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Integrable operator representations of $\mathbb{R}_q^2$ and $SL_q(2,\mathbb{R})$

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1. Introduction

Unbounded self-adjoint or normal operators in Hilbert space which satisfy some algebraic relations appear in representation theory of Lie algebras and in mathematical physics. To be more precise, one deals with certain "well-behaved" representations of the relations. There is no canonical way to define these well-behaved representations for a given set of relations, but in many cases a reasonable candidate for such a definition is easy to guess. For instance, it is quite natural to define the well-behaved representations of the relation $ab-ba=-i$ for self-adjoint operators a and b by the requirement that a and b fulfill the Weyl relation $e^{ita}\,e^{isb}=e^{its}\,e^{isb}\,e^{ita}$, $t,s\in\mathbb{R}$. More generally, for a representation of a Lie algebra by skew-adjoint operators in Hilbert space (or for the corresponding commutation relations of a set of Lie algebra generators) one would take those which come from a unitary representation of the associated simply connected and connected Lie group. In representation theory of Lie algebras, such representations are usuallly called integrable (see e.g. [S1], Definition 10.1.7). Following this terminology, we shall also call the well-behaved representations of an arbitrary set of relations (not necessarily of Lie type) "integrable". Before we come to quantum groups, let us briefly mention another general example: Consider the relation $ab=b\,F(a)$ for a self-adjoint operator a and a normal operator b, where F is a given measurable function on the real line. The integrable representations for this relation can be defined by requiring that $f(a)b\subseteq bf(F(a))$ for all L^∞-functions f on $\mathbb{R}$. This approach has been studied by [OS] and in various other papers of the Kiev school.

Let us turn to quantum groups now. In what follows we suppose that q is a fixed complex number of modulus one. Let $\mathbb{R}_q^2$ denote the free $*$-algebra with unit element $\mathbf{1}$ which is generated by two hermitean elements x and y satisfying the relation

$$xy=qyx\ . \tag{1}$$

In quantum group theory (cf. [M]), $\mathbb{R}_q^2$ is usually called the *real quantum plane.* For $q=1$, $\mathbb{R}_q^2$ is the commutative polynomial algebra $\mathbb{C}[x,y]$ in two hermitean indeterminats x and y. How do we get the points of the real plane $\mathbb{R}^2$ from the $*$-algebra $\mathbb{C}[x,y]$? The answer is easy and well-known: The points of $\mathbb{R}^2$ are (in one-to-one correspondence to) the equivalence classes of irreducible pairs $\{x,y\}$ of strongly commuting self-adjoint operators x and y on a Hilbert space. (Strong commutativity of two self-adjoint operators means that their spectral projections commute. Note that it is not sufficient that the self–adjoint operators commute pointwise on a common invariant core, because there exist so-called "Nelson couples", cf. [S1], Section 9.3). If we try to proceed in a similar way in case of arbitrary q, we first have to define strong commutativity (or integrable representations in our prefered terminology) for the relation (1). The equivalence classes of irreducible integrable representations $\{x, y\}$ of (1) may be thought of as the "points" of the real quantum plane $\mathbb{R}_q^2$.

It might be worth to note that for $q^2 \neq 1$ the relation (1) for self-adjoint operators x and y has interesting operator-theoretic properties: If one of the operators x, y is bounded, then we are in the trivial case where $x = x_1 \oplus 0$ and $y = 0 \oplus y_1$. Further, in contrast to Lie algebra relations, non-zero analytic vectors for x *and* y do not exist except in trivial cases. To be precise, if (1) is satisfied on an invariant dense domain $\mathcal{D}$ of a Hilbert space and if $\ker x = \ker y = \{0\}$, then the only analytic vector for a and b in the domain $\mathcal{D}$ is the null vector.

In the following two sections we propose a definition of integrable representations for the real quantum plane $\mathbb{R}_q^2$ and for the real form $SL_q(2, \mathbb{R})$ of $SL_q(2)$ and we classify the irreducible integrable representations up to unitary equivalence. The material is taken from the authors paper [S2] which contains detailed treatments and proofs of all results discussed in what follows.

