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Tables for an effective enumeration of real representations of quasi-simple Lie groups

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The purpose of this paper is to provide the data which are necessary to calculate the (real) dimension, the centralizer and the kernel of a finite-dimensional irreducible real representation of a quasi-simple Lie group.

The authors have implemented a software package based on the LiE system described in [7] which allows the user to access these data easily. For further information on this software package see the last section of this paper.

The main source for our tables in this paper is J. Tits' Springer Lecture Note 40 [11]. This book contains all of the data we shall need. Unfortunately, much of this information is given in a rather implicit way and one has to perform additional calculations to obtain the final results which can be found in the tables below. Part of this data can also be found in [9]. Note that in this paper we use the (more standard) labelling of the Dynkin diagrams by Bourbaki [1], pp. 250–275. This differs from Tits' notation [11] in the case of the exceptional types E_6, E_7, E_8, and F_4. The theory of irreducible real representations of semi-simple or reductive Lie groups can be found in [5], [9], [4], [12], e. g. A summary of the basic facts of this theory is included in [11]. Extensive tables covering the complex representations of complex quasi-simple Lie groups are contained in [8] and [3].

For the sake of completeness and to fix notation we shall start with some basics on the complex representation theory and then proceed with the real case.

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Complex representations of quasi-simple Lie groups

Let G be a quasi-simple complex Lie group with $\mathfrak{g}$ as its (simple) Lie algebra. Every (complex) finite-dimensional continuous representation $\mathsf{P} : G \to \mathrm{GL}(V)$ yields a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ by differentiation.

The *adjoint representation* ad is the (right) regular representation of $\mathfrak{g}$ defined by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(V) : x \mapsto (z \mapsto [z, x])$, where V is $\mathfrak{g}$ considered just as a vector space. A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a nilpotent self-normalizing subalgebra of $\mathfrak{g}$. All Cartan subalgebras of $\mathfrak{g}$ are in fact conjugate and thus have the same dimension r. The integer r is called the *rank* of $\mathfrak{g}$. A *weight* λ of the representation ρ is an element of the dual $\mathfrak{h}^*$ with the property that there is a

$v \in V\backslash\{0\}$ such that $h^\rho(v) = \lambda(h)\cdot v$ holds for every $h \in \mathfrak{h}$. A *root* of $\mathfrak{g}$ is a non-zero weight of the representation ad. Since $\mathfrak{h}$ is abelian, $\mathfrak{h} = \{x \in \mathfrak{g} \mid x^{\mathrm{ad}(\mathfrak{h})} = 0\}$ by Engel's theorem. Thus the set Φ of all roots of $\mathfrak{g}$ generates the dual $\mathfrak{h}^*$ because $\mathfrak{g}$ is simple. Moreover, there is an $\mathbb{R}-$basis $\Delta \subseteq \Phi$ of $\mathfrak{h}^*$ such that every root λ is a linear combination of elements of Δ having all coefficients either ≥ 0 (then λ is called a *positive* root) or ≤ 0 (and λ is called a *negative* root). The elements of Δ are called *fundamental* roots.

Since semisimple Lie algebras are reductive, we may restrict our attention to irreducible representations. For every finite-dimensional irreducible representation ρ there is a unique weight λ_ρ such that the difference $\lambda_\rho - \lambda$ is a positive linear combination of fundamental roots for every weight λ of ρ. This weight λ_ρ is called the *highest* weight of ρ and the representation ρ is uniquely determined by λ_ρ up to equivalence.

The *Killing form* κ on $\mathfrak{g}$ is the symmetric bilinear form defined by $\kappa(x,y) = \mathrm{trace}(\mathrm{ad}(x)\mathrm{ad}(y))$. Since κ is nondegenerate on $\mathfrak{h}\times\mathfrak{h}$, for any $\varphi \in \mathfrak{h}^*$ there is a unique element $h_\varphi \in \mathfrak{h}$ such that $\varphi(h) = \kappa(h_\varphi, h)$ holds for every $h \in \mathfrak{h}$. Setting $(\varphi,\psi) := \kappa(h_\varphi, h_\psi)$, the mapping $(\mathfrak{h}^*, (.\,,.)) \to (\mathfrak{h},\kappa) : \varphi \mapsto h_\varphi$ hence is an isometry.

Now assume that $\Delta = \{\alpha_1,\dots,\alpha_r\}$. For $\lambda \in \mathfrak{h}^*$ and $1 \leq i \leq r$ set $f_i(\lambda) := 2(\alpha_i,\lambda)(\alpha_i,\alpha_i)^{-1}$. Using the linear forms f_i we can decide whether or not an element $\lambda \in \mathfrak{h}^*$ is a weight or a highest weight of an irreducible representation ρ. Namely, λ is a (highest) weight of ρ iff $f_1(\lambda),\dots,f_r(\lambda) \in \mathbb{Z}$ ($\in \mathbb{N}_0$). If we define $\lambda_i \in \langle\Phi\rangle_{\mathbb{R}}$ by $f_j(\lambda_i) = \delta_{ij}$, the highest weight λ_ρ of ρ can be written as

$$\lambda_\rho = \sum_{i=1}^r f_i(\lambda_\rho)\cdot\lambda_i.$$

In particular, the elements λ_i are highest weights of irreducible representations ρ_i which are called the *fundamental representations* corresponding to the fundamental roots α_i. The free semigroup $\Lambda_+(\mathfrak{g},\mathfrak{h}) := \bigoplus_{i=1}^r \mathbb{N}_0\lambda_i$ is called the *weight space of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$. Since all Cartan subalgebras of $\mathfrak{g}$ are conjugate, we may abbreviate $\Lambda_+(\mathfrak{g}) := \Lambda_+(\mathfrak{g},\mathfrak{h})$. Summing up, we obtain (see [10], 3.2, [13], Thm. 4.7.1, [6], VII.3, or [4], 44.1)

Theorem 1. *The elements of the weight space* $\Lambda_+(\mathfrak{g})$ *and the equivalence classes of irreducible finite-dimensional (complex) representations of* $\mathfrak{g}$ *are in a one-to-one correspondence.*

The complex dimension $\dim_{\mathbb{C}}\rho$ can be calculated by using Weyl's formula (cp. [10], 3.8, [13], Thm. 4.14.6, [6], p. 257, or [4], 47.8):

$$\dim_{\mathbb{C}}\rho = \prod_{\varphi\in\Phi_+}\frac{(\lambda_\rho+\delta,\varphi)}{(\delta,\varphi)} \quad\text{with}\quad \delta = \sum_{i=1}^r \lambda_i,$$

where $\Phi_+ \subseteq \Phi$ denotes the set of all positive roots in Φ. Expressing $\dim_{\mathbb{C}}\rho$ in terms of $f_i(\lambda_\rho)$ and (λ_i,φ) we have

$$\dim_{\mathbb{C}}\rho = \prod_{\varphi\in\Phi_+}\frac{\sum_{i=1}^r (f_i(\lambda_\rho)+1)(\lambda_i,\varphi)}{\sum_{i=1}^r(\lambda_i,\varphi)}$$

and if $\varphi = \sum_{j=1}^{r} k_j \alpha_j \in \Phi_+$ we moreover have

$$(\lambda_i, \varphi) = \sum_{j=1}^{r} k_j (\lambda_i, \alpha_j) = k_i (\lambda_i, \alpha_i) = \frac{1}{2} k_i (\alpha_i, \alpha_i).$$

The sets Φ_+ of positive roots are listed in [11]. An algorithm for computing Φ_+ is presented in [6], Chapt. IV, Thm. XVI. The values (α_i, α_i) can be found in [6], Chapt. IV, §5, or in [10], (2.14), e. g.

