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Rigidly symmetric L^1–group algebras

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A Banach algebra $\mathcal{B}$ with isometric involution $a \mapsto a^*$ is called *symmetric*, if for each $a \in \mathcal{B}$ the spectrum $\operatorname{Sp} a^*a$ is contained in $\mathbb{R}^+ = [0, \infty)$. The theorem of FORD and SHIRALI tells us that this is equivalent to $\mathcal{B}$ being *Hermitian*, i.e., that $\operatorname{Sp} b \subset \mathbb{R}$ for each $b \in \mathcal{B}$ with $b = b^*$. The notion of symmetry was invented in the forties when people were looking for characterizations of closed involutive subalgebras of the algebra of bounded operators on a Hilbert space, i.e., of C^*–algebras in today's terminology. C^*–algebras are always symmetric. In the last decades several authors have investigated the question for which locally compact groups G the L^1–convolution algebra $L^1(G)$ is symmetric, where $L^1(G)$ is formed w.r.t. a, say, left invariant measure. To explain the notion of symmetry let me give an interpretation in the context of amenable groups. Each $f \in L^1(G)$ defines a convolution operator $\lambda_p(f)$ on $L^p(G)$, $1 \le p < \infty$, by $\lambda_p(f)(g) = f * g$. If G is amenable then $L^1(G)$ is symmetric if and only if $\operatorname{Sp} \lambda_1(f) = \operatorname{Sp} \lambda_2(f)$ for all $f = f^* \in L^1(G)$, which in turn by the usual interpolation arguments implies that $\operatorname{Sp} \lambda_1(f) = \operatorname{Sp} \lambda_p(f)$ for all p and all $f = f^* \in L^1(G)$. The case of amenable groups is of particular importance because for connected Lie groups G the symmetry of $L^1(G)$ implies that G is amenable which here means that the semisimple part is compact, [4, § 4]. It is known for which connected solvable groups the corresponding L^1–algebras are symmetric, see [11]. In that paper there is also given a conjecture for general connected groups which is very likely true.

There are several equivalent characterizations of symmetry, see [12, 2, 6, 9, 13]. Below I shall discuss and use one of them. But before doing so I want to explain briefly the result to be proved in this note. Apparently it is not known whether the projective tensor products of two symmetric algebras is again symmetric. There are even no general sufficient criteria, except for trivialities like one factor being commutative. In particular, it is unknown if it is sufficient that one factor is a C^*–algebra. This observation and some technical reasons led us in [7] to the following definition.

Definition 1. An involutive Banach algebra $\mathcal{B}$ is called *rigidly symmetric* if the projective tensor product $\mathcal{B} \widehat{\otimes} \mathcal{A}$ with a C^*–algebra $\mathcal{A}$ is always symmetric.

Let me point out once more that no example is known of a symmetric Banach algebra which is not rigidly symmetric.

In [7] it was shown that for any *discrete* nilpotent G the algebra $\ell^1(G)$ is rigidly symmetric. This implies in particular that $\ell^1(G)$ is symmetric which was proved earlier by A. HULANICKI using complicated combinatorial arguments to estimate spectral radii, see [3]. Our proof is much simpler. It proceeds by induction on the nilpotent length of G, at a crucial point we have to have that groups of lower nilpotent length have rigidly symmetric algebras—this was meant above by "technical reasons".

In this note it will be proved that $L^1(G)$ is rigidly symmetric for any nilpotent locally compact group G by reducing the problem to the case of discrete groups with a very simple trick, which might be useful in other circumstances as well. Again this implies that $L^1(G)$ is symmetric, which was known before, see [8], or [10] for the case of connected Lie groups.

Let us return to a general involutive Banach algebra $\mathcal{B}$. I am going to explain one of the various characterizations of symmetry. Let $b \in \mathcal{B}$ and let z be a non-zero complex number in the left spectrum of b, i.e., $\mathcal{B}(b-z) = \{xb - zx \mid x \in \mathcal{B}\}$ is a proper left ideal. Actually, $\mathcal{B}(b-z)$ is a modular left ideal with right modular unit $u = \frac{1}{z}b$, i.e., $xu \equiv x \mod \mathcal{B}(b-z)$ for all $x \in \mathcal{B}$. By Zorn's Lemma, $\mathcal{B}(b-z)$ is contained in a maximal left ideal Λ with right modular unit u. Then $F := \mathcal{B}/\Lambda$ is a simple left $\mathcal{B}$–module. The coset $\xi = [u] \in F$ is different from zero and $b\xi = z\xi$, hence z is in the point spectrum of the operator $\eta \mapsto b\eta$ on F.