Throughout the following, we assume that $q^4 \neq 1$ and we write $q = e^{-i\varphi}$ with $|\varphi| < \pi$ and $\varphi_n := \varphi - \pi n$ for $n \in \mathbb{Z}$. If x is a self-adjoint operator, we write $x > 0$ (resp. $x < 0$) if $\ker x = \{0\}$ and $x \geq 0$ (resp. $x \leq 0$).

2. Integrable representations of $\mathbb{R}_q^2$

Let x and y be self-adjoint operators acting on the same Hilbert space $\mathcal{H}$. We want to define what we shall mean by saying that the couple $\{x, y\}$ is an integrable representation of $\mathbb{R}_q^2$.

First let us suppose that $x > 0$ or $x < 0$. If the relation (1) is fulfilled for elements x and y of an algebra with unit, then we have $f(x)\, y = y\, f(qx)$ for any complex polynomial f. It seems to be natural to define integrability by requiring the latter relation for the self-adjoint operators x and y and for certain "nice" functions f, say $f(s) = |s|^{it}$, $s, t \in \mathbb{R}$. We shall do this and we interpret $|qx|^{it}$ as $e^{(\varphi - 2\pi k)t} |x|^{it} \equiv e^{\varphi_{2k} t} |x|^{it}$. Since we have asssumed that $x > 0$ or $x < 0$, this might be justified. The precise formulation of the preceding is given in part (i) of the following definition.

Definition 2.1. (i) Suppose that $x > 0$ or $x < 0$. We shall say that the pair $\{x, y\}$ is an *integrable representation* of $\mathbb{R}_q^2$ if there exists an integer k such that

$$|x|^{it} y = e^{\varphi_{2k} t} y |x|^{it} \quad \text{for} \quad t \in \mathbb{R}\,. \tag{2}$$

(ii) Suppose that $x \geq 0$ or $x \leq 0$. Set $\mathcal{H}_0 := \ker x$ and $x_1 := x | \mathcal{H}_0^\perp$. The pair $\{x, y\}$ is called an *integrable representation* of $\mathbb{R}_q^2$ if there are self-adjoint operators y_0 on $\mathcal{H}_0$ and y_1 on $\mathcal{H}_o^\perp$ such that $y = y_0 \oplus y_1$ on $\mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ and $\{x_1, y_1\}$ is an integrable represenation of $\mathbb{R}_q^2$ according to (i).

If (2) is satisfied in part (i) of Definition 2.1, we shall write $\{x,y\} \in \mathcal{C}_{2k}(q)$. In part (ii) we also write $\{x,y\} \in \mathcal{C}_{2k}(q)$ if $\{x_1,y_1\} \in \mathcal{C}_{2k}(q)$.

Next we turn to the general case where x is neither positive nor negative in general. Throughout the following discussion we shall assume that x has a trivial kernel. Formally, (1) yields $x^2y = q^2yx^2$. It might be natural to require that integrability for $xy = qyx$ implies integrability for $x^2y = q^2yx^2$. Since $x^2 > 0$ because of $\ker x = \{0\}$, integrability is already defined for the latter relation. By Definition 2.1, (i), it means that $(x^2)^{it}y = e^{(2\varphi-2\pi k)t}y(x^2)^{it}$, $t \in \mathbb{R}$, for some integer k. That is, $|x|^{is}y = e^{\varphi_k s}y|x|^{is}$ for $s \in \mathbb{R}$ or equivalently, $\{|x|,y\}$ is an integrable representation of $\mathbb{R}^2_{\varepsilon q}$, where $\varepsilon = (-1)^k$. This is our first condition in Definition 2.2 below. To derive the second condition, we use the polar decomposition $x = u_x|x|$ of the self-adjoint operator x. From the first condition that $\{|x|,y\}$ is an integrable representation of $\mathbb{R}^2_{\varepsilon q}$ it can be shown that there is a core $\mathcal{D}$ for $|x|$ and y such that $|x|\mathcal{D} = \mathcal{D}$ and $|x|y\psi = \varepsilon qy|x|\psi$, $\psi \in \mathcal{D}$. Another reasonable requirement for the integrability of (1) might be to assume that $xy\psi = qyx\psi$ for $\psi \in \mathcal{D}$. Then we obtain