Real representations of quasi-simple Lie groups

By integration, every Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ gives rise to a homomorphism $\Psi : \widetilde{G} \to \widetilde{G'}$ between the associated simply connected groups $\widetilde{G}$ and $\widetilde{G'}$. We shall always write the corresponding uppercase letters for the integrated homomorphisms. In particular, by P_i we denote the representation of $\widetilde{G}$ which corresponds to the fundamental representation ρ_i of $\mathfrak{g}$.

If $\mathfrak{g}$ is a simple real Lie algebra, then either

I) its complexification $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a simple complex Lie algebra (and $\mathfrak{g}$ is called a *real form* of $\mathfrak{g}_{\mathbb{C}}$) or

II) it is the realification of a simple complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, i. e. $\mathfrak{g}$ is the algebra $\mathfrak{g}_{\mathbb{C}}$ considered as an $\mathbb{R}$-algebra.

Now let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible real representation. Then $\rho_{\mathbb{C}} := \rho \otimes \mathrm{id}_{\mathbb{C}}$ is a complex representation of $\mathfrak{g}_{\mathbb{C}}$ on $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ which may not be irreducible, see the cases (2), (3) of the next section. Via the natural embedding $v \mapsto v \otimes 1$ of V into $V_{\mathbb{C}}$ we consider $\mathfrak{gl}(V)$ as a subspace of $\mathfrak{gl}(V_{\mathbb{C}})$ and so $\mathrm{GL}(V)$ is a subset of $\mathrm{GL}(V_{\mathbb{C}})$.

The following commutative diagram displays the situation.

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\rho"] \arrow[dr, "\iota"] \arrow[dd, "\exp"'] & & \mathfrak{gl}(V) \arrow[dr, "\subseteq"] \arrow[dd, "\exp"' near end] & \\
& \mathfrak{g}_{\mathbb{C}} \arrow[rr, "\rho_{\mathbb{C}}" near start] \arrow[dd, "\exp" near start] & & \mathfrak{gl}(V_{\mathbb{C}}) \arrow[dd, "\exp"] \\
\widetilde{G} \arrow[rr, "\mathsf{P}" near end] \arrow[dr, "\mathsf{I}"] & & \mathrm{GL}(V) \arrow[dr, "\subseteq"] & \\
& \widetilde{G}_{\mathbb{C}} \arrow[rr, "\mathsf{P}_{\mathbb{C}}"] & & \mathrm{GL}(V_{\mathbb{C}})
\end{tikzcd}$$

Case I) $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$.
The algebra $\mathfrak{g}$ is embedded in $\mathfrak{g}_{\mathbb{C}}$ via the injection $\iota : g \mapsto g \otimes 1$. The integrated map $\mathsf{l} : \widetilde{G} \to \widetilde{G'}$ is not necessarily injective, but its kernel must lie in the center of $\widetilde{G}$ since $\widetilde{G}$ is quasi-simple. The kernel $\ker \mathsf{l}$, which is called the *linearizer* of $\widetilde{G}$, turns out to be the intersection of the kernels of all representations lP_i of $\widetilde{G}$. The quotient $\widetilde{G}/_{\ker \mathsf{l}}$ is the maximal linearly representable group which is locally isomorphic to $\widetilde{G}$. To determine the group $\widetilde{G}^{\mathsf{P}} = \widetilde{G}^{\mathsf{lP}_{\mathbb{C}}}$ which is represented by P, we thus have to know how the center $\mathrm{Z}(\widetilde{G})$ is mapped via l to $\mathrm{Z}(\widetilde{G}_{\mathbb{C}})$. To identify the elements of $\mathrm{Z}(\widetilde{G})$, table 1 shows how $\mathrm{Z}(\widetilde{G})$ is embedded in the group $\mathrm{Z}(\widetilde{K})$, where $\widetilde{K}$ denotes the universal covering of a maximal compact subgroup K of $\widetilde{G}/_{\mathrm{Z}(\widetilde{G})}$. Table 2 lists the image of $\mathrm{Z}(\widetilde{G})$ under l as well as the linearizer. We shall use the symbol $\langle x \rangle_n$ to express that the element x generates a cyclic group of order n, where $n = \infty$ means that $\langle x \rangle_\infty \cong \mathbb{Z}$.

Table 1. The embedding of $\mathrm{Z}(\widetilde{G})$ in $\mathrm{Z}(\widetilde{K})$.

	type of $\widetilde{K}$	$\mathrm{Z}(\widetilde{K})$	$\mathrm{Z}(\widetilde{G})$
type $\mathbf{A}_n^{\mathbb{R}}$			
$n=1$	$\mathbb{R}$	$\mathbb{R}$	$\langle z \rangle_\infty$ for some $z \in \mathbb{R}$
$n=2l$	$\mathrm{B}_l^{\mathbb{R},0}$	$\langle y \rangle_2$	$\langle z \rangle_2$, $z=y$
$n=4l+1$	$\mathrm{D}_{2l+1}^{\mathbb{R},0}$	$\langle y \rangle_4$	$\langle z \rangle_4$, $z=y$
$n=4l+3$	$\mathrm{D}_{2l+2}^{\mathbb{R},0}$	$\langle y \rangle_2 \times \langle y' \rangle_2$	$\langle z \rangle_2 \times \langle z' \rangle_2$, $z=y$, $z'=y'$
type $\mathbf{A}_n^{\mathbb{C},i}$			
$i=0$	$\mathrm{A}_n^{\mathbb{C},0}$	$\langle y \rangle_{n+1}$	$\langle z \rangle_{n+1}$, $z=y$
$i>0$	$\mathrm{A}_{n-i}^{\mathbb{C},0}\times$ $\mathrm{A}_{i-1}^{\mathbb{C},0}\times$ $\mathbb{R}$	$\langle y \rangle_r \times \langle y' \rangle_i \times \mathbb{R}$ with $r=n-i+1$	$\langle z \rangle_g \times \langle z' \rangle_\infty$ $z = y^{r/g} y'^{i/g}$ $z' = x y^{a+b} y'^{-a}$, $x \in \mathbb{R}$ $g = \gcd(n+1, i)$ $= (n+1)a + ib$
type $\mathbf{A}_n^{\mathbb{H}}$			
$n=2l+1$	$\mathrm{C}_{l+1}^{\mathbb{H},0}$	$\langle y \rangle_2$	$\langle z \rangle_2$, $z=y$
type $\mathbf{B}_n^{\mathbb{R},i}$			
$i=0$	$\mathrm{B}_n^{\mathbb{R},0}$	$\langle y \rangle_2$	$\langle z \rangle_2$, $z=y$
$i=1$, $2 \mid n$	$\mathrm{D}_n^{\mathbb{R},0}$	$\langle y \rangle_2 \times \langle y' \rangle_2$	$\langle z \rangle_2$, $z=yy'$
$i=1$, $2 \nmid n$	$\mathrm{D}_n^{\mathbb{R},0}$	$\langle y \rangle_4$	$\langle z \rangle_2$, $z=y^2$
$i=2$	$\mathrm{B}_{n-1}^{\mathbb{R},0} \times \mathbb{R}$	$\langle y \rangle_2 \times \mathbb{R}$	$\langle z \rangle_2 \times \langle z' \rangle_\infty$, $z=y$, $z'=x \in \mathbb{R}$