Now suppose that $\mathcal{B}$ has the following property:

(U) For each simple left $\mathcal{B}$–module E there exists a bounded, topologically irreducible, involutive representation π of $\mathcal{B}$ in some Hilbert space $\mathfrak{H}$ and a non-zero $\mathcal{B}$–intertwining operator $T\colon E \to \mathfrak{H}$.

By a simple $\mathcal{B}$–module E we do not only mean that the only $\mathcal{B}$–invariant subspaces are the obvious ones, but we also exclude that $b\xi = 0$ for all $b \in \mathcal{B}$, $\xi \in E$ (such a degenerate module also has only trivial $\mathcal{B}$–invariant subspaces in case that $\dim E \leq 1$). To remind the reader of this agreement we add occasionally "non–trivial" in brackets.

Applying property (U) to the above constructed module F one finds a non-zero intertwining operator $T\colon F \to \mathfrak{H}$. The equation $b\xi = z\xi$ leads to $\pi(b)T\xi = zT\xi$, hence z is in the point spectrum of $\pi(b)$; observe that T is injective as F is simple. If b is of the form $b = a^*a$ with some $a \in \mathcal{B}$ then $\pi(b) = \pi(a)^*\pi(a)$ is a positive operator, hence $z \in \mathbb{R}^+$. We conclude that the left spectrum of elements of the form a^*a is contained in $\mathbb{R}^+$. Since the right spectrum of such elements (in fact, of any elements b with $b^* = b$) is just the complex conjugate of the left spectrum it follows that $\mathrm{Sp}(a^*a) \subset \mathbb{R}^+$ for all $a \in \mathcal{B}$. In other words, property (U) implies that $\mathcal{B}$ is symmetric. Indeed, symmetry of $\mathcal{B}$ is equivalent to property (U), see e.g. [6, 9]. The other implication is more difficult: Using symmetry one has to construct positive definite functionals, for these techniques see e.g. the relevant sections in [2, 13].

Using criterion (U) one can easily deduce that the L^1–algebras of abelian and of compact groups are symmetric. Better still, they are rigidly symmetric. Of course, this result can also be obtained by other methods.

Criterion (U) shows that one needs some information about simple modules over Banach algebras. Let G be any locally compact group and let $\mathcal{A}$ be any Banach algebra, for a while the involution plays no role. The projective tensor product $\mathcal{B} = L^1(G)\widehat{\otimes}\mathcal{A}$ is a Banach algebra in the obvious manner. It can be identified with the algebra $L^1(G, \mathcal{A})$ of integrable $\mathcal{A}$–valued functions on

G, which may be considered as the completion of $C_c(G,\mathcal{A})$ w.r.t. the norm $\| \|_1$. The multiplication is given by

$$(f * g)(x) = \int_G f(xy)g(y^{-1})\,dy,$$

where dy denotes a left invariant measure on G.

Let E be a (non-trivial) simple left $\mathcal{B}$–module. We now define $\rho\colon \mathcal{B} \to \mathrm{End}(E)$ by $\rho(b)\xi = b\xi$. If ξ_0 is any non-zero vector in E, one gets a norm on E by putting

$$\|\xi\| = \inf\{\|f\|_1 \mid f \in \mathcal{B},\ \rho(f)\xi_0 = \xi\}.$$

This way, E becomes a Banach space, in fact, a Banach $\mathcal{B}$–module, i.e.,