$$xy\psi = u_x|x|y\psi = \varepsilon q u_x y|x|\psi = qyx\psi = qyu_x|x|\psi \quad \text{for} \quad \psi \in \mathcal{D}\,,$$

so that $u_x y\eta = \varepsilon y u_x \eta$ for $\eta \in |x|\mathcal{D}$. Since $\mathcal{D}$ is a core for y, we get $u_x y \subseteq \varepsilon y u_x$. This is our second condition in the following definition.

Definition 2.2. We shall say that a couple $\{x,y\}$ of self-adjoint operators is an *integrable representation* of $\mathbb{R}^2_q$ if for $\varepsilon = 1$ or $\varepsilon = -1$ the following two conditions are fulfilled:
(D.1) $\{|x|,y\}$ is an integrable representation of $\mathbb{R}^2_{\varepsilon q}$.
(D.2) $u_x y \subseteq \varepsilon y u_x$.

From (D.1), there is a $k \in \mathbb{Z}$ such that $\{|x|,y\} \in \mathcal{C}_{2k}(q)$. We write $\{x,y\} \in \mathcal{C}_{2k}(q)$ if $\varepsilon = 1$ and $\{x,y\} \in \mathcal{C}_{2k+1}(q)$ if $\varepsilon = -1$ in Definition 2.2.

Despite of the above discussion, our Definitions 2.1 and 2.2 might still look a bit artificial. We state a few facts which show that integrable representations have indeed very nice properties.

1.) A couple $\{x,y\}$ of self-adjoint operators is an integrable representation of $\mathbb{R}^2_q$ if and only if $u_x|y| \subseteq |y|u_x$, $u_y|x| \subseteq |x|u_y$, $u_xu_y = \varepsilon u_yu_x$ and $\{|x|,|y|\}$ is an integrable representation of $\mathbb{R}^2_{\varepsilon q}$ for $\varepsilon = 1$ or for $\varepsilon = -1$.
In contrast to Definition 2.2, the preceding conditions are symmetric in x and y. Therefore, $\{x,y\}$ is an integrable representation of $\mathbb{R}^2_q$ if and only if $\{y,x\}$ is an integrable representation of $\mathbb{R}^2_{\bar{q}}$.

2.) If $\{x,y\}$ is an integrable representation of $\mathbb{R}^2_q$, then there exists a linear subspace $\mathcal{D}$ of $\mathcal{D}(x) \cap \mathcal{D}(y)$ which is invariant under x and y and a core for both operators such that $xy\psi = qyx\psi$ for vectors ψ in $\mathcal{D}$.

3.) Suppose that $x > 0$ and $y > 0$. Then $\{x,y\}$ is an integrable representation of $\mathbb{R}^2_q$ if and only if there is a $k \in \mathbb{Z}$ such that the unitary groups x^{it} and y^{is} satisfy the Weyl relation $x^{it}\,y^{is} = e^{i\varphi_{2k}ts}\,y^{is}\,x^{it}$ for $t,s \in \mathbb{R}$.

The last assertion is the main technical ingredient for the classification of irreducible integrable representations. As usual, we denote by P and Q the momentum operator resp. the position operator from quantum mechanics, i.e. P is the differential operator $-i\frac{d}{dt}$ and Q is the multiplication operator by t on the Hilbert space $L^2(\mathbb{R})$.

Theorem 2.3. *Each irreducible integrable representation $\{x,y\}$ of $\mathbb{R}_q^2$ is unitarily equivalent to one of the following list:*

$(\mathrm{I})_{\varepsilon_1,\varepsilon_2,k} : x = \varepsilon_1 e^Q, y = \varepsilon_2 e^{\varphi_{2k}P}$ on $\mathcal{H} = L^2(\mathbb{R}) : \varepsilon_1, \varepsilon_2 \in \{1,-1\}, k \in \mathbb{Z}$.

