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A Survey of Noncommutative Geometry Methods for Group Algebras

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Abstract

In this survey we shall report about a K-theoretic approach to study group algebras. Following the example of the group of affine transformations of the straight line, the method consists of: 1. Construction of irreducible group representations (orbit method, category $\mathcal{O}$), 2. Decomposition of the group algebra into a sequence of repeated extensions, and finally 3. Computation of the extension invariants by the methods from noncommutative geometry (KK-theory, cyclic theories).

1. Introduction

The main problem we are interested in is how to characterize groups and their group algebras. For finite and compact groups this problem is well solvable with complex representation theory. It seems to be quite difficult for locally compact groups. We focus our attention, in this survey only on the topological method of characterizing the group C*-algebras.

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2. The Scope and an Example

2.1. The Problem

Let us first of all consider a finite group G, $|G| < \infty$. It is easy to see that the group G can be included in some (co-)algebras, more precisely some Hopf bialgebra, for example in its complex group (Hopf bi-)algebra, $G \hookrightarrow \mathbb{C}[G]$,

$$g \in G \mapsto \sum_{g' \in G} \delta_g(g')g' \in \mathbb{C}[G],$$

which consists of the formal linear combinations of the form $\sum_{g\in G} c_g g$, and where δ_g is the Kronecker symbol corresponding to g. It is well-known that the group representation theory of G is equivalent to the algebra representation theory of $\mathbb{C}[G]$. The last one is practically more flexible to describe. With each representation π of this group one considers the Fourier-Gel'fand transform of the algebra $\mathbb{C}[G]$ to the matrix algebra, corresponding to the representation π,

$$\sum_{g\in G} c_g g \mapsto \sum_{g\in G} c_g \pi(g).$$

Let us denote $\hat{G}$ the dual of G, i.e. the set of equivalence classes of irreducible representations of G. We normally identify it with some set of representatives of equivalence classes. It is well-known that:

a) The set $\hat{G}$ is finite, i.e., there is only a finite number of nonequivalent irreducible representations, say $\pi_1, \dots, \pi_N$;

b) Each irreducible representation is finite dimensional, say of dimension $n_i, i = 1, 2, \dots, N$;

c) This algebra $\mathbb{C}[G]$, by using the Fourier-Gel'fand transform, is isomorphic to the finite Cartesian product of matrix algebras

$$\mathbb{C}[G] \cong \prod_{i=1}^{N} \mathrm{Mat}_{n_i}(\mathbb{C}).$$

This means that the structure of the group algebra $\mathbb{C}[G]$ and therefore of the group G is well defined, if we obtain for G:

1) a good construction of all the irreducible representations $\pi_i, i = 1, \dots, n$ of G and

2) the Fourier-Gel'fand transform, realizing the above cited isomorphism.

The problem is to extend this machinery to infinite, say locally compact, topological groups.

Let us from now on, consider a locally compact group G and some appropriate group algebras. The group algebra $\mathbb{C}[G]$ for G as an abstract group is not enough to define the structure of G. We must find a more effective group algebra.

It is well known that for any locally compact group G one must in place of the general linear representations consider the unitary ones. It is related to the fact that in the general case one must consider also the infinite dimensional representations, which are not always completely reducible. The unitary representations however are completely reducible. In the case of a locally compact group G there is a natural left-(right-)invariant Haar measure dg. The space $L^2(G) := L^2(G, dg)$ of the square-integrable functions plays an important role in harmonic analysis. If the group is of type I, $L^2(G)$ admits a spectral decomposition with respect to the left and right regular representations into a sum of the direct sum (the so called discrete series) and/or the direct integral (the continuous series) of irreducible unitary representations. The space $L^1(G) = L^1(G, dg)$ of the functions with integrable module plays a crucial role. With the well-defined convolution product,

$$\varphi, \psi \in L^1(G) \mapsto \varphi * \psi \in L^1(G);$$
$$(\varphi * \psi)(x) := \int_G \varphi(y)\psi(y^{-1}x)dy$$

it becomes a Banach algebra. There is also a well-defined Fourier-Gel'fand transformation on $L^1(G)$,

$$\varphi \in L^1(G, g) \mapsto \hat{\varphi},$$
$$\hat{\varphi}(\pi) := \pi(\varphi) = \int_G \pi(x)\varphi(x)dx.$$