$$\|\rho(f)\xi\| \le \|f\|_1\|\xi\|$$

for all $\xi \in E$, $f \in \mathcal{B}$. The constructed norm depends on ξ_0, but choosing another non-zero vector leads to an equivalent norm. All these facts are well known and easy to prove. Of course, they have nothing to do with the assumption that $\mathcal{B}$ is a tensor product, but hold true for any Banach algebra. For $x \in G$ and $f \in \mathcal{B}$ define $\varepsilon_x * f \in \mathcal{B}$ by $(\varepsilon_x * f)(t) = f(x^{-1}t)$. Then define an operator $\rho(x)$ on E by requiring that on elements of the form $\rho(f)\xi, f \in \mathcal{B}, \xi \in E$, the operator is given by $\rho(x)(\rho(f)\xi) = \rho(\varepsilon_x * f)\xi$. Since E is simple, each element in E may be written in this form. Then ρ is a strongly continuous representation of G in E, i.e., $x \mapsto \rho(x)\xi$ is continuous for each $\xi \in E$ and ρ is homomorphic. Furthermore, each $\rho(x)$ is an isometry for any of the norms on E as constructed above. Similarily, for $a \in \mathcal{A}$ and $f \in \mathcal{B}$ we define $a*f \in \mathcal{B}$ by $(a*f)(x) = af(x)$, and $\rho(a) \in \mathrm{End}(E)$ by $\rho(a)(\rho(f)\xi)) = \rho(a * f)(\xi)$. Then $\rho\colon \mathcal{A} \to \mathrm{End}(E)$ is a bounded representation of $\mathcal{A}$, bounded by one.

Moreover, $\rho(x)\rho(a) = \rho(a)\rho(x)$ for all $x \in G$ and $a \in \mathcal{A}$. The representation ρ of $\mathcal{B}$ can be reconstructed from the derived representations of G and $\mathcal{A}$ by

$$\rho(f)\xi = \int_G \rho(x)\rho(f(x))\xi\,dx$$

for $\xi \in E$, $f \in \mathcal{B}$. As was said above, I want to reduce the rigid symmetry of L^1–algebras of arbitrary nilpotent groups to the case of discrete groups. The transition is mainly done by means of the following theorem.

Theorem 2. *Let G be a locally compact group, let $\mathcal{A}$ be a Banach algebra and let E be a (non-trivial) simple left $\mathcal{B}$–module where $\mathcal{B} = L^1(G,\mathcal{A})$. The associated representations of G and $\mathcal{A}$ (see above) define a bounded representation ρ_d of $\mathcal{B}_d := \ell^1(G,\mathcal{A})$ in E by*

$$\rho_d(g)\xi = \sum_{x\in G} \rho(x)\rho(g(x))\xi.$$

Then E is also simple when considered as a $\mathcal{B}_d$–module.

Remark 3. The theorem applies in particular to $\mathcal{A} = \mathbb{C}$, hence each simple module over the ordinary L^1–group algebra $L^1(G)$ is a simple $\ell^1(G)$–module in a canonical fashion.

Proof of Theorem 2. We have to show that for each non-zero $\xi_0 \in E$ and each ξ in E there exists $h \in \ell^1(G, \mathcal{A})$ such that $\rho_d(h)\xi_0 = \xi$. We fix a certain ξ_0 once and for ever. This ξ_0 is used to define a norm on E as above. First we claim

(∗) For each $\eta \in E$ and each $\varepsilon > 0$ there exists an $g \in \ell^1(G, \mathcal{A})$ such that

$$\|g\|_{\ell^1} < \varepsilon + \|\eta\| \text{ and } \|\rho_d(g)\xi_0 - \eta\| < \varepsilon.$$

In order to prove (∗) let δ be a small positive real number to be determined later. By definition of the norm on E there exists $f \in L^1(G, \mathcal{A})$ such that $\rho(f)\xi_0 = \eta$ and $\|f\|_{L^1} < \|\eta\| + \delta$. Since $C_c(G, \mathcal{A})$ is dense in $L^1(G, \mathcal{A})$ there exists $f_1 \in C_c(G, \mathcal{A})$ with $\|f - f_1\|_{L^1} < \delta$. Then

$$\|f_1\|_{L^1} < 2\delta + \|\eta\|, \text{ and}$$
$$\|\rho(f_1)\xi_0 - \eta\| = \|\rho(f_1)\xi_0 - \rho(f)\xi_0\| \leq \|f - f_1\|_{L^1}\|\xi_0\| < \delta\|\xi_0\|.$$