$(\mathrm{II})_k : x = \begin{pmatrix} e^Q & 0 \\ 0 & -e^Q \end{pmatrix}, \ y = \begin{pmatrix} 0 & e^{\varphi_{2k+1}P} \\ e^{\varphi_{2k+1}P} & 0 \end{pmatrix}$

on $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) : \ k \in \mathbb{Z}$.

$(\mathrm{III})_{\alpha,0} : x = \alpha\,,\ y = 0$ on $\mathcal{H} = \mathbb{C} : \ \alpha \in \mathbb{R}$.

$(\mathrm{III})_{0,\alpha} : x = 0\,,\ y = \alpha$ on $\mathcal{H} = \mathbb{C} : \ \alpha \in \mathbb{R}$.

All pairs of this list are irreducible integrable representations of $\mathbb{R}_q^2$.

We conclude this section by the following

Example 2.4. Suppose that A and B are self-adjoint operators on a Hilbert space $\mathcal{G}$. Let $k,l,n \in \mathbb{Z}$. Let x denote the self-adjoint operator $e^Q \otimes I \oplus -(e^Q \otimes I)$ on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathcal{G} \oplus L^2(\mathbb{R}) \otimes \mathcal{G}$. It is not difficult to show that the operator matrix

$$\begin{pmatrix} e^{\varphi_{2k}P} \otimes A & e^{\varphi_{2n+1}P} \otimes I \\ e^{\varphi_{2n+1}P} \otimes I & e^{\varphi_{2l}P} \otimes B \end{pmatrix}$$

defines an essentially self-adjoint operator on $\mathcal{H}$. We denote its closure by y. Then the self-adjoint operators x and y satisfy the relation (1) pointwise on a common invariant core for both operators. The couple $\{x,y\}$ on $\mathcal{H}$ is irreducible if and only if the couple $\{A,B\}$ on $\mathcal{G}$ is irreducible. Further, two couples $\{x,y\}$ of this kind are unitarily equivalent if and only if the corresponding couples $\{A,B\}$ on $\mathcal{G}$ are unitarily equivalent and the corresponding triples $\{k,l,n\}$ of integers coincide. That is, our example gives a wealth of irreducible representations of (1) by self-adjoint operators. But such a couple $\{x,y\}$ is an integrable representation of $\mathbb{R}_q^2$ only in the obvious case where $A = 0$ and $B = 0$.

3. Integrable representations of $SL_q(2,\mathbb{R})$

First let us recall that $SL_q(2,\mathbb{R})$ is the free $*$-algebra with unit element $\mathbf{1}$ generated by four hermitean elements a,b,c,d which satisfy the following seven relations:

$$ab = qba\,,\ ac = qca\,,\ bd = qdb\,,\ cd = qdc\,,\ bc = cb\,, \tag{3}$$

$$ad - da = (q - \overline{q})\, bc\,, \tag{4}$$

$$ad - qbc = \mathbf{1}\ . \tag{5}$$

(A little more in detail, this definition means the following: Let $\mathbb{C}\langle a,b,c,d\rangle$ be the free complex algebra with unit generated by four elements a,b,c,d and let $\mathcal{I}$ denote the two-sided ideal of this algebra generated by the seven elements $ab-qba$, $ac-qca$, $bd-qdb$, $cd-qdc$, $bc-cb$, $ad-da-(q-\overline{q})bc$, $ad-qbc-\mathbf{1}$. Then the quotient algebra $\mathbb{C}\langle a,b,c,d\rangle/\mathcal{I}$ is denoted by $SL_q(2)$, cf. [M]. There is a unique involution on the algebra $\mathbb{C}\langle a,b,c,d\rangle$ for which a,b,c and d are hermitean elements. Since $|q|=1$ by assumption, the ideal $\mathcal{I}$ is invariant under this involution. Therefore, $SL_q(2)$ becomes a $*$-algebra which is denoted by $SL_q(2,\mathbb{R})$.)