If G is unimodular, one can also define an involution $\varphi \mapsto \varphi^*$ on $L^1(G)$ by

$$\varphi^*(g) := \overline{\varphi(g^{-1})}.$$

There is a one-to-one correspondence between the (irreducible) unitary representations of G and the nondegenerate (irreducible) *-representations of the involutive Banach algebra $L^1(G)$. The general theorems of the spectral theory of the representations of G are then proved with the help of an appropriate translation into the corresponding theory for $L^1(G)$, for which one can use more tools from functional analysis and topology. However the norm of the involutive Banach algebra $L^1(G)$ is nonregular, i.e. in general

$$\|a^*a\|_{L^1(G)} \neq \|a\|^2_{L^1(G)} .$$

It is therefore more useful to consider the corresponding regular norm $\|.\|_{C^*(G)}$,

$$\|\varphi\|_{C^*(G)} := \sup_{\pi \in \hat{G}} \|\pi(\varphi)\|$$

and its completion $C^*(G)$. The spectral theory of unitary representations of G is equivalent to the spectral theory of nondegenerate *-representations of the C*-algebra $C^*(G)$. The general theorems of harmonic analysis say that the structure of G can be completely defined by the structure of $C^*(G)$. One poses therefore the problem of the description of the structure of the C*-algebras of locally compact groups. This means that we must answer the questions:

1) How to realize the irreducible unitary representations of the locally compact group G?

2) How to describe the image of the Fourier-Gel'fand transform and in particular, of the inclusion of $C^*(G)$ into some "continuous" product of the algebras $\mathcal{L}(\mathcal{H}_\pi)$, $\pi \in \hat{G}$ of bounded operators in the separable Hilbert space $\mathcal{H}_\pi$ of the representation π.

To see that this is a good setting to extend the problem for finite group to the class of locally compact groups, let us consider these questions for the case of compact groups. Consider for the moment a compact group G. For a compact group all irreducible representations are unitarizable, i.e., equivalent to some unitary ones. It is also well-known that:

a) The set of equivalence classes of irreducible unitary representations is not more than countable.

b) Each irreducible unitary representation is finite dimensional, say of dimension $n_i, i = 1, \ldots, \infty$ and there is some good realization of these representations, say in tensor spaces, or last time, in cohomologies.

c) The Fourier-Gel'fand transformation gives us an isomorphism

$$C^*(G) \cong \prod_{i=1}^{\infty} \mathrm{Mat}_{n_i}(\mathbb{C}).$$

This means that in the compact group case the group C*-algebra plays the same role as the group algebra of a finite group.

Let us now return to the general case of a locally compact group.

The main problem is to describe the group algebra in general, and in particular the C-algebra $C^*(G)$.*

In general the problem of describing the structure of C*-algebras of noncompact groups rests open up-to-date. This review outlines only the well-known cases, where there is a nice interaction of the methods from noncommutative geometry, say Orbit Method, category $\mathcal{O}$, KK-theory, deformation quantization, cyclic theories,... We restrict our attention mainly to Lie groups.

2.1.1. Analytic Methods

The first nontrivial example is the group $\mathrm{SL}_2(\mathbb{C})$. Its C*-algebra was studied by J. M. G. FELL in 1961 in [F]. He described exactly the Fourier-Gel'fand transforms of $C^*(G)$ as some C*-algebra of sections of a continuous field of operator algebras over the dual. Many other mathematicians attemped to generalize his beautiful but complicate analytic result to other groups. Nevertheless, until the moment the only groups, the structure of whose C*-algebras were described explicitely are: the abelian or compact groups and a few semi-simple Lie groups, say $\mathrm{SL}_2(\mathbb{R})$ and its univeral covering $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, the de Sitter group $\mathrm{Spin}(4,1)$ and recently a family $G(p,q,\alpha)$ of two step solvable Lie groups (see [De], [F], [M], [KM], [BM] and [W1]). A fair amount is known about the C*-algebras of nilpotent Lie groups (see [P]), including the Heisenberg groups. The C*-algebra of the Euclidean motion group was studied by B. EVANS [E]. Also P. GREEN [Gr] proposed another analytic method for studying the C*-algebras of several solvable Lie groups. The results are given very slowly and spectacularly.

One therefore has to develop another method, say to obtain some topological invariants, which will be described in the rest of this paper.

2.1.2. The K-Theory Approach

The very useful K-functor for our approach is the operator KK-functor of G. G. KASPAROV [K1], generalizing the BDF K-functor [BDF], which characterizes the isomorphy classes of short exact sequences of C*-algebras.

We are trying to decompose our C*-algebras into some towers of ideals and step-by-step define the associated extensions by KK-functors or their generalizations. The resulting invariants form just our index. This idea was proposed in [D1] and developed in [D2] for a large class of type I C*-algebras. Hence, there are two general problems:

(1) Find the C*-algebras which can be characterized by the well-known K-functors, say by the operator K-functors.

(2) Generalize the theory of K-functors in such a way that they are applicable for a larger class of C*-algebras.