■

Denote by S the support of f_1. Choose an open relatively compact neighborhood U of the identity in G such that $\|\rho(u)\xi_0 - \xi_0\| < \delta$ for $u \in U$ and $|f(xu) - f(x)| < \delta|S|^{-1}$ for all $x \in G$ and $u \in U$, where $|\ |$ denotes the Haar measure of measurable subsets of G. Clearly, such an U exists because ρ is strongly continuous and f_1 is uniformly continuous as a compactly supported function. There exist $x_1, \ldots, x_m \in G$ such that S is covered by $\{x_jU \mid 1 \leq j \leq m\}$. To get a disjoint covering of S let $M_1 := x_1U \cap S$ and define inductively M_j by $M_j = (x_jU \cap S) \setminus \bigcup_{k<j} M_k$. Clearly, the M_j are measurable subsets of G. The function $f_2 \in L^1(G, \mathcal{A})$ is defined by

$$f_2 = \sum_{j=1}^{m} a_j \chi_{M_j}, \text{ where } a_j := f_1(x_j) \in \mathcal{A}$$

and χ_{M_j} denotes the characteristic function of M_j. Then

$$\|f_1 - f_2\|_{L^1} = \sum_{j=1}^{m} \int_{M_j} \|f_1(x) - a_j\|_{\mathcal{A}}\, dx \leq \sum_{j=1}^{m} \delta|S|^{-1}|M_j| = \delta,$$

hence $\|f_2\|_{L^1} < \|\eta\| + 3\delta$. Moreover,

$$\|\rho(f_2)\xi_0 - \eta\| = \|\rho(f_2)\xi_0 - \rho(f_1)\xi_0 + \rho(f_1)\xi_0 - \eta\| < \delta\|\xi_0\| + \delta\|\xi_0\| = 2\delta\|\xi_0\|.$$

The desired function $g \in \ell^1(G, \mathcal{A})$ is defined by

$$g = \sum_{j=1}^{m} a_j |M_j| \varepsilon_j$$

where ε_j denotes the point measure at x_j. Then clearly

$$\|g\|_{\ell^1} = \|f_2\|_{L^1} = \sum_{j=1}^{m} \|a_j\|_{\mathcal{A}} |M_j| < \|\eta\| + 3\delta.$$

Furthermore,

$$\|\rho_d(g)\xi_0 - \eta\| = \|\rho_d(g)\xi_0 - \rho(f_2)\xi_0 + \rho(f_2)\xi_0 - \eta\| < \|\rho_d(g)\xi_0 - \rho(f_2)\xi_0\| + 2\delta\|\xi_0\|,$$

and

$$\begin{aligned}\|\rho_d(g)\xi_0 - \rho(f_2)\xi_0\| &\le \sum_{j=1}^{m} \|\rho(a_j)|M_j|\rho(x_j)\xi_0 - \int_{M_j} \rho(a_j)\rho(s)\xi_0\, ds\| \\ &\le \sum_{j=1}^{m} \|a_j\| \| |M_j|\rho(x_j)\xi_0 - \int_{M_j} \rho(s)\xi_0\, ds\|,\end{aligned}$$

because $\rho\colon \mathcal{A} \to \mathrm{End}(E)$ is bounded by 1. Writing $|M_j|\rho(x_j)\xi_0$ as an integral over M_j it follows that

$$\|\rho_d(g)\xi_0 - \rho(f_2)\xi_0\| \le \sum_{j=1}^{m} \|a_j\| \int_{M_j} \|\rho(x_j)\xi_0 - \rho(s)\xi_0\|\, ds.$$

Since $s \in M_j$ is of the form $x_j u$ with $u \in U$ and since $\rho(G)$ consists of isometries, the quantity $\|\rho(x_j)\xi_0 - \rho(s)\xi_0\|$ is less than δ, hence $\|\rho_d(g)\xi_0 - \rho(f_2)\xi_0\|$ is estimated by $\delta \sum_{j=1}^{m} \|a_j\||M_j| = \delta\|f_2\| < \delta(\|\eta\| + 3\delta)$. Altogether we find that

$$\|\rho_d(g)\xi_0 - \xi\| < 2\delta\|\xi_0\| + \delta(\|\eta\| + 3\delta) \quad \text{and} \quad \|g\|_{\ell^1} < \|\eta\| + 3\delta.$$

Choosing δ small enough the inequalities claimed in $(*)$ are satisfied for the given ε.