Table 1. The embedding of $\mathrm{Z}(\widetilde{G})$ in $\mathrm{Z}(\widetilde{K})$, continued.

	type of $\widetilde{K}$	$\mathrm{Z}(\widetilde{K})$	$\mathrm{Z}(\widetilde{G})$
type $\mathbf{B}_n^{\mathbb{R},i}$			
$i=4k$	$\mathrm{D}_{i/2}^{\mathbb{R},0}\times\mathrm{B}_{n-i/2}^{\mathbb{R},0}$	$\langle y\rangle_2\times\langle y'\rangle_2\times\langle y''\rangle_2$	$\langle z\rangle_2\times\langle z'\rangle_2$, $z=yy'$, $z'=y''$
$i=4k+2$	$\mathrm{D}_{i/2}^{\mathbb{R},0}\times\mathrm{B}_{n-i/2}^{\mathbb{R},0}$	$\langle y\rangle_4\times\langle y'\rangle_2$	$\langle z\rangle_2\times\langle z'\rangle_2$, $z=y^2$, $z'=y'$
$i=2k+1$ $2\mid n-k$	$\mathrm{D}_{n-k}^{\mathbb{R},0}\times\mathrm{B}_{k}^{\mathbb{R},0}$	$\langle y\rangle_2\times\langle y'\rangle_2\times\langle y''\rangle_2$	$\langle z\rangle_2\times\langle z'\rangle_2$, $z=yy'$, $z'=y''$
$2\nmid n-k$	$\mathrm{D}_{n-k}^{\mathbb{R},0}\times\mathrm{B}_{k}^{\mathbb{R},0}$	$\langle y\rangle_4\times\langle y'\rangle_2$	$\langle z\rangle_2\times\langle z'\rangle_2$, $z=y^2$, $z'=y'$
type $\mathbf{C}_n^{\mathbb{R}}$			
$n=2l$	$\mathrm{A}_{n-1}^{\mathbb{C},0}\times\mathbb{R}$	$\langle y\rangle_n\times\mathbb{R}$	$\langle z\rangle_2\times\langle z'\rangle_\infty$, $z=y^l$, $z'=xy$, $x\in\mathbb{R}$
$n=2l+1$	$\mathrm{A}_{n-1}^{\mathbb{C},0}\times\mathbb{R}$	$\langle y\rangle_n\times\mathbb{R}$	$\langle z\rangle_\infty$, $z=xy$, $x\in\mathbb{R}$
type $\mathbf{C}_n^{\mathbb{H},i}$			
$i=0$	$\mathrm{C}_n^{\mathbb{H},0}$	$\langle y\rangle_2$	$\langle z\rangle_2$, $z=y$
$i>0$	$\mathrm{C}_i^{\mathbb{H},0}\times\mathrm{C}_{n-i}^{\mathbb{H},0}$	$\langle y\rangle_2\times\langle y'\rangle_2$	$\langle z\rangle_2$, $z=yy'$
type $\mathbf{D}_n^{\mathbb{R},i}$			
$i=0$ $2\mid n$	$\mathrm{D}_n^{\mathbb{R},0}$	$\langle y\rangle_2\times\langle y'\rangle_2$	$\langle z\rangle_2\times\langle z'\rangle_2$, $z=y$, $z'=y'$
$2\nmid n$	$\mathrm{D}_n^{\mathbb{R},0}$	$\langle y\rangle_4$	$\langle z\rangle_4$, $z=y$
$i=1$	$\mathrm{B}_{n-1}^{\mathbb{R},0}$	$\langle y\rangle_2$	$\langle z\rangle_2$, $z=y$
$i=2$ $2\mid n$	$\mathrm{D}_{n-1}^{\mathbb{R},0}\times\mathbb{R}$	$\langle y\rangle_4\times\mathbb{R}$	$\langle z\rangle_2\times\langle z'\rangle_\infty$, $z=y^2$, $z'=xy$, $x\in\mathbb{R}$
$2\nmid n$	$\mathrm{D}_{n-1}^{\mathbb{R},0}\times\mathbb{R}$	$\langle y\rangle_2\times\langle y'\rangle_2\times\mathbb{R}$	$\langle z\rangle_2\times\langle z'\rangle_\infty$, $z=yy'$, $z'=xy$, $x\in\mathbb{R}$
$i=4k$ $2\mid n$	$\mathrm{D}_{2k}^{\mathbb{R},0}\times\mathrm{D}_{n-2k}^{\mathbb{R},0}$	$\langle y\rangle_2\times\langle y'\rangle_2\times$ $\langle y'''\rangle_2\times\langle y''''\rangle_2$	$\langle z\rangle_2\times\langle z'\rangle_2\times\langle z''\rangle_2$, $z=yy'$, $z'=y'''y''''$, $z''=yy'''$
$2\nmid n$	$\mathrm{D}_{2k}^{\mathbb{R},0}\times\mathrm{D}_{n-2k}^{\mathbb{R},0}$	$\langle y\rangle_2\times\langle y'\rangle_2\times\langle y''\rangle_4$	$\langle z\rangle_2\times\langle z'\rangle_4$, $z=yy'$, $z'=yy''$

Table 1. The embedding of $\mathrm{Z}(\widetilde{G})$ in $\mathrm{Z}(\widetilde{K})$, continued.

