n$. Whence V is a semisimple M-module and therefore we may assume that it is simple. Then we may even assume that $V \subseteq P(M)$ and now the semisimplicity of $P(M)$ shows that $V = \sum_{\chi \in V} \mathbb{K}\chi$.
(3) $\Rightarrow$ (1): Since the action of M on $P(M)$ is locally finite ([7, p.20]), we find a finite dimensional M-invariant subspace $V \subseteq P(M)$ which generates the algebra $P(M)$. According to (3), the space V is spanned by characters. These characters separate the points of M. This proves that M is diagonalizable. ∎

For the notion of the group of units of an algebraic monoid M we refer to [7, p.24]. This is an algebraic $\mathbb{K}$-group $H(M)$ for which there exists an inclusion

$$j_H : H(M) \to M$$

with the universal property that for every morphism $\alpha\colon G \to M$ of algebraic monoids, where G is an algebraic group, there exists a unique morphism $\alpha'\colon G \to H(M)$ of algebraic groups such that $j_H \circ \alpha' = \alpha$. It is clear that this property determines the algebraic group $H(M)$ up to isomorphy.

Moreover, it follows from [7, p.24] that there exists a character $\chi \in X(M)$ with the property that $H(M) = \{m \in M\colon \chi(m) \neq 0\}$, and

$$P\big(H(M)\big) = \langle j_H^*\big(P(M)\big) \cup j_H^*(\chi)^{-1}\rangle, \tag{3.4}$$

i.e., the algebra $P\big(H(M)\big)$ of polynomials on $H(M)$ is generated by the restrictions of functions on M and the inverse of the character χ. If, f.i., $M = \mathbb{M}(n, \mathbb{K})$, then $H(M) = \mathrm{Gl}(n, \mathbb{K})$ and $\chi = \det$.

Corollary III.5. *Let M be a diagonalizable $\mathbb{K}$-monoid, $H(M)$ its group of units, and $j_H\colon H(M) \to M$ its inclusion. Then the following assertions hold:*

(i) *$H(M)$ is a diagonalizable $\mathbb{K}$-group. More precisely, it is a direct product of a finite diagonalizable $\mathbb{K}$-group and a $\mathbb{K}$-torus.*

(ii) *If $S \subseteq M$ is a closed submonoid, then S is also diagonalizable and every character of S has an extension to a character of M.*

(iii) *Let Y be a finitely generated monoid. Then*

$$\operatorname{Hom}_{monoid}\big(Y, X(M)\big) \cong \operatorname{Hom}_{\mathbb{K}-monoid}(M, \widehat{Y}).$$

(iv) *If $G\big(X(M)\big)$ is the free group over the monoid $X(M)$, then*

$$G\big(X(M)\big)\widehat{\ } \cong H(M).$$

Beweis. (i) Since $j_H^*\big(X(M)\big) \subseteq X\big(H(M)\big)$, it follows from (3.4) that the algebra $P\big(H(M)\big)$ is generated by $X\big(H(M)\big)$, hence $H(M)$ is diagonalizable. The second part of (i) is a direct consequence of the Structure Theorem for finitely generated abelian groups and the fact that

$$(S_1 \oplus S_2)\widehat{\ } \cong \widehat{S}_1 \times \widehat{S}_2.$$

(ii) We consider the surjective $\mathbb{K}$-algebra morphism

$$P(M) \to P(S), \quad \chi \mapsto \chi|_S.$$

Then the image of $X(M)$ generates $P(S)$. Thus S is diagonalizable. Moreover, the image of $X(M)$ consists precisely of $X(S)$ since $X(S)$ is linearly independent. Whence every character of S entends to M because this mapping is surjective.

(iii) Let $\alpha: M \to \widehat{Y}$ be a $\mathbb{K}$-monoid morphism. Then $\alpha^*: P(\widehat{Y}) \to P(M) \cong \mathbb{K}[X(M)]$ is a morphism of $\mathbb{K}$-bialgebras. Let $\eta: \mathbb{K}[Y] \to P(\widehat{Y})$ denote the quotient morphism. Then $\alpha^* \circ \eta: \mathbb{K}[Y] \to \mathbb{K}[X(M)]$ is a morphism of $\mathbb{K}$-bialgebras, hence $\delta_Y(y) = y \otimes y$ for every $y \in Y$ leads to the same relations for the elements in $\alpha^* \circ \eta(Y)$. It follows that

$$\alpha^* \circ \eta(Y) \subseteq X(M).$$

Hence $\alpha^* \circ \eta|_Y: Y \to X(M)$ is a morphism of monoids.

If, conversely, $\beta: Y \to X(M)$ is a morphism of monoids, then it extends to a morphism $\mathbb{K}[\beta]: \mathbb{K}[Y] \to \mathbb{K}[X(M)]$ of $\mathbb{K}$-bialgebras which induces a morphism $\mathbb{K}[\beta]_*: M \to \widehat{Y}$.

Now the relations

$$\mathbb{K}[\alpha^* \circ \eta|_Y]_* = (\alpha^* \circ \eta)_* = \alpha$$

and

$$(\mathbb{K}[\beta]_*)^* \circ \eta|_Y = \mathbb{K}[\beta]|_Y = \beta$$

imply the assertion.

(iv) Let $i: X(M) \to G\big(X(M)\big)$ denote the free group over the monoid $X(M)$. Since $X(M)$ is a finitely generated semigroup, the group $G\big(X(M)\big)$ is finitely generated and abelian. According to the universal property, we find a group homomorphism $j\colon G\big(X(M)\big) \to X\big(H(M)\big)$ such that $j_H^*|_{X(M)} = j \circ i$.

On the level of algebraic monoids the diagram

$$X(M) \xrightarrow[i]{} G\big(X(M)\big) \xrightarrow[j]{} X\big(H(M)\big)$$

induces morphisms

$$H(M) \xrightarrow[j_*]{} G\big(X(M)\big)\widehat{\ } \xrightarrow[i_*]{} M.$$

Now it is immediate that $i_*\colon G\big(X(M)\big)\widehat{\ } \to M$ has the universal property of the unit group of M, hence j_* is an isomorphism of algebraic groups. ■

To get a feeling for the meaning of Corollary III.5(iv), we consider the example $M = \{0,1\}^n$. Then $X(M) \cong \{0,1\}^n$ and $G\big(X(M)\big) \cong \{\mathbf{1}\}$. This means in particular that every morphism of $X(M)$ into a group is constant which is related to the fact that $H(M) = \{\mathbf{1}\}$ is the one-element group.