Now suppose that any $\xi \in E$ is given as in the beginning of the proof. Applying $(*)$ to $\eta = \xi$ and $\varepsilon = \frac{1}{4}$ we find $g_1 \in \ell^1(G, \mathcal{A})$ such that

$$\|\rho_d(g_1)\xi_0 - \xi\| < \tfrac{1}{4} \quad \text{and} \quad \|g_1\|_{\ell^1} < \|\xi\| + \tfrac{1}{4}.$$

Applying $(*)$ to $\varepsilon = \frac{1}{8}$ and $\eta = \xi - \rho_d(g_1)\xi_0$ one gets $g_2 \in \ell^1(G, \mathcal{A})$ such that

$$\|\rho_d(g_2)\xi_0 + \rho_d(g_1)\xi_0 - \xi\| < \tfrac{1}{8} \quad \text{and} \quad \|g_2\|_{\ell^1} < \tfrac{1}{8} + \|\xi - \rho_d(g_1)\xi_0\| < \tfrac{1}{2}$$

Inductively, we find a sequence (g_n) in $\ell^1(G, \mathcal{A})$ such that $\|g_n\|_{\ell^1} < \frac{1}{2^{n-1}}$ and $\|\sum_{j=1}^{n} \rho_d(g_j)\xi_0 - \xi\| < \frac{1}{2^{n+1}}$ for $n \ge 2$: If $g_1, \ldots, g_n$ are already constructed then apply $(*)$ to $\eta = \xi - \sum_{j=1}^{n} \rho_d(g_j)\xi_0$ and $\varepsilon = \frac{1}{2^{n+2}}$ to obtain an $g_{n+1} \in \ell^1(G, \mathcal{A})$ such that

$$\|g_{n+1}\|_{\ell^1} < \tfrac{1}{2^n} \quad \text{and} \quad \|\sum_{j=1}^{n+1} \rho_d(g_j)\xi_0 - \xi\| < \tfrac{1}{2^{n+2}}.$$

The desired h is defined as $h = \sum_{n=1}^{\infty} g_n$. Clearly, h exists in $\ell^1(G, \mathcal{A})$ and the equation $\rho_d(h)\xi_0 = \xi$ is satisfied. ∎

Remark 4. Instead of discretisizing G totally one may make it partially discrete in the following sense. If N is a given closed normal subgroup in G then one obtains a new locally compact group topology on G by declaring N to be open in G while the topology on N coincides with the original one. Denote this new locally compact group by G^N. If again E is a simple left $L^1(G,\mathcal{A})$–module with associated representations ρ of G and $\mathcal{A}$, then one obtains an $L^1(G^N,\mathcal{A})$–module structure on E by putting

$$\rho^N(f)\xi = \int_{G^N} \rho(x)\rho(f(x))\xi \, d\mu^N(x),$$

where $d\mu^N$ denotes a left Haar measure on G^N. This $L^1(G^N,\mathcal{A})$–module is simple as well.

Proof. For ξ in E let $E(\xi) := \rho^N(L^1(G^N,\mathcal{A}))\xi$. We have to show that $E(\xi) = E$ if $\xi \neq 0$. For any $f \in L^1(G^N,\mathcal{A})$ and $h \in \ell^1(G,\mathcal{A})$ one computes that

$$\rho_d(h)\rho^N(f)\xi = \rho^N(h * f)\xi$$

where $h * f \in L^1(G^N,\mathcal{A})$ is given by

$$(h * f)(x) = \sum_{y\in G} h(y)f(y^{-1}x), \text{ or}$$

$$h * f = \sum_{y\in G} h(y)(\varepsilon_y * f).$$

Hence $E(\xi)$ is $\ell^1(G,\mathcal{A})$–invariant. By the theorem, $E(\xi) = E$ as claimed or $E(\xi) = 0$. To exclude $E(\xi) = 0$ if $\xi \neq 0$, let $F := \{\xi \in E | E(\xi) = 0\}$. For $\xi \in E$, $f \in L^1(G^N,\mathcal{A})$ and $h \in \ell^1(G,\mathcal{A})$ one finds that