	type of $\widetilde{K}$	$\mathrm{Z}(\widetilde{K})$	$\mathrm{Z}(\widetilde{G})$
type $\mathbf{D}_n^{\mathbb{R},i}$			
$i = 4k+2$			
$2 \mid n$	$\mathrm{D}^{\mathbb{R},0}_{2k+1} \times$ $\mathrm{D}^{\mathbb{R},0}_{n-2k-1}$	$\langle y\rangle_4 \times$ $\langle y'\rangle_4$	$\langle z\rangle_2 \times \langle z'\rangle_4$, $z = {y'}^2$, $z' = yy'$
$2 \nmid n$	$\mathrm{D}^{\mathbb{R},0}_{2k+1} \times$ $\mathrm{D}^{\mathbb{R},0}_{n-2k-1}$	$\langle y\rangle_4 \times$ $\langle y'\rangle_2 \times \langle y''\rangle_2$	$\langle z\rangle_2 \times \langle z'\rangle_4$, $z = y'y''$, $z' = yy'$
$i = 2k+1$	$\mathrm{B}^{\mathbb{R},0}_{k} \times \mathrm{B}^{\mathbb{R},0}_{n-k-1}$	$\langle y\rangle_2 \times \langle y'\rangle_2$	$\langle z\rangle_2 \times \langle z'\rangle_2$, $z = y$, $z' = y'$
type $\mathbf{D}_n^{\mathbb{H}}$			
$n = 2l$	$\mathrm{A}^{\mathbb{C},0}_{n-1} \times \mathbb{R}$	$\langle y\rangle_n \times \mathbb{R}$	$\langle z\rangle_2 \times \langle z'\rangle_\infty$, $z = y^l$, $z' = xy$, $x \in \mathbb{R}$
$n = 2l+1$	$\mathrm{A}^{\mathbb{C},0}_{n-1} \times \mathbb{R}$	$\langle y\rangle_n \times \mathbb{R}$	$\langle z\rangle_\infty$, $z = xy$, $x \in \mathbb{R}$
exceptional types			
$\mathrm{E}_{6(-78)}$	$\mathrm{E}_{6(-78)}$	$\langle y\rangle_3$	$\langle z\rangle_3$, $z = y$
$\mathrm{E}_{6(-26)}$	$\mathrm{F}_{4(-52)}$	$\mathbb{1}$	$\mathbb{1}$
$\mathrm{E}_{6(-14)}$	$\mathrm{D}^{\mathbb{R},0}_{5} \times \mathbb{R}$	$\langle y\rangle_4 \times \mathbb{R}$	$\langle z\rangle_\infty$, $z = xy$, $x \in \mathbb{R}$
$\mathrm{E}_{6(2)}$	$\mathrm{A}^{\mathbb{C},0}_{5} \times \mathrm{A}^{\mathbb{C},0}_{1}$	$\langle y\rangle_6 \times \langle y'\rangle_2$	$\langle z\rangle_6$, $z = yy'$
$\mathrm{E}_{6(6)}$	$\mathrm{C}^{\mathbb{H},0}_{4}$	$\langle y\rangle_2$	$\langle z\rangle_2$, $z = y$
$\mathrm{E}_{7(-133)}$	$\mathrm{E}_{7(-133)}$	$\langle y\rangle_2$	$\langle z\rangle_2$, $z = y$
$\mathrm{E}_{7(-25)}$	$\mathrm{E}_{6(-78)} \times \mathbb{R}$	$\langle y\rangle_3 \times \mathbb{R}$	$\langle z\rangle_\infty$, $z = xy$, $x \in \mathbb{R}$
$\mathrm{E}_{7(-5)}$	$\mathrm{D}^{\mathbb{R},0}_{6} \times \mathrm{A}^{\mathbb{C},0}_{1}$	$\langle y\rangle_2 \times \langle y'\rangle_2 \times \langle y''\rangle_2$	$\langle z\rangle_2 \times \langle z'\rangle_2$, $z = yy'$, $z' = y''$
$\mathrm{E}_{7(7)}$	$\mathrm{A}^{\mathbb{C},0}_{7}$	$\langle y\rangle_8$	$\langle z\rangle_4$, $z = y^2$
$\mathrm{E}_{8(-248)}$	$\mathrm{E}_{8(-248)}$	$\mathbb{1}$	$\mathbb{1}$
$\mathrm{E}_{8(-24)}$	$\mathrm{E}_{7(-133)} \times \mathrm{A}^{\mathbb{C},0}_{1}$	$\langle y\rangle_2 \times \langle y'\rangle_2$	$\langle z\rangle_2$, $z = yy'$
$\mathrm{E}_{8(8)}$	$\mathrm{D}^{\mathbb{R},0}_{8}$	$\langle y\rangle_2 \times \langle y'\rangle_2$	$\langle z\rangle_2$, $z = y$
$\mathrm{F}_{4(-52)}$	$\mathrm{F}_{2(-52)}$	$\mathbb{1}$	$\mathbb{1}$
$\mathrm{F}_{4(-20)}$	$\mathrm{B}^{\mathbb{R},0}_{4}$	$\langle y\rangle_2$	$\mathbb{1}$
$\mathrm{F}_{4(4)}$	$\mathrm{C}^{\mathbb{H},0}_{3} \times \mathrm{A}^{\mathbb{C},0}_{1}$	$\langle y\rangle_2 \times \langle y'\rangle_2$	$\langle z\rangle_2$, $z = yy'$
$\mathrm{G}_{2(-14)}$	$\mathrm{G}_{2(-14)}$	$\mathbb{1}$	$\mathbb{1}$
$\mathrm{G}_{2(2)}$	$\mathrm{A}^{\mathbb{C},0}_{1} \times \mathrm{A}^{\mathbb{C},0}_{1}$	$\langle y\rangle_2 \times \langle y'\rangle_2$	$\langle z\rangle_2$, $z = yy'$

Table 2. Description of $\mathsf{I}|_{\mathrm{Z}(\widetilde{G})}$ and linearizer.

	$\mathsf{I}\|_{\mathrm{Z}(\widetilde{G})}$	linearizer
type $\mathbf{A}_n^{\mathbb{R}}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x\rangle_{n+1}$	
$n=1$	$z \mapsto x$	$\langle z^2\rangle$
$n=2l$	$z \mapsto 1$	Z
$n=4l+1$	$z \mapsto x^{2l+1}$	$\langle z^2\rangle$
$n=4l+3$	$z \mapsto x^{2l+2},\ z' \mapsto x^{2l+2}$	$\langle zz'\rangle$
type $\mathbf{A}_n^{\mathbb{C},i}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x\rangle_{n+1}$	
$i=0$	$z \mapsto x$	$\mathbb{1}$
$i>0$	$z \mapsto x^{(n+1)/g},\ z' \mapsto x^b$	$\langle z^{g-b} z'^{(n+1)/g}\rangle$
	$g = \gcd(n+1,i) = (n+1)a+ib$	
type $\mathbf{A}_n^{\mathbb{H}}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x\rangle_{n+1}$	
$n=2l+1$	$z \mapsto x^{l+1}$	$\mathbb{1}$
type $\mathbf{B}_n^{\mathbb{R},i}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x\rangle_2$	
$i \le 1$	$z \mapsto x$	$\mathbb{1}$
$i \ge 2$	$z \mapsto x,\ z' \mapsto x$	$\langle zz'\rangle$
type $\mathbf{C}_n^{\mathbb{R}}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x\rangle_2$	
$2\|n$	$z \mapsto x,\ z' \mapsto 1$	$\langle z'\rangle$
$2\nmid n$	$z \mapsto x$	$\langle z^2\rangle$
type $\mathbf{C}_n^{\mathbb{H},i}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x\rangle_2$	
	$z \mapsto x$	$\mathbb{1}$
type $\mathbf{D}_n^{\mathbb{R},i}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x\rangle_2 \times \langle x'\rangle_2,$ if $2\|n$	
	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x\rangle_4,$ if $2\nmid n$	
$i=0$		
$2\|n$	$z \mapsto x,\ z' \mapsto x'$	$\mathbb{1}$
$2\nmid n$	$z \mapsto x$	$\mathbb{1}$
$i=1$		
$2\|n$	$z \mapsto xx'$	$\mathbb{1}$
$2\nmid n$	$z \mapsto x^2$	$\mathbb{1}$
$i=2$		
$2\|n$	$z \mapsto xx',\ z' \mapsto x$	$\langle z'^2\rangle$
$2\nmid n$	$z \mapsto x^2,\ z' \mapsto x$	$\langle zz'^2\rangle$
$i=4k$		
$2\|n$	$z \mapsto xx',\ z' \mapsto xx',\ z'' \mapsto x$	$\langle zz'\rangle$
$2\nmid n$	$z \mapsto x^2,\ z' \mapsto x$	$\langle zz'^2\rangle$
$i=4k+2$		
$2\|n$	$z \mapsto xx',\ z' \mapsto x$	$\langle z'^2\rangle$
$2\nmid n$	$z \mapsto x^2,\ z' \mapsto x$	$\langle zz'^2\rangle$

Table 2. Description of $\iota|_{\mathrm{Z}(\widetilde{G})}$ and linearizer, continued.