Proposition III.6. *If M is an algebraic diagonalizable monoid over $\mathbb{K}$ and $H(M)$ its group of units, then the character monoid of $\overline{H(M)}$ is isomorphic to the image of $X(M)$ in the group $X\big(H(M)\big)$.*

Proof. It is clear that $M' := \overline{H(M)}$ is a closed submonoid and the sequence of inclusions

$$H(M) \to M' \to M$$

induces a sequence of finitely generated monoids

$$X(M) \to X(M') \to X\big(H(M)\big),$$

where the morphism on the right hand side is injective and the morphism on the left hand side is surjective. ■

To study semigroups which posess dense subgroups, the preceding proposition tells us that we have to look at finitely generated subsemigroups of finitely generated groups. The easiest, but nevertheless important, special case is when the groups involved are free.

Proposition III.7. *Let G be a diagonalizable $\mathbb{K}$-group. Then the following are equivalent:*

(i) $X(G) \cong \mathbb{Z}^n$.

(ii) $G \cong \widehat{\mathbb{Z}}^n \cong (\mathbb{K}^*)^n$, *i.e., G is a $\mathbb{K}$-torus and $\mathbb{K}$ is infinite.*

Proof. (i) $\Rightarrow$ (ii): If $X(G) \cong \mathbb{Z}^n$, then

$$G = \widehat{X(G)} \cong \widehat{\mathbb{Z}}^n \cong (\mathbb{K}^*)^n.$$

It follows in particular that the characters of $\mathbb{Z}$ separate the points. Hence $\mathbb{K}$ must be infinite.

(ii) $\Rightarrow$ (i): If $\mathbb{K}$ is infinite, then $X(\mathbb{K}^*) \cong \mathbb{Z}$. Hence $G \cong (\mathbb{K}^*)^n$ implies $X(G) \cong \mathbb{Z}^n$. ■

IV. Toric monoids

In this section $\mathbb{K}$ denotes an infinite field. We need this assumption to ensure that $P(\mathbb{K}^*) \cong \mathbb{K}[X]$ and that $\mathbb{K}^*$ is Zariski dense in $\mathbb{K}$.

Definition IV.1. A *toric algebraic monoid* is an irreducible algebraic $\mathbb{K}$-monoid M such that its group of units $H(M)$ is dense and a $\mathbb{K}$-torus, i.e., isomorphic to $(\mathbb{K}^*)^n$, where $\mathbb{K}^*$ is the multiplicative group of $\mathbb{K}$. Note that an algebraic monoid M is toric if and only if it is irreducible and diagonalizable. ■

Lemma IV.2. *If M is a toric algebraic monoid, then $X(M)$ is a finitely generated subsemigroup of the free abelian group $X\big(H(M)\big)$ and*

$$X(M) - X(M) = X\big(H(M)\big).$$

Proof. Since $H(M)$ is dense in M, $X(M)$ is isomorphic to a submonoid of $X\big(H(M)\big)$. The second assertion follows from Proposition III.6. ■

Remark IV.3. Let $T = \mathbb{K}^*$ be the one-dimensional torus. Then $X(T) \cong \mathbb{Z}$ is generated by $\chi := \mathrm{id}_{\mathbb{K}^*}$. We consider some monoids associated with T.

a) Let $C := \langle \chi^2 \rangle \cong 2\mathbb{N}$. Then $M := \widehat{C}$ is a monoid and $X\big(H(M)\big) \cong 2\mathbb{Z} \subseteq X(T)$. The morphism $T \to \widehat{C}$ induced by the inclusion $C \to X(T)$ factors over the square mapping of T and therefore it is not injective if $\mathrm{char}(\mathbb{K}) \neq 2$.

b) Let $C := \langle \chi^2, \chi^3 \rangle$. Then the inclusion $T \to \widehat{C}$ is injective and $X(\widehat{C}) \cong C$ is not saturated in $X(T)$. ■

Having these examples in mind, one may think that $X(M)$ may be a somewhat pathological subsemigroup of $X\big(H(M)\big)$. The situation looks completely different from the dual point of view.

Let M be an algebraic $\mathbb{K}$-monoid. Then an algebraic monoid morphism $\mathbb{K}_m \to M$ is called a (*multiplicative*) *one-parameter semigroup*. The set of all one parameter semigroups is denoted $Y(M)$. Suppose that M is diagonalizable and write $G(S)$ for the free group on a monoid S. Then, in view of Corollary III.5(iii) and Proposition III.6,

$$\begin{aligned} Y(M) &= \mathrm{Hom}_{\mathbb{K}-monoid}(\mathbb{K}_m, M) \\ &\cong \mathrm{Hom}_{\mathbb{K}-monoid}\big(\widehat{\mathbb{N}_0}, \widehat{X(M)}\big) \\ &\cong \mathrm{Hom}_{monoid}\big(X(M), \mathbb{N}_0\big) \\ &\subseteq \mathrm{Hom}_{\mathbb{K}-group}\big(\mathbb{K}^*, H(M)\big). \end{aligned}$$

If G is an algebraic $\mathbb{K}$-group, then let $X_*(G)$ denote the set

$$\mathrm{Hom}_{\mathbb{K}-group}\big(\mathbb{K}^*, H(M)\big)$$

of all one-parameter groups. Thus we can identify $Y(M)$ with a submonoid of the group $X_*\big(H(M)\big)$. More precisely,

$$Y(M) = \{\gamma \in X_*\big(H(M)\big) : \gamma\big(X(M)\big) \subseteq \mathbb{N}_0\}. \tag{4.1}$$

Let us reinforce the global assumption that M is a toric algebraic monoid.

Proposition IV.4. *For every toric algebraic monoid the set $Y(M)$ of one-parameter semigroups is an integral polyhedral cone in the free abelian group $X_*\big(H(M)\big)$.*

Proof. Since the finitely generated semigroup $Y(M)$ sits in the group

$$X_*\big(H(M)\big) \cong \operatorname{Hom}\Big(X\big(H(M)\big), \mathbb{Z}\Big) \cong \mathbb{Z}^n,$$

this follows immediately from (4.1) and the definitions. ∎

Definition IV.5. For every one-parameter semigroup $\gamma \in Y(M)$ we have an idempotent $\gamma(0) \in M$. Let

$$\omega : Y(M) \to E(M), \quad \gamma \mapsto \gamma(0)$$

denote the corresponding evaluation mapping. ∎

First we study the semigroup $E(M)$ of idempotents.