Note this $SL_q(2,\mathbb{R})$ is even a Hopf $*$-algebra, but we do not use this additional structure in what follows. For our purposes it is convenient to replace (4) by

$$da - \overline{q}bc = \mathbf{1}\ . \tag{6}$$

It is clear that (3), (4), (5) and (3), (5), (6) are equivalent families of relations.

We begin with a simple lemma which is the algebraic background for the operator-theoretic considerations below.

Lemma 3.1. *Let $a,b,c,$ and d be hermitean elements of a $*$-algebra $\mathcal{A}$ with unit.*
(i) Suppose that a is an invertible element of $\mathcal{A}$. Then a,b,c,d fulfill the relations (3), (4), (5), if and only if

$$ab = qba\ ,\ ac = qca\ ,\ bc = cb \tag{7}$$

and

$$d = (\overline{q}bc + 1)\, a^{-1}\ . \tag{8}$$

(ii) Suppose that b resp. c is an invertible element of $\mathcal{A}$. Then $z := b^{-1}c$ (resp. $z := c^{-1}b$) is a hermitean element of $\mathcal{A}$ which permutes with $a,b,c,$ and d.

Slightly simplified, the main idea behind our integrability definition given below is as follows: We assume that the self-adjoint operator a has trivial kernel and we express d in terms of a^{-1},b,c via (8).

On the purely algebraic level, the seven equations (3), (4), (5) are equivalent to the three equations of (7). For the first two of the three relations of (7) integrable representations have been already defined in the preceding section. For the third relation such a definition is obvious: A couple $\{b,c\}$ of two self-adjoint operators is said to be an integrable representation of $\mathbb{R}^2$ if b and c strongly commute (i.e. the spectral projections of b and c commute).

Definition 3.2. Let a,b,c,d be self-adjoint operators acting on the same Hilbert space $\mathcal{H}$ such that $\ker a = \{0\}$. We shall say that the quadruple $\{a,b,c,d\}$ is an *integrable representation* of $SL_q(2,\mathbb{R})$ if the following three conditions are satisfied:

(D.1) Three is an $n\in\mathbb{Z}$ such that $\{a,b\}\in\mathcal{C}_n(q)$ and $\{a,c\}\in\mathcal{C}_n(q)$.

(D.2) b and c strongly commute.

(D.3) d is the closure of the operator $(\overline{q}bc+1)a^{-1}$.

Let us discuss the preceding by a number of remarks.

1.) In [S2] another definition of integrable representations of $SL_q(2,\mathbb{R})$ has been used. That it is equivalent to the above Definition 3.2 is stated in Theorem 4.12, (i), of [S2].

2.) Formally, $ab = qba$ and $ac = qca$ imply that $a^{-1}bc = \overline{q}^2 bca^{-1}$. From condition (D.1) it follows easily that the pair $\{a^{-1}, \overline{bc}\}$ is an integrable representation of $\mathbb{R}^2_{\overline{q}^2}$. Having this, it can be shown that the operator $(\overline{q}bc+1)a^{-1}$ is symmetric and densely defined. Since d is self-adjoint, $(\overline{q}bc+1)a^{-1}$ has to be essentially self-adjoint by (D.3). This suggests the following view of our Definition 3.2: The integrable representations of $SL_q(2,\mathbb{R})$ are in one-to-one correspondence to triples $\{a,b,c\}$ of self-adjoint operators on a Hilbert space $\mathcal{H}$ with $\ker a = \{0\}$ such that (D.1) and (D.2) are fulfilled and the operator $(\overline{q}bc+1)a^{-1}$ is essentially self-adjoint.