Concerning the first problem, we propose in [D8] a general construction and some reduction procedure of the K-theory invariant Index $C^*(G)$ of group C*-algebras. Using the orbit method [Ki], [D4] - [D7], we reduce Index $C^*(G)$ to a family of A. CONNES' foliation C*-algebra indices Index $C^*(V_{2n_i},\ \mathcal{F}_{2n_i})$, see [C1]-[C2], by a family of KK-theory invariants. Using some generalization of the Kasparov type condition (treated by G.G. KASPAROV in the nilpotent Lie group case [K2]), we reduce every Index $C^*(V_{2n_i}, \mathcal{F}_{2n_i})$ to a family of KK-theory invariants of the same type valuated in KK(X,Y) type groups. The last ones are in some sense computable by using the cup-cap product realizing the Fredholm operator indices.

To demonstrate the idea, we consider the C*-algebra of the group of affine transformations of the real straight line, but first of all we need some new K-functor tool. It is described in the next two subsections.

2.2. BDF K-Homology functor

Let us recall in this subsection the well-known BDF K-functor $\mathcal{E}xt$. The main reference is [BDF]. Denote by $C(X)$ the C*-algebra of continuous complex-valued functions over a fixed metrizable compact space X, $\mathcal{H}$ a fixed separable Hilbert space over the complex numbers, $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ the C*-algebras of bounded and respectively, compact linear operators in $\mathcal{H}$. An extension of C*-algebras means a short exact sequence of C*-algebras and *-homomorphisms of special type

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{E} \longrightarrow C(X) \longrightarrow 0.$$

Two extensions are by definition equivalent iff there exists an isomorphism $\psi : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ and its restriction $\psi|_{\mathcal{K}(\mathcal{H}_1)} : \mathcal{K}(\mathcal{H}_1) \longrightarrow \mathcal{K}(\mathcal{H}_2)$ such that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(\mathcal{H}_1) & \longrightarrow & \mathcal{E}_1 & \longrightarrow & C(X) & \longrightarrow & 0 \\ & & \downarrow \psi|. & & \downarrow \psi & & \| & & \\ 0 & \longrightarrow & \mathcal{K}(\mathcal{H}_2) & \longrightarrow & \mathcal{E}_2 & \longrightarrow & C(X) & \longrightarrow & 0 \end{array}$$

There is a canonical universal extension of C*-algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{A}(\mathcal{H}) \longrightarrow 0,$$

the quotient algebra $\mathcal{A}(\mathcal{H}) \cong \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is well-known as the Calkin algebra. By the construction of fiber products, there is one-to-one correspondence between the extensions of type

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{E} \longrightarrow C(X) \longrightarrow 0$$

and the unital monomorphisms of type

$$\varphi : C(X) \hookrightarrow \mathcal{A}(\mathcal{H}).$$

Thus we can identify the extensions with the inclusions of $C(X)$ into $\mathcal{A}(\mathcal{H})$. Because all separable Hilbert spaces are isomorphic [Ki] and the automorphisms of $\mathcal{K}(\mathcal{H})$ are inner and

$$\operatorname{Aut} \mathcal{K}(\mathcal{H}) \cong \mathcal{PU}(\mathcal{H}),$$

the projective unitary group, where $\mathcal{U}(\mathcal{H})$ denotes the unitary operator group, we can identify the equivalences classes of extensions with the unitary conjugacy classes of unital inclusions of $C(X)$ into the Calkin algebra: Two extensions τ_1 and τ_2 are equivalent iff there exists a unitary operator $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, such that $\tau_2 = \alpha_U \circ \tau_1$, where by definition $\alpha_U : \mathcal{A}(\mathcal{H}_1) \longrightarrow \mathcal{A}(\mathcal{H}_2)$ is the isomorphism obtained from the inner isomorphism

$$U.(-).U^{-1} : \mathcal{L}(\mathcal{H}_1) \longrightarrow \mathcal{L}(\mathcal{H}_2).$$

The extension $\tau : C(X) \hookrightarrow \mathcal{A}(\mathcal{H})$ is called trivial iff there exists a unital inclusion $\sigma : C(X) \hookrightarrow \mathcal{L}(\mathcal{H})$ such that $\tau = \pi \circ \sigma$, where $\pi : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{A}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the canonical quotient map. This inclusion τ corresponds to the split short exact sequence. The sum of two extensions $\tau_i : C(X) \hookrightarrow \mathcal{A}_i, i = 1, 2$ is defined as the extension

$$\tau_1 \oplus \tau_2 : C(X) \hookrightarrow \mathcal{A}(\mathcal{H}_1) \oplus \mathcal{A}(\mathcal{H}_2) \hookrightarrow \mathcal{A}(\mathcal{H}_1 \oplus \mathcal{H}_2).$$

This definition is also compatible with the equivalence classes of extensions. In [BDF] the authors proved that:

1) The equivalence class of the trivial extension is the identity element with respect to this sum.