Lemma IV.6. *Let M be a toric algebraic monoid. Then the following assertions hold:*

(i) *The semigroup $E(M)$ is finite.*

(ii) *The prescription*

$$e \leq f \qquad \Longleftrightarrow \qquad ef = e$$

defines a partial order on $E(M)$ and $E(M)$ is a lattice with respect to this partial order, where the meet operation is defined by

$$e \wedge f = \min\{e, f\} = ef.$$

(iii) *The mapping*

$$E(M) \to \operatorname{Fa}\big(E(M)\big), \quad e \mapsto \uparrow\! e := \{f \in E(M) : e \leq f\}$$

is a lattice isomorphism.

Proof. (i) Let $\chi_1, \ldots, \chi_n$ be a system of generators of the semigroup $X(M)$. Then we obtain a realization of M as a closed submonoid of the monoid of diagonal matrices $\mathbb{D}(n) \subseteq \mathbb{M}(n, \mathbb{K})$ by

$$\pi(m) := \big(\chi_1(m), \ldots, \chi_n(m)\big).$$

Since $E\big(\mathbb{D}(n)\big) \cong \mathbb{Z}_2^n$, the set $E(M)$ is finite as a subsemigroup of this finite semigroup.

(ii) Since $E(M)$ is abelian, this is the standard partial order on $E(M)$. If $e \leq f \leq e$, then $e = ef = f$ and if $e \leq f \leq g$, then

$$eg = efg = ef = e$$

shows that $\leq$ defines a partial order.

Let $e, f \in E(M)$. Then $ef \leq e, f$ and if $g \leq e, f$, then

$$g(ef) = (ge)f = gf = g$$

shows that $g \leq ef$. Hence $ef = \min\{e, f\}$ exists. Thus $E(M)$ is an inf-semilattice and since it is finite, arbitrary infs exist. Hence arbitrary sups exist, too. Thus $E(M)$ is a lattice.

(iii) Let $e \in E(M)$. We claim that $\uparrow e$ is a face. Let $f, g \in E(M)$ with $e \leq fg$. Then $(fg)e = e$ and therefore

$$fe = f(fg)e = f^2 ge = fge = e,$$

i.e., $f \in \uparrow e$. The relation $g \in \uparrow e$ follows by symmetry.

Now let F be a face of $E(M)$. Since $E(M)$ is finite, the product e of all elements in F exists and is a minimal element in F. Hence $e \in F$ and $F \subseteq \uparrow e$. If $f \in \uparrow e$, then $fe = e \in F$ implies that $f \in F$. Thus $\uparrow e = F$. So we have shown that the mapping $E(M) \to \mathrm{Fa}\,\big(E(M)\big)$ is surjective and well defined. That it is order preserving follows immediately from the definition, and injectivity follows from the fact that the order $\leq$ is partial. ∎

Note that (ii) and (iii) above hold in every finite semilattice. In view of Lemma IV.6, we have an isomorphism $E(M) \cong \mathrm{Fa}\,\big(E(M)\big)$. On the other hand, we know from Proposition II.9 that we also have an antiisomorphism

$$\mathrm{op} : \mathrm{Fa}\,\big(X(M)\big) \to \mathrm{Fa}\,\big(Y(M)\big)$$

because $X(M)$ is finitely generated and $Y(M)$ is the dual integral polyhedral cone. Next we want to link these isomorphisms to obtain isomorphisms or antiisomorphisms of these four lattices. First we exploit the relation between $\mathrm{Fa}\,\big(Y(M)\big)$ and $E(M)$.

Lemma IV.7. *The function ω has the following properties:*

(i) *ω is a morphism of monoids.*

(ii) *For every $F \in \mathrm{Fa}\,\big(E(M)\big)$, the set*

$$\omega^{-1}(F) = \{\gamma \in Y(M) : \gamma(0) \in F\}$$

is a face of $Y(M)$.

Proof. (i) This follows from

$$(\gamma_1\gamma_2)(0) = \gamma_1(0)\gamma_2(0).$$

(ii) It is a general fact that a monoid morphism $\alpha\colon S \to T$ induces a morphism

$$\mathrm{Fa}(T) \to \mathrm{Fa}(S), \quad F \mapsto \alpha^{-1}(F)$$

of complete inf-semilattices. Since it is clear that this mapping preserves arbitrary infs, i.e., intersections, we only have to show that $\alpha^{-1}(F)$ is a face whenever F is a face. If $xy \in \alpha^{-1}(F)$, then

$$\alpha(xy) = \alpha(x)\alpha(y) \in F$$

and therefore $\alpha(x), \alpha(y) \in F$. Thus $x, y \in \alpha^{-1}(F)$. ∎

Lemma IV.8. *The mapping*

$$\mathrm{op}_E : \mathrm{Fa}\big(X(M)\big) \to \mathrm{Fa}\big(E(M)\big), \quad F \mapsto F^{\perp} := \{e \in E(M) : (\forall \chi \in F)\chi(e) = 1\}$$

is surjective and inclusion reversing.

Proof. If F is a face in $X(M)$, and $e, f \in E(M)$ with $ef \in F^{\perp}$, then $\chi(ef) = \chi(e)\chi(f) = 1$ holds for all $\chi \in F$. Hence $\chi(e) = \chi(f) = 1$ and therefore $e, f \in F^{\perp}$. It follows that $\mathrm{op}_E(F) = F^{\perp} \in \mathrm{Fa}\big(E(M)\big)$.