	$\iota\|_{\mathrm{Z}(\widetilde{G})}$	linearizer
type $\mathbf{D}_n^{\mathbb{R},i}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x \rangle_2 \times \langle x' \rangle_2$, if $2\|n$ $\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x \rangle_4$, if $2\nmid n$	
$i = 2k+1$		
$2\|n$	$z \mapsto xx'$, $z' \mapsto xx'$	$\langle zz' \rangle$
$2\nmid n$	$z \mapsto x^2$, $z' \mapsto x^2$	$\langle zz' \rangle$
type $\mathbf{D}_n^{\mathbb{H}}$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x \rangle_2 \times \langle x' \rangle_2$, if $2\|n$ $\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x \rangle_4$, if $2\nmid n$	
$2\|n$	$z \mapsto x$, $z' \mapsto xx'$	$\langle z'^2 \rangle$
$2\nmid n$	$z \mapsto x$	$\langle z^4 \rangle$
type $\mathbf{E}_6$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x \rangle_3$	
$\mathrm{E}_{6(-78)}$	$z \mapsto x$	$\mathbb{1}$
$\mathrm{E}_{6(-26)}$	—	$\mathbb{1}$
$\mathrm{E}_{6(-14)}$	$z \mapsto x$	$\langle z^3 \rangle$
$\mathrm{E}_{6(2)}$	$z \mapsto x$	$\langle z^3 \rangle$
$\mathrm{E}_{6(6)}$	$z \mapsto 1$	Z
type $\mathbf{E}_7$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \langle x \rangle_2$	
$\mathrm{E}_{7(-133)}$	$z \mapsto x$	$\mathbb{1}$
$\mathrm{E}_{7(-25)}$	$z \mapsto x$	$\langle z^2 \rangle$
$\mathrm{E}_{7(-5)}$	$z \mapsto 1$, $z' \mapsto x$	$\langle z \rangle$
$\mathrm{E}_{7(7)}$	$z \mapsto x$	$\langle z^2 \rangle$
type $\mathbf{E}_8$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \mathbb{1}$	
$\mathrm{E}_{8(-248)}$	—	$\mathbb{1}$
$\mathrm{E}_{8(-24)}$	$z \mapsto 1$	Z
$\mathrm{E}_{8(8)}$	$z \mapsto 1$	Z
type $\mathbf{F}_4$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \mathbb{1}$	
$\mathrm{F}_{4(-52)}$	—	$\mathbb{1}$
$\mathrm{F}_{4(-20)}$	—	$\mathbb{1}$
$\mathrm{F}_{4(4)}$	$z \mapsto 1$	Z
type $\mathbf{G}_2$	$\mathrm{Z}(\widetilde{G_{\mathbb{C}}}) = \mathbb{1}$	
$\mathrm{G}_{2(-14)}$	$z \mapsto 1$	Z
$\mathrm{G}_{2(2)}$	—	$\mathbb{1}$

Case II) $\mathfrak{g}$ is the realification of a simple complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$.
The complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_{\mathbb{C}} \oplus \overline{\mathfrak{g}_{\mathbb{C}}}$ via the mapping $g \otimes c \mapsto (g, 0)$, $g \otimes \overline{c} \mapsto (0, g)$, where $c = \frac{1}{2}(1+i)$. The embedding ι of $\mathfrak{g}$ into $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is now given by $\iota : g \mapsto g \otimes 1 = g \otimes c + g \otimes \overline{c} \mapsto (g, g) : \mathfrak{g} \to \mathfrak{g}_{\mathbb{C}} \oplus \overline{\mathfrak{g}_{\mathbb{C}}}$. Every real representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is induced by a pair of irreducible complex representations $\rho_1, \rho_2 : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V_{\mathbb{C}})$. The representation (ρ_1, ρ_2) of $\mathfrak{g}_{\mathbb{C}} \oplus \overline{\mathfrak{g}_{\mathbb{C}}}$ is equivalent to the representation (ρ_1, ρ_2^*) of $\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$, where ρ_2^* is the contragradient representation of ρ_2, cp. [5], §11 and §12. Every complex representation $\tau : \mathfrak{g} \to \mathfrak{gl}(V)$ arises as the pair $(\tau, 0)$, which is equivalent to $(0, \tau^*)$. The integrated map $\mathsf{I} : \widetilde{G} \to \widetilde{G}_{\mathbb{C}} \times \widetilde{G}_{\mathbb{C}}$ is injective and the image of I is diagonal in $\widetilde{G}_{\mathbb{C}} \times \widetilde{G}_{\mathbb{C}}$. Thus the I-image of $\mathrm{Z}(\widetilde{G})$ is obvious and therefore we have omitted tables for this case.

The contragradient representation of the fundamental representation ρ_j is always ρ_j, except for the following cases: If $\mathfrak{g}_{\mathbb{C}}$ is of type A_n, then $\rho_j^* = \rho_{n+1-j}$. If $\mathfrak{g}_{\mathbb{C}}$ is of type D_n, then $\rho_{n-1}^* = \rho_n$ and $\rho_n^* = \rho_{n-1}$ for n odd. If $\mathfrak{g}_{\mathbb{C}}$ is of type E_6, then $\rho_1^* = \rho_6$, $\rho_6^* = \rho_1$, $\rho_3^* = \rho_5$, $\rho_5^* = \rho_3$.

Extend one (of the mutually conjugate [2], 20.9(ii)) maximal $\mathbb{R}$-split toral subalgebras of $\mathfrak{g}$ to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Then $\mathfrak{t}_{\mathbb{C}} := \mathfrak{t} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Now complex conjugation acts on $\mathfrak{g}_{\mathbb{C}}$ via $\sigma : g \otimes c \mapsto g \otimes \overline{c}$ and thus also on $\Lambda_+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, see [12], 3.1. This action is explicitly given in Table 3 for case I. Note that the action of $\langle \sigma \rangle$ on $\Lambda_+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ may be different for $\mathfrak{h}$ not conjugate to $\mathfrak{t}$ in $\mathfrak{g}$. For case II the action is simply the exchange of the two components. We have ([5], [9], p. 290/291, [12], 7.2, 8.2, or [4], section 55)

Theorem 2. *The orbits of $\langle \sigma \rangle$ on $\Lambda_+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and the equivalence classes of finite-dimensional irreducible real representations of $\mathfrak{g}$ are in a one-to-one correspondence.*

Centralizers and real dimensions

Let $\mathfrak{g}$ be a simple real Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ an irreducible real representation. Then the centralizer $\mathrm{Cs}(\rho)$ of the image $\mathfrak{g}^\rho$ in $\mathfrak{gl}(V)$ is a skew field by Schur's lemma. Moreover, $\mathrm{Cs}(\rho)$ obviously contains $\mathbb{R} = \mathbb{R} \cdot \mathrm{id}_V$ in its center. Thus $\mathrm{Cs}(\rho) \cong \mathbb{R}, \mathbb{C}, \mathbb{H}$ by a result of Frobenius. Let λ be the highest weight of $\rho_{\mathbb{C}}$. We consider the subsemigroup $\Sigma_+(\mathfrak{g})$ of $\Lambda_+(\mathfrak{g}_{\mathbb{C}})$ of those weights that are fixed by σ. By §4 and §9 of [5] there exists an additive mapping $\alpha_{\mathfrak{g}} : \Sigma_+(\mathfrak{g}) \to \mathbb{Z}/2\mathbb{Z}$, the *index* of $\mathfrak{g}$, such that either