2) For every metrizable compact space X, the set $\mathcal{E}xt_1(X)$ of the equivalence classes of extensions is an abelian group. One defines the higher groups by $\mathcal{E}xt_{1+n}(X) := \mathcal{E}xt_1(\mathbb{S}^n \wedge X)$, $n = 0, 1, 2, \ldots,$

3) $\mathcal{E}xt_*$ is a generalized K-homology. In particular, the group $\mathcal{E}xt_1(X)$ depends only on the homotopy type of X and there is a homomorphism

$$Y_\infty : \mathcal{E}xt_1(X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K^{-1}(X), \mathbb{Z})$$

which will be an isomorphism if $X \subset \mathbb{R}^3$.

This K-homology is well developed and fruitfully applicable. It has many applications in operator theory and in our problem of characterizing the group C*-algebras. Let us demonstrate this in the first example of the group of affine transformations of the real straight line.

2.3. Topological Invariant Index

Let us in this subsection denote by G the group of all affine transformations of the real straight line.

Theorem 2.1. *Every irreducible unitary representation of the group G is unitarily equivalent to one of the following mutually nonequivalent representations:*

a) the representation S, realized in the space $L^2(\mathbb{R}^*, \frac{dx}{|x|})$, where $\mathbb{R}^* := \mathbb{R} \setminus (0)$, and G acts according to the formula

$$(S_g f)(x) = e^{\sqrt{-1}bx} f(\alpha x), \qquad \textit{where } g = \begin{pmatrix} \alpha & b \\ 0 & 1 \end{pmatrix}.$$

b) the representation U_λ^ε, realized in $\mathbb{C}^1$ and given by the formula

$$U_\lambda^\varepsilon(g) = |\alpha|^{\sqrt{-1}\lambda} \cdot (\operatorname{sgn} \alpha)^\varepsilon, \qquad \textit{where } \lambda \in \mathbb{R}; \varepsilon = 0, 1.$$

Proof. See [GN].

This list of all the irreducible unitary representations gives the corresponding list of all the irreducible nondegenerate unitary *-representations of the group C*-algebra $C^*(G)$. In [D1] it was proved that

Theorem 2.2. *The group C^*-algebra with formally adjoint unit $C^*(G)^\sim$ can be included in a short exact sequence of C^*-algebras and *-homomorphisms*

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(G)^\sim \longrightarrow C(\mathbb{S}^1 \vee \mathbb{S}^1) \longrightarrow 0,$$

i.e. the C^*-algebra $C^*(G)^\sim$, following the BDF theory, is defined by an element, called the index and denoted by Index $C^*(G)^\sim$, *of the groups $\mathcal{E}xt_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. See [D1].

The infinite dimensional representation S realizes the inclusion mentioned above. Since

$$\mathcal{E}xt(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathrm{Hom}_{\mathbb{Z}}(\pi^1(\mathbb{S}^1 \vee \mathbb{S}^1), \mathbb{Z}^1)$$

is realized by a homomorphism from $\pi^1(\mathbb{S}^1 \vee \mathbb{S}^1)$ to $\mathbb{C}^*$. Since the isomorphism

$$Y_\infty : \mathcal{E}xt(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathrm{Hom}_{\mathbb{Z}}(\pi^1(\mathbb{S}^1 \vee \mathbb{S}^1), \mathbb{Z}^1)$$

is obtained by means of computing the indices and because the general type of elements of $\pi^1(\mathbb{S}^1 \vee \mathbb{S}^1)$ is $g_{k,l} = [g_{0,1}]^k [g_{1,0}]^l$, $k, l \in \mathbb{Z}$, we have

$$\mathrm{Ind}\,(g_{k,l}) = k\cdot \mathrm{Ind}\; T(g_{1,0}) + l\cdot \mathrm{Ind}\; T(g_{0,1}),$$

where T is the the *-isomorphism corresponding to S. Therefore it is enough to compute the pair of indices $\mathrm{Ind}\; T(g_{1,0})$ and $\mathrm{Ind}\; T(g_{0,1})$. The last ones are directely computed by the indices of the corresponding Fredholm operators.

Theorem 2.3.

$$\mathrm{Index}\; C^*(G) = (1,1) \in \mathcal{E}xt(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Proof. See [D1].

Let us now go to the general situation. To do this we must introduce also some preparation about, first of all, the construction of irreducible unitary representations by means of the orbit method, then a method of decomposing the C*-algebra into a tower of extensions and lastly to compute the index with the help of the general KK-theory.

3. Multidimensional Orbit Methods

Let us in this section consider the problem of realization of irreducible unitary representations of Lie groups. There are two versions of the orbit method; one is the multidimensional quantization, the other is the infinitesimal orbit method, related with the so called category $\mathcal{O}$.

3.1. Multidimensional Quantization

The orbit method can be constructed from the point of view of the theory of holomorphically induced representations and also from the point of view of the ideas of quantization from physics.