We show that op_E is surjective. Let $F' \in \mathrm{Fa}\big(E(M)\big)$. According to Lemma IV.6, there exists a minimal idempotent $e \in F'$ with $F' = \uparrow e$. Let

$$F := e^{\perp} := \{\chi \in X(M) : \chi(e) = 1\}.$$

Since the mapping

$$e^* : X(M) \to \mathbb{K}_m, \quad \chi \mapsto \chi(e)$$

is a morphism of monoids, it follows that e^* maps $X(M)$ into $E(\mathbb{K}_m) = \{0, 1\}$. Thus

$$e^{\perp} = e^{*-1}(1) = e^{*-1}\big(H(\{0,1\})\big) \in \mathrm{Fa}\big(X(M)\big).$$

We claim that $\mathrm{op}_E(F) = \uparrow e = F'$. Suppose that this is false. Since $F' \subseteq \mathrm{op}_E(F)$, there exists $f \in \mathrm{op}_E(F) \setminus F'$. Then $f \notin \uparrow e$, i.e., $ef \neq e$. Now the fact that $X(M)$ separates the points of M yields $\chi \in X(M)$ with $\chi(ef) = \chi(e)\chi(f) \neq \chi(e)$. It follows that $\chi(f) \neq 1$, i.e., $\chi(f) = 0$ and $\chi(e) = 1$. Whence $\chi \in e^{\perp} = F$ and $f \notin \mathrm{op}_E(F)$. Thus $\mathrm{op}_E(F) = F'$. ∎

Putting the pieces together, we get the following diagram:

$$(D) \qquad \begin{array}{ccc} \mathrm{Fa}\big(X(M)\big) & \xrightarrow{\ \mathrm{op}\ } & \mathrm{Fa}\big(Y(M)\big) \\ \big\downarrow \mathrm{op}_E & & \big\uparrow \omega^{-1} \\ E(M) \cong \mathrm{Fa}\big(E(M)\big) & \xrightarrow{\ \mathrm{id}\ } & \mathrm{Fa}\big(E(M)\big) \end{array}$$

The horizontal morphisms are (anti)isomorphisms and the vertical morphism on the left is surjective (Lemma IV.8).

We show that the diagram commutes. Let $F \in \mathrm{Fa}\big(X(M)\big)$. Then

$$\begin{aligned} \mathrm{op}(F) &= \{\gamma \in Y(M) : (\forall \chi \in F)\langle \chi, \gamma\rangle = 0\} \\ &= \{\gamma \in Y(M) : (\forall \chi \in F)\chi\big(\gamma(0)\big) = 1\} \\ &= \{\gamma \in Y(M) : (\forall \chi \in F)\chi\big(\omega(\gamma)\big) = 1\} \\ &= \omega^{-1}\big(\mathrm{op}_E(F)\big). \end{aligned}$$

Now the bijectivity of op shows that ω^{-1} is surjective and that op_E is injective. Since op_E was already seen to be surjective, it follows that op_E is bijective and thus ω^{-1} is bijective, too. We have proved the following theorem:

Theorem IV.9. *All morphisms in the diagram (D) are bijective and isomorphisms or antiisomorphisms of finite lattices.* ∎

This result is the basic ingredient in the structure theory of toric monoids. Next we give an explicit description of the inverses of the mappings in the diagrams.

Proposition IV.10. *The following assertions hold:*

(i) *If $F \in \mathrm{Fa}\big(E(M)\big)$, then*

$$\mathrm{op}_E^{-1}(F) = F^\perp = \{\chi \in X(M) : (\forall f \in F)\chi(f) = 1\}.$$

(ii) *Let $F \in \mathrm{Fa}\big(Y(M)\big)$. Then $\omega(F) \in \mathrm{Fa}\big(E(M)\big)$ and $\omega^{-1}\big(\omega(F)\big) = F$. Moreover, if the face F is generated by γ, then $\gamma(0)$ is the unique minimal idempotent in $\omega(F) = \uparrow\gamma(0)$.*

(iii) *The mapping $\omega: Y(M) \to E(M)$ is surjective, i.e., every idempotent is reachable by a one-parameter semigroup.*

Proof. (i) Let $F = \uparrow e$ (Lemma IV.6). We have seen in the proof of Lemma IV.8 that $F = \mathrm{op}_E(e^\perp)$. If $e \leq f$, then $ef = e$ shows that $\chi(e) = 1$ implies that $\chi(e)\chi(f) = 1$, hence that $\chi(f) = 1$. Thus

$$\mathrm{op}_E^{-1}(F) = e^\perp = (\uparrow e)^\perp = F^\perp.$$

(ii),(iii) Since ω^{-1} is bijective, we find $e \in E(M)$ such that $F = \omega^{-1}(\uparrow e)$ (Lemma IV.6). We claim that $e \in \omega(F)$. Suppose that this is false. Since $\omega(F)$ is a finite subsemigroup of $E(M)$, there exists a minimal element $f \in \omega(F)$ (take the product of all elements). Then

$$F \subseteq \omega^{-1}(\uparrow f) \subseteq \omega^{-1}(\uparrow e) = F$$

shows that $\omega^{-1}(\uparrow f) = \omega^{-1}(\uparrow e)$. Since ω^{-1} is injective, we find that $e = f \in \omega(F)$. This proves in particular that the mapping $\omega: Y(M) \to E(M)$ is surjective. If $\omega(\alpha) = g \in \uparrow e$, then $\gamma \in \omega^{-1}(\uparrow e) = F$. Whence ω maps F onto $\uparrow e$.

Suppose that γ generates the face F and that $\alpha \in F$. Then $\alpha \in \mathbb{N}\gamma - Y(M)$ and we find $\beta \in Y(M)$ with $\alpha + \beta = n\gamma$. It follows that

$$\alpha(0)\beta(0) = \gamma^n(0) = \gamma(0).$$

Thus $\alpha(0)\beta(0) \in \uparrow\gamma(0)$ and consequently $\alpha(0), \gamma(0) \in \uparrow\gamma(0)$. It follows that $\gamma(0)$ is minimal in $\omega(F) = \uparrow e$. Therefore $e = \gamma(0)$. ∎

So far we have a complete description of the lattice of idempotents in $E(M)$ if M is a toric monoid. In concrete case one can compute this lattice by taking a finite set $D \subseteq X(M)$ generating the extremal rays of the real polyhedral cone $X(M)_{\mathbb{R}}$. Modding out the group of units of $X(M)$ does not effect the lattice $\mathrm{Fa}\big(X(M)\big)$. So we may assume that $H\big(X(M)\big)$ is trivial. Passing to $X\big(H(M)\big)_{\mathbb{R}}$, one obtains a realization of D as a finite subset of a real vector space. Let $P := \operatorname{conv}(D)$ denote the convex polytope obtained as the convex hull of D. This polytope generates the cone $X(M)_{\mathbb{R}} = \mathbb{R}^+ P$ and the lattice of faces of $X(M)$, or $X(M)_{\mathbb{R}}$ respectively, is therefore isomorphic to the lattice of faces of P if we add the empty set as a minimal element. Thus one has reduced the problem to compute $E(M)$ to a completely combinatorial problem.