(1) $\lambda \in \Sigma_+(\mathfrak{g})$, $\alpha_{\mathfrak{g}}(\lambda) = 0$, $\mathrm{Cs}(\rho) \cong \mathbb{R}$, $\dim_{\mathbb{R}}(\rho) = \dim_{\mathbb{C}}(\rho_{\mathbb{C}})$, and $\rho_{\mathbb{C}}$ is irreducible,

or (2) $\lambda \in \Sigma_+(\mathfrak{g})$, $\alpha_{\mathfrak{g}}(\lambda) = 1$, $\mathrm{Cs}(\rho) \cong \mathbb{H}$, $\dim_{\mathbb{R}}(\rho) = 2\dim_{\mathbb{C}}(\rho_{\mathbb{C}})$, and $\rho_{\mathbb{C}}$ is the direct sum of two equivalent irreducible complex representations corresponding to λ,

or (3) $\lambda \notin \Sigma_+(\mathfrak{g})$, $\mathrm{Cs}(\rho) \cong \mathbb{C}$ and $\dim_{\mathbb{R}}(\rho) = 2\dim_{\mathbb{C}}(\rho_{\mathbb{C}})$, and $\rho_{\mathbb{C}}$ is the direct sum of two non-equivalent irreducible complex representations corresponding to λ and λ^σ.

Note that the semigroup $\Sigma_+(\mathfrak{g})$ is generated by the set $\{\lambda_i, \lambda_j + \lambda_j^\sigma \mid 1 \leq i, j \leq \operatorname{rank} \mathfrak{g}_\mathbb{C}, \lambda_i^\sigma = \lambda_i\}$. Since $\alpha_\mathfrak{g}(\mu + \mu^\sigma) = 0$, see [5], Lemma 11, we can extend the definition of $\alpha_\mathfrak{g}$ to $\Lambda_+(\mathfrak{g}_\mathbb{C})$ by setting $\alpha \equiv 0$ on $\Lambda_+(\mathfrak{g}_\mathbb{C}) \setminus \Sigma_+(\mathfrak{g})$. Thus, the mapping $\alpha_\mathfrak{g}$ is uniquely determined by the images of the fundamental weights λ_i. Table 3 lists the index for the real forms, whereas for case II we have $\alpha_\mathfrak{g} \equiv 0$ on the whole weightspace $\Lambda_+(\mathfrak{g}_\mathbb{C})$. The canonical projection of $\mathbb{Z}$ to $\mathbb{Z}/_{2\mathbb{Z}}$ is denoted by par.

Table 3. The action of σ on $\Lambda_+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and the index $\alpha_\mathfrak{g}$.

type	action of σ on $\Lambda_+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$	$\alpha_\mathfrak{g}$
$\mathrm{A}_n^\mathbb{R}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto 0$
$\mathrm{A}_n^{\mathbb{C},i}$	$\lambda_j^\sigma = \lambda_{n+1-j}$	$\lambda_j \mapsto \begin{cases} \operatorname{par}(\frac{n+1}{2} - i) & \text{if } j = \frac{(n+1)}{2} \\ 0 & \text{otherwise} \end{cases}$
$\mathrm{A}_n^\mathbb{H}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto \operatorname{par}(j)$
$\mathrm{B}_n^{\mathbb{R},i}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto \begin{cases} 0 & \text{if } j<n \text{ or } n-i \not\equiv 1,2\,(4) \\ 1 & \text{otherwise} \end{cases}$
$\mathrm{C}_n^\mathbb{R}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto 0$
$\mathrm{C}_n^{\mathbb{H},i}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto \operatorname{par}(j)$
$\mathrm{D}_n^{\mathbb{R},i}$	$\lambda_j^\sigma = \lambda_j$ $(j<n-1$ or $2 \mid n-i)$, $\lambda_{n-1}^\sigma = \lambda_n, \lambda_n^\sigma = \lambda_{n-1} (2 \nmid n-i)$	$\lambda_j \mapsto \begin{cases} 0 & \text{if } j<n-1 \text{ or } 4 \mid n-i \\ 1 & \text{otherwise} \end{cases}$
$\mathrm{D}_n^\mathbb{H}$	$\lambda_j^\sigma = \lambda_j$ $(j<n-1$ or $2 \mid n)$, $\lambda_{n-1}^\sigma = \lambda_n, \lambda_n^\sigma = \lambda_{n-1} (2 \nmid n-i)$	$\lambda_j \mapsto \begin{cases} \operatorname{par}(j) & \text{if } j<n-1 \\ 0 & \text{if } j=n-1 \text{ and } 2 \mid n \\ 1 & \text{otherwise} \end{cases}$
$\mathrm{E}_{6(-78)}$	$\lambda_1^\sigma = \lambda_6, \lambda_2^\sigma = \lambda_2, \lambda_3^\sigma = \lambda_5$ $\lambda_4^\sigma = \lambda_4, \lambda_5^\sigma = \lambda_3, \lambda_6^\sigma = \lambda_1$	$\lambda_j \mapsto \begin{cases} 0 & \text{if } j=2,4 \\ 1 & \text{otherwise} \end{cases}$
$\mathrm{E}_{6(-26)}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto 0$
$\mathrm{E}_{6(-14)}$	see $\mathrm{E}_{6,-78}$	see $\mathrm{E}_{6,-78}$
$\mathrm{E}_{6(2)}$	see $\mathrm{E}_{6,-78}$	see $\mathrm{E}_{6,-78}$
$\mathrm{E}_{6(6)}$	see $\mathrm{E}_{6,-26}$	see $\mathrm{E}_{6,-26}$
$\mathrm{E}_{7(-133)}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto \begin{cases} 0 & \text{if } j=1,3,4,6 \\ 1 & \text{otherwise} \end{cases}$
$\mathrm{E}_{7(-25)}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto 0$
$\mathrm{E}_{7(-5)}$	see $\mathrm{E}_{7,-133}$	see $\mathrm{E}_{7,-133}$
$\mathrm{E}_{7(7)}$	see $\mathrm{E}_{7,-25}$	see $\mathrm{E}_{7,-25}$
$\mathrm{E}_{8(i)}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto 0$
$\mathrm{F}_{4(i)}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto 0$
$\mathrm{G}_{2(i)}$	$\lambda_j^\sigma = \lambda_j$	$\lambda_j \mapsto 0$

Kernels of representations

Let $\mathfrak{g}$ be a simple real Lie algebra and let $\mathfrak{g}_{\mathbb{C}}$ be its complexification, with corresponding simply connected groups $\widetilde{G}$ and $\widetilde{G}_{\mathbb{C}}$, resp. Consider a non-trivial irreducible representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ with highest weight λ.