3.1.1. Construction of Partially Invariant Holomorphically Induced Representations

Let us consider a connected and simply connected Lie group G with Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$. Denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. The complex conjugation in the Lie algebra will also be denoted by an overline sign. Consider the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$. The group G acts on itself by the inner automorphisms

$$A(g) := g(\cdot)g^{-1} : G \longrightarrow G,$$

for each $g \in G$, conserving the identity element e as some fixed point. It follows therefore that the associated adjoint action $A(g)_*$ maps $\mathfrak{g} = T_eG$ into itself and the coadjoint action $K(g) := A(g^{-1})^*$ maps the dual space $\mathfrak{g}^*$ into itself. The orbit space $\mathcal{O}(G) := \mathfrak{g}^*/G$ is in general a bad topological space, namely non- Hausdorff. Consider one orbit $\Omega \in \mathcal{O}(G)$ and an element $F \in \mathfrak{g}^*$ in it. The stabilizer of

F is denoted by G_F, its connected componenet by $(G_F)_0$ and its Lie algebra by $\mathfrak{g}_F := \mathrm{Lie}(G_F)$. It is well-known that

$$\begin{array}{ccc} G_F & \hookrightarrow & G \\ & & \downarrow \\ & & \Omega_F \end{array}$$

is a principal bundle with the structural group G_F. Let us fix some *connection in this principal bundle,* i.e. some *trivialization* of this bundle, see [SW]. We want to construct representations in some cohomology spaces with coefficients in the sheaf of sections of some vector bundle associated with this principal bundle. It is well known [SW] that every homogeneous vector bundle is an induced one with respect to some representation of the structural group in the typical fibre. It is natural to fix some unitary representation $\tilde{\sigma}$ of G_F such that its kernel contains $(G_F)_0$, the character χ_F of the connected component of the stabilizer is given by

$$\chi_F(\exp X) := \exp\left(2\pi\sqrt{-1} < F, X >\right)$$

and therefore the differential $D(\tilde{\sigma}\chi_F) = \tilde{\rho}$ is some representation of the Lie algebra $\mathfrak{g}_F$. We suppose that the representation $D(\tilde{\sigma}\chi_F)$ was extended to the complexification $(\mathfrak{g}_F)_{\mathbb{C}}$. The whole space of all sections seems to be too large for the construction of irreducible unitary representations. One considers the invariant subspaces with the help of some so called polarizations.

Definition 3.1. We say that a triple $(\mathfrak{p}, \rho, \sigma_0)$ is some $(\tilde{\sigma}, F)$-*polarization* iff :

a) $\mathfrak{p}$ is some subalgebra of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, containing $\mathfrak{g}_F$.

b) The subalgebra $\mathfrak{p}$ is invariant under the action of all the operators of type $Ad_{\mathfrak{g}_{\mathbb{C}}} x, x \in G_F$.

c) The vector space $\mathfrak{p} + \overline{\mathfrak{p}}$ is the complexification of some real subalgebra $\mathfrak{m} = (\mathfrak{p} + \overline{\mathfrak{p}}) \cap \mathfrak{g}$.

d) All the subgroups M_0, H_0, M, H are closed, where by definition M_0 (resp., H_0) is the connected subgroup of G with the Lie algebra $\mathfrak{m}$ (resp., $\mathfrak{h} := \mathfrak{p} \cap \mathfrak{g}$) and $M := G_F.M_0$, $H := G_F.H_0$.

e) σ_0 is an irreducible representation of H_0 in some Hilbert space V such that: 1. the restriction $\sigma_0|_{G_F \cap H_0}$ is some multiple of the restriction $\chi_F.\tilde{\sigma}|_{G_F \cap H_0}$, where by definition $\chi_F(\exp X) := \exp\left(2\pi\sqrt{-1} < F, X >\right)$; 2. under the action of G_F on the dual $\hat{H}_0$, the point σ_0 is fixed.

f) ρ is some representation of the complex Lie algebra $\mathfrak{p}$ in V , which satisfies the E. Nelson conditions for H_0 and $\rho|_{\mathfrak{h}} = D\sigma_0$.

Let us recall that R. BLATTNER introduced the notion of mixed manifold of type (k, l), see for example [Ki]. We consider the fiber bundle, the base of which is some type (k, l) mixed manifold and the fibers of which are smooth m dimensional manifold. We say that this fiber bundle is some *mixed manifold of type* (k, l, m).