If, f.i., the semigroup $X(M)$ is generated by a set D of linearly independent elements in $X\big(H(M)\big)_{\mathbb{R}}$, then $\operatorname{Fa}\big(X(M)\big) \cong 2^D$ is a boolean lattice. This happens for example if $H(M)$ is a maximal $\mathbb{K}$-torus in the adjoint group of a Borel subgroup of a semisimple algebraic $\mathbb{K}$-group.

Other motivations for the investigation of toric monoids come from invariant theory, where Hilbert's criterion can be used to describe orbit closures of reductive groups via orbit closures of torus subgroups, i.e., orbits of toric monoids (cf. Ch. 2 in [6]). We will see in the following sections how Hilbert's criterion is linked with the reachability of idempotents by one-parameter semigroups (Proposition IV.10(iii)).

More specialized results on $E(M)$ in more concrete situations and results on automorphisms of M can be found in [7, Ch. 8].

Applications to polynomial group actions

Let V be an affine $\mathbb{K}$-variety such that the torus T acts algebraically on V, i.e., the mapping $T \times V \to V$ is polynomial. Let $v \in V$ and $W := \overline{T.v}$. To study W, we may assume that $T.v$ is dense in V, i.e., $V = W$. Let

$$\pi : T \to V, \quad t \mapsto t.v$$

denote the orbit mapping. Then $\pi^* \colon P(V) \to P(T)$ maps $P(V)$ injectively onto a finitely generated subalgebra of $P(T)$. Let $C := X(T) \cap \pi^*\big(P(V)\big)$. Then

$$\pi^*\big(P(V)\big) = \sum_{\chi \in C} \mathbb{K}\chi$$

(Proposition III.4) and C is a finitely generated submonoid of the free abelian group $X(T)$. Let $M := \widehat{C}$ denote the character monoid of C. Then the sequence

$$P(V) \xrightarrow{\quad\cong\quad} P(M) = \mathbb{K}[C] \xrightarrow{\qquad} P(T)$$

induces morphisms of algebraic $\mathbb{K}$-varieties

$$T \xrightarrow{\qquad} M \xrightarrow{\quad\cong\quad} V,$$

where the mapping $M \to V$ is an isomorphism of affine varieties which is T-equivariant and the mapping $T \to M$ is the inclusion of the group of units.

This shows that one understands T-orbit closures if one understands the action of T on M, where $H(M) \cong T$.

Lemma IV.11. *Let $M \subseteq \mathbb{M}(n, \mathbb{K})$ be an algebraic $\mathbb{K}$-monoid and $m \in M$ a semisimple element. Then m is contained in a subgroup of M.*

Proof. (cf. [7, 3.18]) Let e denote the projection onto the image of m. Then $e^2 = e$, e commutes with m, and there exists $a \in \mathbb{M}(n, \mathbb{K})$ with

$$ae = ea = e \quad \text{and} \quad am = ma = e.$$

Let $M_e := \{x \in M : xe = ex = x\}$. This is a closed submonoid of M containing m. Moreover, for $i \in \mathbb{N}$,

$$m^i M_e = \{x \in M_e : a^i x \in M_e\}$$

are closed subsemigroups of M_e. Since the sequence

$$M_e \supseteq mM_e \supseteq \ldots \supseteq m^i M_e \supseteq \ldots$$

is a decreasing sequence of closed subsets of the Noetherian space M_e, there exists $j \in \mathbb{N}$ such that $m^j M_e = m^{j+1} M_e$. Then

$$M_e = eM_e = a^j m^j M_e = a^j m^{j+1} M_e = emM_e = mM_e.$$

Similarly $M_e m = M_e$. Whence there exists $c \in M_e$ with $cm = m$. Then

$$e = ma = cma = ce = c.$$

Thus $e \in M_e \subseteq M$ and $e \in E(M)$. There exist $x, y \in M_e$ with $xm = my = e$. Then $x = xe = xmy = ey = y$ and therefore m is contained in the group of units of $M_e = eMe$. ∎

Theorem IV.12. (Structure Theorem for Toric Monoids) *Let $\mathbb{K}$ be an infinite field and M a toric algebraic $\mathbb{K}$-monoid. Then the following assertions hold:*

(i) *M is the disjoint union of the subgroups $H(eM)$, $e \in E(M)$.*

(ii) *The mapping $H(M) \to H(eM)$, $m \mapsto em$ is a morphism of algebraic groups with dense image.*

(iii) *The sets eM, $e \in E(M)$ are precisely the closures of $H(M)$-orbits in M. It follows in particular that the lattice of $H(M)$-orbit closures in M is isomorphic to the lattice $E(M)$.*

(iv) *If $\mathbb{K}$ is algebraically closed, then $M = H(M)E(M)$ and the groups $H(eM)$ are precisely the $H(M)$-orbits in M.*

Proof. (i) Since M may be viewed as a monoid of diagonal matrices, every element of M is semisimple and the preceding lemma shows that M is a union of groups. Since every group contains exactly one idempotent, (i) follows.
(ii) The fact that M is abelian yields that

$$\lambda_e : M \to eM, \quad m \mapsto em$$

is a morphism of algebraic monoids. Since $H(M)$ is dense in M, if follows that $\lambda_e\big(H(M)\big) \subseteq H(eM)$ is also dense because $H(eM)$ carries the Zariski topology induced by M.

(iii) Since $eM = \{m \in M : em = m\}$ is a closed subsemigroup of M, it is clear that $eM = \lambda_e(M) = \overline{\lambda_e(H(M))} = \overline{H(M)e}$. If, conversely, $m \in M$ and $m \in H(eM)$, then $e \in mM \subseteq \overline{H(M)m}$. Hence the $H(M)$-orbit closure of m contains e, so that $eM = \overline{H(M)m}$ follows from $m \in \overline{H(M)e}$. This shows that the sets $H(eM)$ are precisely the $H(M)$-orbit closures in M.
(iv) (cf. [7, p.23]) If $\mathbb{K}$ is algebraically closed, then the homomorphism $H(M) \mapsto H(eM)$ of algebraic groups has closed image. Hence $H(eM) = eH(M)$ by (i) and the assertion follows from (i). ■

Note that Theorem IV.12 gives a complete picture of the action of the torus $H(M)$ on M. The orbit closures are parameterized via $E(M)$, and the dimension of the orbits may be computed as

$$\dim \overline{eH(M)} = \dim H(eM) = \operatorname{rank} X(eM) = \operatorname{rank} X(M) - \operatorname{rank} e^{\perp},$$

where the rank of a finitely generated monoid means the rank of the corresponding free group.