3.) Suppose $\{a,b,c,d\}$ is an integrable representation of $SL_q(2,\mathbb{R})$. Then, by condition (D.1), the pairs $\{a,b\}$ and $\{a,c\}$ are integrable representations of $\mathbb{R}^2_q$. It should be emphasized that (D.1) requires more, namely that both couples $\{a,b\}$ and $\{a,c\}$ belong to the same class $\mathcal{C}_n(q)$. The reason for this stronger requirement in (D.1) is that it has the following consequence: If $\ker b = \{0\}$, then the self-adjoint operator $z := \overline{b^{-1}c}$ commutes *strongly* with a (and obviously also with b, c and d).

4.) The relations (3), (4), (5) defining $SL_q(2,\mathbb{R})$ imply that

$$ad - q^2 da = \left(1-q^2\right)\mathbf{1}\,. \tag{9}$$

and

$$da - \overline{q}^2 ad = \left(1-\overline{q}^2\right)\mathbf{1}\,. \tag{10}$$

Integrable representations for this type of relations have been also defined in [S2]. We shall not repeat this definition here.

5.) The above Definition 3.2 may look as the weakest possible variant of a reasonable definition of integrability. However, it hides our original motivation for such a definition: We wanted to select the "best possible well-behaved" representations of the $*$-algebra $SL_q(2,\mathbb{R})$ in the sense that they give integrable representations of all subrelations and that they behave nicely under algebraic manipulations. It turns out that our Definition 3.2 achieves these aims. To be precise, each integrable representation $\{a,b,c,d\}$ of $SL_q(2,\mathbb{R})$ has the following properties:

– The couples $\{a,b\}$, $\{a,c\}$, $\{b,d\}$, $\{c,b\}$, $\{b,c\}$, $\{a,d\}$ and $\{d,a\}$ are integrable representations of the corresponding relations in (3), (9) and (10).

– $\{d,b,c,a\}$ is an integrable representation of $SL_{\overline{q}}(2,\mathbb{R})$. (In particular, this means that $\ker d = \{0\}$ and a is the closure of the operator $(qbc+1)d^{-1}$.)

– There is a dense domain $\mathcal{D}$ of the underlying Hilbert space $\mathcal{H}$ which is an invariant core for each operator a, a^{-1}, b, c, d such that all relations (3)–(10) are pointwise fulfilled for vectors in $\mathcal{D}$. (In particular, this implies that each integrable representation of $SL_q(2,\mathbb{R})$ defines indeed a $*$-representation of the

$*$-algebra $SL_q(2,\mathbb{R})$ on the domain $\mathcal{D}$ in the sense of unbounded representation theory, cf. Section 8.1 of [S1].)

The following theorem gives a complete classification of all irreducible integrable representations of $SL_q(2,\mathbb{R})$ up to unitary equivalence.

Theorem 3.3. *Put $k=0$ if $\varphi>0$ and $k=-1$ if $\varphi<0$. For arbitrary $\lambda\in(0,+\infty)$, $\varepsilon_1,\varepsilon_2\in\{1,-1\}$ and $\alpha\in\mathbb{R}\backslash\{0\}$, the following quadruples $\{a,b,c,d\}$ are irreducible integrable representations of $SL_q(2,\mathbb{R})$:*

$$\begin{aligned}(\mathrm{I})_{\lambda,\varepsilon_1,\varepsilon_2}: a&=\varepsilon_1\, e^{Q}\,,\ b=\varepsilon_2\, e^{\varphi P}\,,\ c=\lambda b\,,\\ d&=\varepsilon_1\left(\overline{q}\lambda e^{2\varphi P}+1\right)e^{-Q}\quad \text{on}\quad \mathcal{H}=L^2(\mathbb{R})\,.\end{aligned}$$

$$\begin{aligned}(\mathrm{I})_{\infty,\varepsilon_1,\varepsilon_2}: a&=\varepsilon_1\, e^{Q}\,,\ b=0\,,\ c=\varepsilon_2\, e^{\varphi P}\,,\ d=\varepsilon_1\, e^{Q}\\ &\text{on}\quad \mathcal{H}=L^2(\mathbb{R})\,.\end{aligned}$$