Theorem 3.2. *Suppose that Ω_F, $\tilde{\sigma}$, G_F, etc. are as introduced above, and let us denote by χ_F the character of the group G_F such that $D\chi_F = 2\pi\sqrt{-1}F|_{\mathfrak{g}_F}$. Then:*

1) *On the K-orbit Ω_F there exists a structure of some mixed manifold of type (k,l,m), where*

$$k = \dim G - \dim M,$$

$$l = \frac{1}{2}(\dim M - \dim H),$$

$$m = \dim H - \dim G_F.$$

2) *There exists some irreducible unitary representation σ of the group H such that its restriction $\sigma|_{G_F}$ is some multiple of the representation $\chi_F.\tilde{\sigma}$ and $\rho|_{\mathfrak{h}} = D\sigma$.*

3) *On the G-fiber bundle $\mathcal{E}_{\sigma|_{G_F}} = G \times_{G_F} V$ associated with the representation $\sigma|_{G_F}$, there exists a structure of a partially invariant and partially holomorphic Hilbert vector G-bundle $\mathcal{E}_{\sigma,\rho}$ such that the natural representation of G on the space of (partially invariant and partially holomorphic) sections is equivalent to the representation by right translations of G in the space $C^\infty(G;\mathfrak{p},\rho,F,\sigma_0)$ of V-valued C^∞-functions on G satisfying the equations*

$$f(hx) = \sigma(h)f(x), \forall h \in H, \forall x \in G,$$

$$L_X f + \rho(X)f = 0, \forall X \in \overline{\mathfrak{p}},$$

where L_X denotes the Lie derivative along the vector field ξ_X on G, corresponding to X.

Proof. The first assertion is clear. The second one can be deduced from the remark that the formula

$$(x,h) \mapsto (I_{V'} \otimes \chi_F.\tilde{\sigma})(x).\sigma_0(h)$$

defines an irreducible representation of the semidirect product $G_F \times H_0$ which is trivial on the kernel of the surjection

$$G_F \times H_0 \longrightarrow G_F.H_0.$$

This point is essential in the sense that with the assumption about the fixed point property of σ_0 we can ignore the Mackey obstruction, which appears when we take the representations which are multiples of some representations $\tilde{\sigma}\chi_F$ at the restriction to some normal subgroup. M. Duflo [Du] considered two-fold coverings to avoid this obstacle. See also [D5].

One can than apply the construction of unitarization $\overline{\mathcal{E}}_{\sigma,\rho}$ in order to obtain the corresponding unitary representation, which is noted by $\mathrm{Ind}(G;\mathfrak{p},F,\rho,\sigma_0)$. One can define also the representations in cohomologies with coefficients in this sheaf of partially invariant and partially holomorphic sections, which will be noted by $(L^2 - Coh)\,\mathrm{Ind}(G;\mathfrak{p},F,\rho,\sigma_0)$.

Remark 3.3. One introduces some order in the set of all $(\tilde{\sigma}, F)$-polarizations

$$(\mathfrak{p}, \rho, \sigma_0) \leq (\mathfrak{p}', \rho', \sigma_0') \Longleftrightarrow \mathfrak{p} \subseteq \mathfrak{p}', \sigma_0'|_{H_0} \simeq \sigma_0, \rho'|_{\mathfrak{p}} \simeq \rho.$$

To have some irreducible resresentation, one must take the maximal polarizations in this construction. It is interesting that this representations coincide with the representations that appear via geometric quantization.

3.1.2. Multidimensional Geometric Quantization

Let us now consider the general concept of multidimensional geometric quantization. Consider a symplectic manifold (M, ω), i.e. a smooth manifold equipped with a nondegenerate closed skew-symmetric differential 2-form ω. The vector space $C^\infty(M, \omega)$, with respect to the Poisson brackets

$$f_1, f_2 \in C^\infty \mapsto \{f_1, f_2\} \in C^\infty(M, \omega)$$

becomes an infinite dimensional Lie algebra.

Definition 3.4. A *procedure of quantization* is a correspondence associating to each *classical quantity* $f \in C^\infty(M)$ a *quantum quantity* $Q(f) \in \mathcal{L}(\mathcal{H})$, i.e. a closed, possibly unbounded, normal operator, which is autoadjoint if f is a real-valued function, in some Hilbert space $\mathcal{H}$, such that

$$Q(\{f_1, f_2\}) = \frac{i}{\hbar}[Q(f_1), Q(f_2)],$$
$$Q(1) = \mathrm{Id}_{\mathcal{H}},$$

where $\hbar := h/2\pi$ is the normalized Planck constant, and h is the unnormalized Planck constant.

Let us denote by $\mathcal{E}$ a fiber bundle of Hilbert spaces over M, Γ a fixed connection preserving the Hilbert structure on the fibers; in other words, if γ is a curve connecting two points x and x', the parallel transport along the way γ provides a unitary isomorphism from the fiber $\mathcal{E}_x$ onto the fiber $\mathcal{E}_{x'}$. In this case we can define the corresponding covariant derivative ∇_ξ, $\xi \in \mathrm{Vect}(M) := Der\, C^\infty(M)$ in the space of smooth sections. One considers the invariant Hilbert space $L^2(\mathcal{E}_{\rho,\sigma})$, which is the completion of the space $\Gamma(\mathcal{E}_{\rho,\sigma})$ of square-integrable partially invariant and partially holomorphic sections.