To visualize the difficulties which arise if the field $\mathbb{K}$ is not algebraically closed, let $\mathbb{K} = \mathbb{R}$, $T := \mathbb{R}^* \times \mathbb{R}^*$, and

$$\alpha: T \to (\mathbb{R}_m)^3, \quad (t_1, t_2) \mapsto (t_1, t_1 t_2, t_2^2).$$

We set $M := \overline{\alpha(T)}$. Identifying $X(T)$ with $\mathbb{Z}^2$, the semigroup $\alpha^*(Y(M)) \cong Y(M)$ is generated by $\{(1,0),(1,1),(0,2)\}$ and $Y(M)$ is not saturated in $\mathbb{Z}^2$. Furthermore $Y(M) = (\mathbb{N}_0)^2 \subseteq \mathbb{Z}^2 \cong X_*(T)$. Let $\gamma \in Y(M)$ be given by $\gamma(t) = (t,t,1)$. Then $e := \gamma(0) = (0,0,1)$ and

$$H(eM) = \{(0,0)\} \times \mathbb{R}^* \neq eH(M) = \{(0,0)\} \times (\mathbb{R}^*)_+.$$

Orbits on projective spaces

We have already seen that the orbit closures of tori on affine varieties may be completely described as algebraic monoids. In some cases this works also for orbits in projective spaces.

Let T be a torus acting linearly on a finite dimensional $\mathbb{K}$-vector space V. Then we obtain an action of T on the projective space $\mathbb{P}(V)$ by setting

$$t.[v] := [t.v],$$

where $[v]$ denotes the one-dimensional subspace spanned by the non-zero vector v. We are interested in the orbit closures $\overline{T.[v]}$ in $\mathbb{P}(V)$.

Let $T' := \mathbb{K}^* \times T$. This is a $\mathbb{K}$-torus whose dimension is $\dim T + 1$. Moreover, we obtain an action of T' on V by the prescription

$$T' \times V \to V, \qquad ((\lambda, t), v) \mapsto \lambda(t.v).$$

For $[v] \in \mathbb{P}(V)$ we get that

$$T'.[v] = [T'.v] = [\mathbb{K}^* T.v] = [T.v] = T.[v],$$

so that the orbits on $\mathbb{P}(V)$ remain the same. On the other hand, the orbit closures in $\mathbb{P}(V)$ may now be described by

$$\overline{T.[v]} = [\overline{\mathbb{K}^* T.v} \setminus \{0\}] = [\overline{T'.v} \setminus \{0\}].$$

More concretely, let

$$\pi : T \to \mathbb{D}(n, \mathbb{K}), \qquad t \mapsto \big(\chi_1(t), \chi_2(t), \ldots, \chi_n(t)\big)$$

be a representation of T on $V = \mathbb{K}^n$. For every $\chi \in X(T)$ we obtain a character of T' by

$$\chi'(\lambda, t) := \lambda \chi(t).$$

This means that we identify $X(T)$ with the set

$$\{\chi \in X(T') : \chi|_{\mathbb{K}^* \times \mathbf{1}} = \mathrm{id}_{\mathbb{K}^*}\} \cong \{1\} \times X(T)$$

which is an affine integral hyperplane in $X(T')$.

Theorem IV.13. *Let $P \subseteq X(T)_{\mathbb{R}}$ denote the convex hull of the set*

$$\{\chi_1, \ldots, \chi_n\}$$

and $v \in \mathbb{K}^n$ with all components non-zero. Then the following assertions hold:

(i) *The lattice $\mathrm{Fa}(P)$ is isomorphic to the lattice of T-orbit closures in the set $\mathcal{O} := \overline{T.[v]} \subseteq \mathbb{P}(V)$.*

(ii) *The dimension of a T-orbit closure in $\mathcal{O}$ equals the dimension of the corresponding face of P.*

(iii) *Let $F \in \mathrm{Fa}(P)$ and*

$$V_F := \sum_{\chi \in F} \mathbb{K} V_\chi.$$

Then the corresponding orbit closure is contained in $\mathbb{P}(V_F) \subseteq \mathbb{P}(V)$. It coincides with the orbit closure of the projection of v onto this subspace.

Proof. (i) The lattice of T'-orbits in $\overline{T'.v}$ is isomorphic to the lattice of faces of the cone C' spanned by the set $D' := \{\chi_1', \ldots, \chi_n'\}$ in $X(T')$ (Theorems IV.9, IV.12). The one-point orbit $\{0\}$ corresponds to $H(C') = \{0\}$ because C' is pointed (Proposition IV.10(i)).

Since $\mathrm{Fa}(C') \setminus \{0\}$ is order isomorphic to $\mathrm{Fa}(P)$, the assertion follows from Theorem IV.12.

(ii) Let $F \in \mathrm{Fa}(P)$, $F' := \mathbb{R}^+ F \subseteq \mathrm{Fa}(C')_{\mathbb{R}}$ the corresponding face of $C'_{\mathbb{R}}$, and $e' \in E(M')$ the corresponding idempotent, where $M' = \widehat{C'}$ (Proposition IV.10(i)). Then

$$F'^{\perp} = \{f \in E(M') : (\forall \chi \in F')\chi(f) = 1\} = \uparrow e'$$

yields

$$\begin{aligned}\dim F &= \dim F' - 1 = \dim T' - \dim(F'^{\perp}) - 1 \\ &= \dim(\overline{T'e}) - 1 = \dim \overline{T'.v} - 1 = \dim \overline{T.[v]}.\end{aligned}$$