As $\mathfrak{g}$ is simple the representation ρ is faithful and $\ker \mathsf{P} = \ker(\mathsf{IP}_{\mathbb{C}})$ is discrete and therefore central in the group $\widetilde{G}$. Of course $\mathrm{Z}(\widetilde{G})$ is mapped to $\mathrm{Z}(\mathrm{GL}(V_{\mathbb{C}}))$ by P. Note that $\mathrm{Z}(\widetilde{G})^{\mathsf{P}}$ is a cyclic group of roots of unity. Define a group homomorphism $\varepsilon_{\mathsf{P}} : \mathrm{Z}(\widetilde{G}) \to \mathbb{C}^{\times}$ by $\varepsilon_{\mathsf{P}}(z) \cdot \mathrm{id}_{V_{\mathbb{C}}} = z^{\mathsf{P}}$. Then ε_{P} can be written as

$$(*) \qquad \varepsilon_{\mathsf{P}}(z) = \prod_{i=1}^{r} \varepsilon_{\mathsf{P}_i}(z)^{f_i(\lambda)}.$$

Obviously, $\ker \mathsf{P} = \ker \varepsilon_{\mathsf{P}}$. Together with the information on $\mathsf{I}|_{\mathrm{Z}(\widetilde{G})}$ already given above this is enough to compute the kernel of P if one knows the kernels of all fundamental representations of $\widetilde{G}_{\mathbb{C}}$. This is contained in Table 4 below, where ξ_k denotes a kth primitive root of unity. The group $\widetilde{G}/_{\ker \mathsf{P}}$ is called the group which is *represented* by P.

Table 4. The images of the center $\mathrm{Z}(\widetilde{G}_{\mathbb{C}})$ under the fundamental representations P_j of $\widetilde{G}_{\mathbb{C}}$

type	$\mathrm{Z}(\widetilde{G}_{\mathbb{C}})$	$\mathsf{P}_j\big\vert_{\mathrm{Z}(\widetilde{G}_{\mathbb{C}})}$
A_n	$\langle x \rangle_{n+1}$	$x \mapsto \xi_{n+1}^{j}$
B_n	$\langle x \rangle_2$	$x \mapsto \begin{cases} 1 & \text{if } j < n \\ -1 & \text{if } j = n \end{cases}$
C_n	$\langle x \rangle_2$	$x \mapsto (-1)^j$
D_{2l}	$\langle x \rangle_2 \times \langle x' \rangle_2$	$x \mapsto \begin{cases} (-1)^j & \text{if } j < 2l-1 \\ 1 & \text{if } j = 2l-1 \\ -1 & \text{if } j = 2l \end{cases}$
		$x' \mapsto \begin{cases} (-1)^j & \text{if } j < 2l-1 \\ -1 & \text{if } j = 2l-1 \\ 1 & \text{if } j = 2l \end{cases}$
D_{2l+1}	$\langle x \rangle_4$	$x \mapsto \begin{cases} (-1)^j & \text{if } j < 2l \\ \sqrt{-1} & \text{if } j = 2l \\ -\sqrt{-1} & \text{if } j = 2l+1 \end{cases}$
E_6	$\langle x \rangle_3$	$x \mapsto \begin{cases} \xi_3 & \text{if } j = 1,5 \\ \xi_3^2 & \text{if } j = 3,6 \\ 1 & \text{if } j = 2,4 \end{cases}$
E_7	$\langle x \rangle_2$	$x \mapsto \begin{cases} -1 & \text{if } j = 2,5,7 \\ 1 & \text{if } j = 1,3,4,6 \end{cases}$
E_8	$\mathbb{1}$	—
F_4	$\mathbb{1}$	—
G_2	$\mathbb{1}$	—

About the implementation

Our initial motivation was to compute a list of the equivalence classes of all irreducible real representations of a given simple real Lie algebra $\mathfrak{g}$ up to a certain dimension. We found this to be not only tedious but also very fault prone to do by hand.

Obviously one can establish a (lexicographic) ordering on the elements of the weight space $\Lambda_+(\mathfrak{g}_\mathbb{C})$. Moreover the dimension function (Weyl formula) is increasing if the multiplicities of all roots in the weight space but one are fixed. This ensures that there is only a finite number of equivalence classes of irreducible representations for $\mathfrak{g}$ up to a given dimension. Furthermore these representations can be enumerated in that way.

So what is left is to deal with a fixed representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Again let λ be the highest weight of $\rho_\mathbb{C}$ and let $\widetilde{G}$ denote the simply connected group associated with $\mathfrak{g}$.

(1) Determine $\dim_\mathbb{C} \rho_\mathbb{C}$ via the Weyl formula.
(2) Derive $\rho_\mathbb{C}^\sigma$ and the index $\alpha_\mathfrak{g}(\lambda)$ from Table 3. This gives the centralizer $\mathrm{Cs}(\rho)$ and, together with (1), $\dim_\mathbb{R} \rho$.
(3) Compute $\ker \mathsf{P} = \ker \varepsilon_\mathsf{P}$ as follows: Table 4 gives the images of each element of the generating system S for $\mathrm{Z}(\widetilde{G})$ (as given in table 1) under all fundamental representations P_i. Substitute these data and λ in formula $(*)$ to obtain the images x^P for each $x \in S$. Using the Chinese-Remainder-Algorithm one obtains a generating system for $\ker \varepsilon_\mathsf{P}$. By some reduction strategy the result can be transformed into a canonical form such that the name for the represented group $\widetilde{G}/_{\ker \mathsf{P}}$ can be looked up in a list.

The first implementation of the algorithm sketched above has been done in the LiE language [7]. To give the reader an idea of the complexity: this LiE based implementation takes 5 minutes, approximately, on a 486 IBM computer (50 MHz) to determine all irreducible representations of all simple real Lie algebras up to dimension 16. We have to mention that it was not possible to run our present LiE based program with the original LiE 2.0 MS–DOS version. This is mainly due to the fact that this particular LiE version used short integers for its memory management, which did not meet our requirements. However, since the sources were included we were able to do a recompilation of the (slightly modified) source code with the GNU C compiler. A significantly faster ANSI C based version of our representation package is in preparation. This implementation will also be more portable, since it will be independent from LiE. The output (of both versions) is an ASCII file that contains all information computed. Because this file uses some crude format, we have written a TeX back end that transforms the plain ASCII file into a TeX file which produces tables that look like the ones listed in the next section.

Example Tables

$\mathbf{A}_3^{\mathbb{C},1}$ — $\mathfrak{su}_4\mathbb{C}(1)$ — real dimension 15

Real representations from dimension 2 to dimension 300.
Center Z of the universal covering : $Z = \langle z \rangle \cong \mathbb{Z}$
The linearizer is $\langle z^4 \rangle$.
By $\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$ we denote the antiunitary group on $\mathbb{H}^3$.