Suppose from now on that M is a homogeneous G-space. Choose a trivialization Γ of the principal bundle $G_x \rightarrowtail G \twoheadrightarrow M$, where G_x is the stabilizer of the point x on M. Let us denote by L_ξ the Lie derivation corresponding to the vector field $\xi \in \mathrm{Vect}(M)$. Let us denote by $\beta \in \Omega^1(M)$ the connection form of the affine connection on $\mathcal{E}$. It is more comfortable to consider the normalized connection form $\alpha(\xi) = \frac{\hbar}{\sqrt{-1}}\beta(\xi)$, the values of which are skew-adjoint operators on fibers. One has therefore

$$\nabla_\xi = L_\xi + \frac{\sqrt{-1}}{\hbar}\alpha(\xi),$$

see for example [SW] for the finite dimensional case.

For each function $f \in C^\infty(M)$ one denotes ξ_f the corresponding Hamiltonian vector field, i.e.

$$i(\xi_f)\omega + df = 0.$$

Definition 3.5. We define the geometrically quantized operator $Q(f)$ as

$$Q(f) := f + \frac{\hbar}{\sqrt{-1}}\nabla_{\xi_f} = f + \frac{\hbar}{\sqrt{-1}}L_{\xi_f} + \alpha(\xi_f).$$

Theorem 3.6. *The following three conditions are equivalent:*

1)

$$(\forall \xi, \eta) \quad \xi\alpha(\eta) - \eta\alpha(\xi) - \alpha([\xi,\eta]) + \frac{\sqrt{-1}}{\hbar}[\alpha(\xi), \alpha(\eta)] = -\omega(\xi,\eta).\,\mathrm{Id}_{\mathcal{H}}\,.$$

2) The curvature of the affine connection ∇ *is equal to* $-\frac{\sqrt{-1}}{\hbar}\omega(\xi,\eta).Id_{\mathcal{H}}$*, i.e.*

$$(\forall \xi, \eta) \quad [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]} = -\frac{\sqrt{-1}}{\hbar}\omega(\xi,\eta).Id_{\mathcal{H}}.$$

3) The correspondence $f \mapsto Q(f)$ *is a quantization procedure.*

Proof. See [D6].

Suppose that the Lie group G acts on M by symplectomorphisms. Then each element X of the Lie algebra $\mathfrak{g}$ corresponds to a one-parameter subgroup $\exp(tX)$ in G, which acts on M. Let us denote by ξ_X the corresponding strictly hamiltonian vector field. Let us denote also L_X the Lie derivation along this vector field. We have

$$[L_X, L_Y] = L_{[X,Y]},$$

and

$$L_X f = \{f_X, f\}.$$

Suppose that f_X depends linearly on X. One has then a 2-cocycle of the action

$$c(X,Y) := \{f_X, f_Y\} - f_{[X,Y]}.$$

Definition 3.7. We say that the action of G on M is *flat* iff this 2-cocycle is trivial.

In this case we obtain from the quantization procedure a representation $\wedge$ of the Lie algebra $\mathfrak{g}$ by the skew-adjoint operators

$$X \mapsto \frac{\sqrt{-1}}{\hbar}Q(f_X)$$

and also a representation of $\mathfrak{g}$ by the functions

$$X \mapsto f_X.$$

If the E. NELSON conditions are satisfied, we have a unitary representation of the universal covering of the group G.

Theorem 3.8. *The Lie derivative of the partially invariant and holomorphically induced representation* $\mathrm{Ind}(G;\mathfrak{p},F,\rho,\sigma_0)$ *of a connected Lie group* G *is just the representation obtained from the procedure of multidimensional geometric quantization, corresponding to a fixed connection* ∇ *of the partially invariant partially holomorphic induced unitarized bundle* $\overline{\mathcal{E}}_{\sigma,\rho}$*, i.e.*

$$\mathrm{Lie}_X(\mathrm{Ind}(G;\mathfrak{p},F,\rho,\sigma_0)) = \frac{\sqrt{-1}}{\hbar}Q(f_X).$$

Proof. See [D4].

Remark 3.9. The multidimensional version of the orbit method was developed independently by the author in the language of multidimensional quantization [D4] - [D7] and by M. Duflo [Du], see also, [Ki] in the language of Mackey's method of small subgroups. The result shows that for most of the connected Lie groups the construction gives us at least a quantity of irreducible unitary representations, enough to decompose the regular representations of G in $L^2(G)$, i.e. enough to prove the Plancherèl formula [Du].