Whence the dimension of the face F equals the dimension of the corresponding orbit in the projective space.
(iii) Let $\{e_{\chi_i}: i = 1,\dots,n\}$ denote the canonical basis in $V \cong \mathbb{K}^n$, v_{χ_i} the corresponding coordinates of v, and e as in (ii) above. Then

$$(1,\chi)(e') = \begin{cases} 1 & \text{for } \chi \in F \\ 0 & \text{for } \chi \notin F \end{cases},$$

shows that

$$e.v = e.\sum_{i=1}^n v_{\chi_i} e_{\chi_i} = \sum_{i=1}^n v_{\chi_i}(1,\chi_i)(e) e_{\chi_i} = \sum_{\chi\in F} v_\chi e_\chi.$$

This proves (iii). ■

Example IV.14. (a) Let $V = \mathbb{C}^2$, $T = \mathbb{K}^*$, and

$$\pi(t) = (t, t^{-1}).$$

Then $D = \{1,-1\}$ and $P = [-1,1] \subseteq \mathbb{R} \cong X(T)_\mathbb{R}$. It follows that for every $v = (v_1, v_2)$ with $v_1 v_2 \neq 0$ the set $\overline{T.[v]}$ consists of one one-dimensional T-orbit and two fixed point.
(b) Let $V = \mathbb{C}^n$, $T = \mathbb{K}^*$, and

$$\pi(t) = (t^n, t^{n-2}, \dots, t^{-2-n}, t^{-n}).$$

Then $D = \{n, n-2, \dots, 2-n, -n\}$ and $P = [-n,n] \subseteq \mathbb{R} \cong X(T)_\mathbb{R}$. It follows that for every $v = (v_1, v_2, \dots, v_n)$ with $v_i \neq 0$ for $i = 1,\dots,n$ the set $\overline{T.[v]}$ consists of one one-dimensional T-orbit and two fixed points. Note that the whole space $\mathbb{P}(\mathbb{C}^n)$ contains $n+1$ different fixed points of T but that Theorem IV.13(iii) shows that the only possible fixed points in $\overline{T.[v]}$ are given by $[(1,0,\dots,0)]$ and $[(0,0,\dots,1)]$. ■

V. Related compact semigroups

We want to relate the algebraic concepts developed above to topological concepts. So we have to impose a further condition on the fields under consideration. From now on we assume that the field $\mathbb{K}$ is a *local field*, i.e., *locally compact* and non-discrete (every field is locally compact with the discrete topology). Let $m_\mathbb{K}: \mathbb{K} \to \mathbb{R}_+^*$ denote the modular function of $\mathbb{K}$ and

$$\mathbb{B} := \mathbb{B}_\mathbb{K} := \{x \in \mathbb{K} : m_\mathbb{K}(x) \leq 1\}.$$

We recall some basic proporties of locally compact fields (cf. [10]):

The structure of local fields

Proposition V.1.

(i) $\mathbb{B}$ *is a compact submonoid of* $\mathbb{K}$.

(ii) *The sets* $\mathbb{B}_m := \{x \in \mathbb{K}\colon m_{\mathbb{K}}(x) \leq m\}$ *make up a fundamental system of* 0*-neighborhoods in* $\mathbb{K}$.

(iii) $m_{\mathbb{K}}$ *is continuous and its restriction to* $\mathbb{K}^*$ *is an open homomorphism onto a closed subgroup of* $\mathbb{R}^*$.

(iv) $x^n \to 0$ *if and only if* $m_{\mathbb{K}}(x) < 1$.

Proof. [10, pp.4–7] ■

Example V.2. (a) If $\mathbb{K} = \mathbb{R}$, then $m_{\mathbb{R}}(x) = |x|_\infty := |x|$ and $\mathbb{B} = [-1, 1]$.
(b) If $\mathbb{K} = \mathbb{C}$, then $m_{\mathbb{C}}(z) = |z|^2$ and $\mathbb{B}$ is the closed unit disk.
(c) Let p be a prime. We define $|0|_p := 0$ and $|x|_p = p^{-n}$ if $x = p^n \frac{a}{b}$ with a, b relatively prime to p. This function is called the *p-adic absolute value* on $\mathbb{Q}$ and the completion of $\mathbb{Q}$ with respect to the corresponding metric are the *p-adic numbers.* In this case the semigroup $\mathbb{B} = \mathbb{Z}_p$ is called the ring of *p-adic integers.* ■

In the following we set $\mathbb{P} := \{\infty\} \cup \{2, 3, 5, 7, \ldots\}$ for the set of primes together with infinity. We also set $\mathbb{Q}_\infty := \mathbb{R}$.

Theorem V.3. *Let* $\mathbb{K}$ *be a local field. Then either*

(a) $\mathrm{char}(\mathbb{K}) = p > 0$ *and*

$$m_{\mathbb{K}}(n \cdot \mathbf{1}_{\mathbb{K}}) = \begin{cases} 0 & \text{for } m \in p\mathbb{Z} \\ 1 & \text{otherwise} \end{cases}$$

or

(b) $\mathbb{K}$ *is an extension of finite dimension* n *of a field* $\mathbb{Q}_p$, $p \in \mathbb{P}$ *and* $m_{\mathbb{K}}(x) = |x|_p^n$.

Proof. [10, pp.11–13] ■

A local field with $m_{\mathbb{K}}(p \cdot \mathbf{1}_{\mathbb{K}}) < 1$ is called a *p-field.* Suppose that $\mathbb{K}$ is a p-field. Then $\mathbb{K}$ is *ultrametric*, i.e.,

$$m_{\mathbb{K}}(x + y) \leq \max\{m_{\mathbb{K}}(x), m_{\mathbb{K}}(y)\},$$

the group of units of $\mathbb{B}$ is

$$H(\mathbb{B}) = \{x \in \mathbb{K} : m_{\mathbb{K}}(x) = 1\},$$

there exists an element $\pi \in \mathbb{K}$ with $m_{\mathbb{K}}(\pi) = 1$ and

$$\mathbb{B} = H(\mathbb{B}) \dot\cup \pi\mathbb{B}.$$

The residue field $\mathbb{B}/\pi\mathbb{B}$ is a finite field of characteristic p. If it has $q = p^m$ elements, then $m_{\mathbb{K}}(\mathbb{K}^*)$ is the cyclic subgroup generated by $m_{\mathbb{K}}(\pi) = \frac{1}{q}$ ([10, p.13]). Here one sees that the character $\chi\colon \mathbb{K}^* \to \mathbb{K}^*$, $t \mapsto t^{q-1}$ cannot be surjective because $y = x^{q-1}$ and $m_{\mathbb{K}}(y) = 1$ entail that $m_{\mathbb{K}}(x)^{q-1} = m_{\mathbb{K}}(y)$, so that $m_{\mathbb{K}}(x) = 1$, and hence $y + \pi\mathbb{B} = x^{q-1} + \pi\mathbb{B} = \mathbf{1} + \pi\mathbb{B}$. It follows in particular that morphisms of algebraic groups over ultrametric local fields are in general not surjective.