$$\begin{aligned}(\mathrm{II})_{-\lambda}: a&=\begin{pmatrix} e^{Q} & 0\\ 0 & -e^{Q}\end{pmatrix},\ b=\begin{pmatrix} 0 & e^{\varphi_{2k+1}P}\\ e^{\varphi_{2k+1}P} & 0\end{pmatrix},\ c=\lambda b\,,\\ d&=\begin{pmatrix}\left(\overline{q}\lambda e^{2\varphi_{2k+1}P}+1\right)e^{-Q} & 0\\ 0 & -\left(\overline{q}\lambda e^{2\varphi_{2k+1}P}+1\right)e^{-Q}\end{pmatrix}\\ &\text{on}\quad \mathcal{H}=L^2(\mathbb{R})\oplus L^2(\mathbb{R})\,.\end{aligned}$$

$$\begin{aligned}(\mathrm{II})_{\infty}: a&=\begin{pmatrix} e^{Q} & 0\\ 0 & -e^{Q}\end{pmatrix},\ b=0\,,\ c=\begin{pmatrix} 0 & e^{\varphi_{2k+1}P}\\ e^{\varphi_{2k+1}P} & 0\end{pmatrix}\\ d&=a^{-1}\quad \text{on}\quad \mathcal{H}=L^2(\mathbb{R})\oplus L^2(\mathbb{R})\,.\end{aligned}$$

$$(\mathrm{III})_{\alpha}: a=\alpha\,,\ b=c=0\,,\ d=\alpha^{-1}\quad \text{on}\quad \mathcal{H}=\mathbb{C}\,.$$

(The formulaes for d in $(\mathrm{I})_{\lambda,\varepsilon_1,\varepsilon_2}$ and in $(\mathrm{II})_{-\lambda}$ should be read in the sense that d is the closure of the operator on the right hand side.)
Conversely, each irreducible integrable representation of $SL_q(2,\mathbb{R})$ is unitarily equivalent to one and only one representation of this list.

After a first look at this list one may wonder at the restriction to positive values of λ for $(\mathrm{I})_{\lambda,\varepsilon_1,\varepsilon_2}$ and $(\mathrm{II})_{-\lambda}$. Also, one may try to replace φ by φ_{2n}, $n\in\mathbb{Z}$, in (I) or to allow arbitrary integers k in (II). Indeed, all this gives quadruples $\{a,b,c,d\}$ of operators which fulfill the relations (3)–(10) on a suitable invariant core for a,a^{-1},b,c and d. Further, a,b and c are self-adjoint and satisfy the conditions (D.1) and (D.2) of Definition 3.2. But the operator $(\overline{q}bc+1)a^{-1}$ is essentially self-adjoint only in the few cases collected in our list of Theorem 3.3. (The other representations just mentioned are called a-integrable in [S2].)

We close our discussion by two examples.

Example 3.4. The only operator representation of $SL_q(2,\mathbb{R})$ which appeared in the literature (according to my knowledge) is due to L.D. Faddeev and L.A. Takhtajan, cf. [FT]. It occured in their study of the Liouville model on the lattice and it is defined as follows: Set $q = e^i$, $a = e^{\frac{P}{2}}\sqrt{1+e^{2Q}}e^{\frac{P}{2}}$, $b = c = e^Q$ and $d = e^{-\frac{P}{2}}\sqrt{1+e^{2Q}}e^{-\frac{P}{2}}$ on $\mathcal{H} = L^2(\mathbb{R})$. The operators a and d (considered on a suitable domain) have self-adjoint closures which are again denoted by a and d, respectively. It can be shown that $\{a,b,c,d\}$ is an irreducible integrable representation of $SL_q(2,\mathbb{R})$ and unitarily equivalent to the representation $(\mathrm{I})_{1,1,1}$ of our list.

Example 3.5. Retain the notation of Example 2.4 and put $a = x$, $b = y$, $c = 0$, $d = x^{-1}$. From Example 2.4 we then obtain a large family of irreducible non-integrable representations of $SL_q(2,\mathbb{R})$ by self-adjoint operators a, b, c, d which fulfill the relations (3)–(10) on a common invariant core for these operators.

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