$$\rho^N(f)\rho_d(h)\xi = \rho^N(f * h)\xi$$

where $f * h \in L^1(G^N,\mathcal{A})$ is given by $(f * h)(x) = \sum_{y\in G} \Delta^N(y)f(xy)h(y^{-1})$; here Δ^N denotes the modular function of G^N. This equation shows that F is $\ell^1(G,\mathcal{A})$–invariant, hence by the theorem $F = 0$ or $F = E$. If $F = 0$ we are done. If $F = E$ the operator $\rho^N(f)$ is zero for all $f \in L^1(G^N,\mathcal{A})$. We apply this information to particular f's. Let $a \in \mathcal{A}$ be arbitrarily given and let $(\varphi_j)_{j\in J}$ be an approximate identity in the algebra $L^1(G^N)$ of numerical functions on G^N. The φ_j's may be chosen as nonnegative continuous functions with (small) compact support around the origin such that $\|\varphi_j\|_{L^1(G^N)} = 1$. For all j and all $\xi \in E$ the equation

$$0 = \rho^N(\varphi_j \otimes a)\xi = \int_{G^N} \varphi_j(x)\rho(x)\rho(a)\xi \, d\mu^N(x)$$

holds true. Passing to the limit we conclude that $\rho(a)\xi = 0$ for all $\xi \in E$ and all $a \in \mathcal{A}$, which is impossible because E was assumed to be non-trivial.

Observe that in the present proof we did not use explicitly that E is a *simple* $L^1(G,\mathcal{A})$–module. We only used that E is simple as an $\ell^1(G,\mathcal{A})$–module and that ρ^N has the above structure, which means that the group representation ρ is strongly continuous when considered as representation of G^N. This observation applies in particular to $N = G$, i.e., $G^N = G$. ∎

Remark 5. I want to spell out one fact explicitly even though it is evident. The Banach space E above with its original norm derived from the norm on $L^1(G,\mathcal{A})$ was made into a Banach $L^1(G^N,\mathcal{A})$–module. Being a *simple* $L^1(G^N,\mathcal{A})$–module the vector space E can be endowed with the quotient norm derived from the norm on $L^1(G^N,\mathcal{A})$. Actually, the two norms are equivalent; they are only fixed up to equivalence anyway. This is an immediate consequence of the Open Mapping Theorem.

Corollary 6. *Let G be a locally compact group. If the involutive Banach algebra $\ell^1(G)$ is rigidly symmetric then $L^1(G)$ is rigidly symmetric as well. In particular, $L^1(G)$ is rigidly symmetric for any nilpotent locally compact group G.*

Proof. The claim in the third sentence of the corollary follows from the preceding claim because, as was pointed out earlier, it was proved in [7, Theorem 7] that ℓ^1–algebras of discrete nilpotent groups are rigidly symmetric.

In view of 1. and of the criterion (U) we have to consider a simple left $L^1(G,\mathcal{A})$–module E, where now $\mathcal{A}$ denotes a C^*–algebra. The involutions in $L^1(G,\mathcal{A})$ and $\ell^1(G,\mathcal{A})$ are given by

$$f^*(x) = \Delta(x)^{-1} f(x^{-1})^* \text{ and } g^*(x) = g(x^{-1})^*$$

for $f \in L^1(G,\mathcal{A})$ and $g \in \ell^1(G,\mathcal{A})$, where Δ denotes the modular function of G.