Moreover, if $\mathrm{char}(\mathbb{K}) = p$, then $\mathbb{K}$ is isomorphic to a field of formal Laurent series in one indeterminate with coefficients in a finite field $\mathbb{F}$ with characteristic p. In this case

$$\mathbb{B} \cong \{\sum_{n\geq 0} \lambda_n X^n : \lambda_n \in \mathbb{F}\}.$$

Compact semigroups

If $\mathbb{K}$ is a local field, then every affine algebraic monoid M over $\mathbb{K}$ is also a locally compact monoid with respect to the topology induced by a realization in $\mathbb{M}(n, \mathbb{K})$. Then the algebra $P(M)$ consists of continuous functions and therefore every morphism of algebraic monoids is automatically continuous.

Let us return to the setting of the previous section, where M is a toric $\mathbb{K}$-monoid, $X(M)$ the character monoid, and $Y(M)$ the integral polyhedral cone of one-parameter semigroups. Every one-parameter semigroup

$$\gamma : \mathbb{K}_m \to M$$

is a continuous mapping and therefore $\gamma(\mathbb{B})$ is a compact submonoid of M. Since $Y(M)$ is finitely generated (Theorem II.8, Proposition IV.4), the product of all these subsemigroups is a finite product and therefore a compact subsemigroup $\mathbb{B}_M$ of M. As we have seen in Section IV, every idempotent in $E(M)$ is obtained as $\gamma(0)$ for a one-parameter subsemigroup. Thus $E(M) = E(\mathbb{B}_M)$.

This has the following consequences. If $T := H(M)$ acts morphically on an algebraic $\mathbb{K}$-variety and $v \in V$, then we have seen in Section IV that we may describe the orbits closure $\overline{T.v}$ as one orbit of the monoid M.

Theorem V.4. *For every algebraic action of a torus T on an affine $\mathbb{K}$-variety V, there exists a compact semigroup $\mathbb{B}_M$ acting on the orbit closure $\overline{T.v}$ such that the lattice of T-orbit closures in $\overline{T.v}$ is isomorphic to the finite lattice of idempotents of the compact semigroup $E(\mathbb{B}_M)$. This isomorphism is implemented by the fact that every orbit closure can be written as $\overline{Te.v}$ for $e \in E(\mathbb{B}_M)$.* ■

Let us consider the special case $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then $\mathbb{B}_{\mathbb{C}} = [0,1]\mathbb{S}^1$ is the closed unit disc, and $\mathbb{B}_{\mathbb{R}} = [-1,1]$. We set $\mathbb{B}_{\mathbb{R}}^+ := [0,1]$.

Let M be a complex toric monoid with unit group $T \cong (\mathbb{C}^*)^n$. Let $T_a := (\mathbb{S}^1)^n$ denote the maximal anisotropic torus (in the Lie group sense) in T, $T_{\mathbb{R}} \cong (\mathbb{R}^*)^n$, and $T_{\mathbb{R}}^+ \cong (\mathbb{R}_+^*)^n$ the connected component of the identity in $T_{\mathbb{R}}$.

Then $T \cong T_a \times T_{\mathbb{R}}^+$ and $T_a \subseteq \mathbb{B}_M$, so

$$\mathbb{B}_M \cap T \cong T_a \times (\mathbb{B}_M \cap T_{\mathbb{R}}^+).$$

The semigroup $C := (\mathbb{B}_M \cap T_{\mathbb{R}}^+)$ is the product of the real one-parameter semigroups $\gamma(]0,1])$, $\gamma \in Y(M)$. Whence C corresponds to a polyhedral convex cone in the Lie group $T_{\mathbb{R}}^+ \cong \mathbb{R}^n$. It may be described as

$$C = \{t \in T_{\mathbb{R}} : \big(\forall \chi \in X(M)\big)\chi(t) \leq 1\}.$$

This semigroup is maximally compact in the locally compact monoid $\overline{T_{\mathbb{R}}^+}$ obtained as the closure of $T_{\mathbb{R}}$ in M.

Finally we note that the compactness of T_a yields that

$$\mathbb{B}_M = T_a\overline{C},$$

where $\overline{C}$ is a compactification of the real polyhedral cone C.

This observation is a link between the theory of compactification of cones in vector spaces (cf. [9]) and the theory of algebraic monoids.

We conclude with some remarks which illustrate the similarities between Lie semigroups and algebraic monoids. Firstly one should note that the set $Y(M)$ of one-parameter semigroups of M is the algebraic analog of the set

$$\mathbf{L}(S) = \{X \in \mathbf{L}(G) : \exp(\mathbb{R}^+ X) \subseteq S\}$$

of one-parameter subsemigroups of a closed submonoid S of a Lie group G. These two objects correspond via $C \mapsto C_{\mathbb{R}}$ if $S = \mathbb{B}_M$ is the compact subsemigroup of the real algebraic monoid M described above.

There exists another interesting link between the representation theory of Lie groups and representations of tori. Namely, if M is a toric monoid and $T = H(M)$ its group of units, we concider a polynomial representation $\alpha: T \to \mathbb{D}(n, \mathbb{K})$ and we ask whether we can extend this to a representation $\overline{\alpha}: M \to \mathbb{D}(n, \mathbb{K})$. In view of Corollary III.5(iii) this is possible if and only if α^* mapsto $P\big(\mathbb{D}(n, \mathbb{K})\big)$ into $P(M) \subseteq P(T)$. This condition can be read as a positivity condition because it means in particular that $\langle \gamma, \alpha^* \chi \rangle \geq 0$ for all $\gamma \in Y(M)$ and every character χ of $\mathbb{D}(n, \mathbb{K})$. To be more precise, this condition means that we can extend α to the normalization $\big(Y(M)^\star\big)\widehat{}$ of the monoid M (cf. [5, p.5]). Similar conditions occur in the theory of unitary representation of groups G, where they guarantee the existence of extensions to semigroups containing G as the group of units.

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