dimension	centralizer	dominant weight	kernel	represented group
8	$\mathbb{C}$	λ_1	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
12	$\mathbb{H}$	λ_2	$\langle z^2 \rangle$	$\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$
15	$\mathbb{R}$	$\lambda_1 + \lambda_3$	Z	$\mathrm{PSU}_4\mathbb{C}$ (1)
20	$\mathbb{R}$	$2\lambda_2$	Z	$\mathrm{PSU}_4\mathbb{C}$ (1)
20	$\mathbb{C}$	$2\lambda_1$	$\langle z^2 \rangle$	$\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$
40	$\mathbb{C}$	$\lambda_1 + \lambda_2$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
40	$\mathbb{C}$	$3\lambda_1$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
70	$\mathbb{C}$	$4\lambda_1$	Z	$\mathrm{PSU}_4\mathbb{C}$ (1)
72	$\mathbb{C}$	$2\lambda_1 + \lambda_3$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
84	$\mathbb{R}$	$2\lambda_1 + 2\lambda_3$	Z	$\mathrm{PSU}_4\mathbb{C}$ (1)
90	$\mathbb{C}$	$2\lambda_1 + \lambda_2$	Z	$\mathrm{PSU}_4\mathbb{C}$ (1)
100	$\mathbb{H}$	$3\lambda_2$	$\langle z^2 \rangle$	$\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$
105	$\mathbb{R}$	$4\lambda_2$	Z	$\mathrm{PSU}_4\mathbb{C}$ (1)
112	$\mathbb{C}$	$5\lambda_1$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
120	$\mathbb{C}$	$\lambda_1 + 2\lambda_2$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
128	$\mathbb{H}$	$\lambda_1 + \lambda_2 + \lambda_3$	$\langle z^2 \rangle$	$\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$
140	$\mathbb{C}$	$3\lambda_1 + \lambda_3$	$\langle z^2 \rangle$	$\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$
168	$\mathbb{C}$	$3\lambda_1 + \lambda_2$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
168	$\mathbb{C}$	$6\lambda_1$	$\langle z^2 \rangle$	$\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$
175	$\mathbb{R}$	$\lambda_1 + 2\lambda_2 + \lambda_3$	Z	$\mathrm{PSU}_4\mathbb{C}$ (1)
240	$\mathbb{C}$	$4\lambda_1 + \lambda_3$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
240	$\mathbb{C}$	$7\lambda_1$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
252	$\mathbb{C}$	$2\lambda_1 + 2\lambda_2$	$\langle z^2 \rangle$	$\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$
280	$\mathbb{C}$	$\lambda_1 + 3\lambda_2$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
280	$\mathbb{C}$	$2\lambda_1 + \lambda_2 + \lambda_3$	$\langle z^4 \rangle$	$\mathrm{SU}_4\mathbb{C}$ (1)
280	$\mathbb{C}$	$4\lambda_1 + \lambda_2$	$\langle z^2 \rangle$	$\mathrm{S}\alpha\mathrm{U}_3\mathbb{H}$
300	$\mathbb{R}$	$3\lambda_1 + 3\lambda_3$	Z	$\mathrm{PSU}_4\mathbb{C}$ (1)

$\mathbf{D}_4$ — $\mathfrak{o}_8\mathbb{C}$ — real dimension 56

Real representations from dimension 2 to dimension 100.
Center Z of the universal covering : $Z = \langle z, z' \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
The linearizer is 1.

dimension	centralizer	dominant weight	kernel	represented group
16	$\mathbb{C}$	$(\lambda_4, 0)$	$\langle z \rangle$	$\mathrm{SO}_8\mathbb{C}$
16	$\mathbb{C}$	$(\lambda_3, 0)$	$\langle z' \rangle$	$\mathrm{SO}_8\mathbb{C}$
16	$\mathbb{C}$	$(\lambda_1, 0)$	$\langle zz' \rangle$	$\mathrm{SO}_8\mathbb{C}$
56	$\mathbb{C}$	$(\lambda_2, 0)$	Z	$\mathrm{PSO}_8\mathbb{C}$
64	$\mathbb{R}$	(λ_4, λ_4^*)	Z	$\mathrm{PSO}_8\mathbb{C}$
64	$\mathbb{R}$	(λ_3, λ_3^*)	Z	$\mathrm{PSO}_8\mathbb{C}$
64	$\mathbb{R}$	(λ_1, λ_1^*)	Z	$\mathrm{PSO}_8\mathbb{C}$
70	$\mathbb{C}$	$(2\lambda_4, 0)$	Z	$\mathrm{PSO}_8\mathbb{C}$
70	$\mathbb{C}$	$(2\lambda_3, 0)$	Z	$\mathrm{PSO}_8\mathbb{C}$
70	$\mathbb{C}$	$(2\lambda_1, 0)$	Z	$\mathrm{PSO}_8\mathbb{C}$

$\mathbf{E}^{\mathbb{R}}_{6(-78)}$ — real dimension 78

Real representations from dimension 2 to dimension 35000.
Center Z of the universal covering : $Z = \langle z \rangle \cong \mathbb{Z}_3$
The linearizer is 1.

dimension	centralizer	dominant weight	kernel
54	$\mathbb{C}$	λ_1	1
78	$\mathbb{R}$	λ_2	Z
650	$\mathbb{R}$	$\lambda_1 + \lambda_6$	Z
702	$\mathbb{C}$	λ_3	1
702	$\mathbb{C}$	$2\lambda_1$	1
2430	$\mathbb{R}$	$2\lambda_2$	Z
2925	$\mathbb{R}$	λ_4	Z
3456	$\mathbb{C}$	$\lambda_1 + \lambda_2$	1
6006	$\mathbb{C}$	$3\lambda_1$	Z
11648	$\mathbb{C}$	$\lambda_1 + \lambda_3$	Z
14742	$\mathbb{C}$	$\lambda_1 + \lambda_5$	1
15444	$\mathbb{C}$	$2\lambda_1 + \lambda_6$	1
34749	$\mathbb{R}$	$\lambda_1 + \lambda_2 + \lambda_6$	Z

References

[1] Bourbaki, N., "Groupes et algèbres de Lie, Chapitres IV – VI", Paris: Hermann, 1968.

[2] Borel, A., "Linear Algebraic Groups", Berlin–Heidelberg–New York: Springer, 2nd ed., 1991.

[3] Bremner, M. R., R. V. Moody, J. Patera, "Tables of dominant weight multiplicities for representations of simple Lie algebras", Monographs and Textbooks in Pure and Appl. Math. **40**, New York: Dekker, 1985.

[4] Freudenthal, H., H. de Vries, "Linear Lie Groups", New York–London: Academic Press, 1969.

[5] Iwahori, N., *On real irreducible representation of Lie algebra*, Nagoya Math. J. **14** (1959), 59–83.

[6] Jacobson, N., "Lie algebras", New York: Interscience, 1962.

[7] van Leeuwen, M. A. A., A. M. Cohen, B. Lisser, "LiE – A Package for Lie Group Computations", Amsterdam: Computer Algebra Nederland, 1992.

[8] McKay, W. G., J. Patera, "Tables of dimensions, indices and branching rules for representations of simple Lie algebras", Lect. Notes in Pure and Appl. Math. **69**, New York: Dekker, 1981.

[9] Onishchik, A. L., E. B. Vinberg, "Lie Groups and Algebraic Groups", Berlin–Heidelberg–New York: Springer, 1990.

[10] Samelson, H., "Notes on Lie Algebras", Berlin – Heidelberg – New York: Springer, 1990.

[11] Tits, J., "Tabellen zu den einfachen Liegruppen und ihren Darstellungen", Lecture Notes **40**, Berlin–Heidelberg–New York: Springer, 1967.

[12] Tits, J., *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. **247** (1971), 196–220.

[13] Varadarajan, V. S., "Lie groups, Lie algebras, and their representations", Englewood Cliffs (N.J.): Prentice-Hall; Berlin etc.: Springer, 1984.

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