From 2. it follows that E is also a simple left $\ell^1(G,\mathcal{A})$–module in a canonical manner. Criterion (U) tells us that there exist a bounded, topologically irreducible, involutive representation π_d of $\ell^1(G,\mathcal{A})$ in some Hilbert space $\mathfrak{H}$ and a non–zero $\ell^1(G,\mathcal{A})$–intertwining operator $T\colon E \to \mathfrak{H}$. The operator T is necessarily bounded, because for each non–zero $\xi_0 \in E$ the map $g \mapsto \pi_d(g)T\xi_0 = T\rho_d(g)\xi_0$ from $\ell^1(G,\mathcal{A})$ into $\mathfrak{H}$ is bounded (the letters ρ and ρ_d have, of course, the same meaning as in 2.), and because E carries the quotient norm w.r.t. $g \mapsto \rho_d(g)\xi_0$. Like in the case of simple modules the representation π_d delivers a group representation of G in $\mathfrak{H}$ and a bounded representation of the algebra $\mathcal{A}$ in $\mathfrak{H}$, both denoted by the same letter π. Again $\pi(x)\pi(a) = \pi(a)\pi(x)$ for $x \in G$ and $a \in \mathcal{A}$. From the fact that π_d is an involutive representation it follows easily that each $\pi(x), x \in G$, is unitary and that $\pi(a^*) = \pi(a)^*$ for $a \in \mathcal{A}$. Again, π_d can be reconstructed by the formula

$$\pi_d(g)\xi = \sum_{x\in G} \pi(x)\pi(g(x))\xi$$

for $\xi \in \mathfrak{H}$ and $g \in \ell^1(G,\mathcal{A})$. Moreover, T intertwines the actions of G and $\mathcal{A}$ on E and on $\mathfrak{H}$. We would like to define an involutive representation π of $L^1(G,\mathcal{A})$ by the formula

$$\pi(f)\xi = \int_G \pi(x)\pi(f(x))\xi \, dx.$$

To this end, we have to know that π is a strongly continuous representation of the topological group G. But for $\xi \in E$ the function $x \mapsto \pi(x)T\xi = T\rho(x)\xi$ from G

into $\mathfrak{H}$ is continuous, because T is a bounded linear map and because $x \mapsto \rho(x)\xi$ is continuous. At this point we use that E is not an arbitrary simple $\ell^1(G, \mathcal{A})$–module, but that originally E is an $L^1(G, \mathcal{A})$–module. As π_d is irreducible and as T is an intertwining operator, the range of T is a dense subspace of $\mathfrak{H}$. Since each $\pi(x), x \in G$, is a unitary operator, an easy estimate shows that $x \mapsto \pi(x)\eta$ is not only continuous for $\eta \in T(E)$, but rather for all $\eta \in \mathfrak{H}$. Now it is clear that the above formula defines a (topologically irreducible) involutive representation of $L^1(G, \mathcal{A})$ in $\mathfrak{H}$, and that T is an intertwining operator for the actions of $L^1(G, \mathcal{A})$ on E and on $\mathfrak{H}$. We conclude that $L^1(G, \mathcal{A})$ is a symmetric Banach algebra (for all C^*–algebras $\mathcal{A}$), whence $L^1(G)$ is rigidly symmetric. ∎

Concluding remarks

It is very natural to ask if the theorem can be generalized to twisted covariance algebras in the obvious manner. A twisted covariance algebra $L^1(G, \mathcal{A}, \mathcal{T}, \mathcal{P})$ also consists of the $\mathcal{A}$–valued integrable functions on a locally compact group, but the convolution is "perturbed" by an action T of G on $\mathcal{A}$ and by a factor system P. For this notion see H. Leptin [5]. There such algebras are called "Verallgemeinerte L^1–Algebren". I did not thoroughly consider this question, at the first glance I don't see any fundamental difficulties.

The next remark concerns property (U). One should notice, that there to a given E only the *existence* of $(\mathfrak{H}, \pi, T)$ is required. Nothing is said about *uniqueness* (in the appropriate sense). I don't know of any simple module of a symmetric algebra where the "unitarization" $\mathfrak{H}$ is not unique, but it is hard to imagine that a general uniqueness theorem could be true. Nevertheless such a theorem might hold for ℓ^1–algebras of discrete nilpotent groups (it holds true in the case of connected nilpotent groups, but for reasons which definitely don't apply in the discrete case). Uniqueness for discrete nilpotent groups would have some interesting consequences.

Finally, there is a very loose connection between the present article and recent investigations on the relation between the unitary duals of a locally compact group and its discrete version. For more information on the latter subject see the forthcoming paper [1] by M. Bekka and A. Valette and the references given there. In this article the authors prove that the unitary dual $\widehat{G}$ of a connected Lie group G is dense in the unitary dual of the underlying discrete group iff G is solvable.

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