Remark 3.10. There are some reductions of this multidimensional quantization procedure to the radical or nil-radical of stabilisors of type G_F, see [DV], and lifting them to $U(1)$-coverings [Vui1] - [Vui3], [Do1] - [Do2].

Remark 3.11. In [D10] the author proposed some method for common quantization for foliations, the fibers of which are the K-orbits, and its relation with the integral Fourier operators.

3.2. Category $\mathcal{O}$ and globalization of Harish-Chandra modules

The construction of the irreducible unitary representations $(L^2-\mathrm{Coh})\,\mathrm{Ind}(G;\mathfrak{p},F,\rho,\sigma_0)$ in the Hilbert space $L^2(\mathcal{E}_{\rho,\sigma}) \cong L^2(G;\mathfrak{p},F,\rho,\sigma_0)$ can in some cases be considered as a globalization of a so called $(\mathfrak{g},K)$-module, i.e. $L^2(\mathcal{E}_{\rho,\sigma_0})_{(K)}$ itself is some $(\mathfrak{g},K)$-module, where K is some maximal compact subgroup of G. It is therefore interesting to consider these $(\mathfrak{g},K)$-module as some infinitesimal version of the orbit method. Let us see this in this subsection.

3.2.1. Admissible representations

Let us in this subsubsection recall some results about the Borell-Weil-Bott theorem and the construction of admissible representations of finite dimensional semisimple Lie groups as $(\mathfrak{g},K)$-modules, [Wo], [M2].

If G is a compact connected Lie group, and $F\in\mathfrak{g}^*$ is a well-regular integral functional on its Lie algebra, then the stabilizer is a maximal torus T, (If the Harish-Chandra criterion for extistence of discrete series holds, it is a compact Cartan subgroup.) and a choice of a positive system of roots $\Phi^+ = \Phi^+(\mathfrak{g},\mathfrak{t})$ defines a G-invariant complex manifold structure on G/T in such a way that $\sum_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$ represents the holomorphic tangent space. The character $\chi_\lambda, \lambda := \frac{\sqrt{-1}}{\hbar}F$ can be extended to a character of the stabilizer $G_F = T$, if the orbit is as usually supposed to be integral. Let us denote in this case the induced bundle $\mathcal{E}_{\rho,\sigma}$ simply by $\mathbb{E}_\lambda$ as in [Wo]. It is the associated homogeneous holomorphic hermitian line bundle. One writes $\mathcal{O}(\mathbb{E}_\lambda) \longrightarrow G/T$ for the sheaf of germs of holomorphic sections of $\mathbb{E}_\lambda \longrightarrow G/T$. The group G acts everywhere, including the cohomologies $H^q(G/T; \mathcal{O}(\mathbb{E}_\lambda))$. One denotes by $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ the half-sum of positive roots. We cite from [Wo] the Borel-Weil-Bott theorem

Theorem 3.12. *If $\lambda + \rho$ is singular then every $H^q(G/T; \mathcal{O}(\mathbb{E}_\lambda))$ is trivial. If $\lambda + \rho$ is regular, let w denote the unique element of the Weyl group such that*

$$\langle w(\lambda+\rho), \alpha\rangle > 0, \forall\alpha \in \Phi^+$$

and let $\ell(w)$ denote its minimal length as a word in the simple root reflections. Then

i) $H^q(G/T; \mathcal{O}(\mathbb{E}_\lambda)) = 0$ for all $q \neq \ell(w)$, and

ii) the action of G in $H^{\ell(w)}(G/T; \mathcal{O}(\mathbb{E}_\lambda))$ is the representation with heighest weight $w(\lambda+\rho) - \rho$.

This result was then extended for realizing *the discrete series representations of general semisimple Lie groups.* It is well known that one can induce from these discrete series representations of a reductive part of parabolic subgroups to obtain the *tempered admissible representations* of G. It was then remarked that the representation of G in $L^2(\mathcal{E}_{\rho,\sigma})$ can be considered as the globalization of some $(\mathfrak{g}, K)$-module , namely, $L^2(\mathcal{E}_{\rho,\sigma})_{(K)}$.

Lastly the *tempered admissible representations* are decribed in the cohomologies corresponding to $\mathcal{D}$-modules [M2].

3.2.2. Discrete Series for loop groups

Let us now consider the loop groups associated with compact Lie groups. With the help of the Zuckermann's derived functor, we can construct the infinitesimal version of the "discrete series" for loop groups, [D9]. The algebraic realization of these representations are decribed in [D9] as a version of the Borel-Weil-Bott and Kostant theorems. It is very interesting to develop a theory of "tempered representations" for loop groups.

4. KK-theory

4.1. About KK-Functors

We now recall some essential points of Kasparov's setting of the KK-theory. It is an analog of the Brown-Douglas-Fillmore theory, but settled for the general case.

4.1.1. Definitions

The main reference for this subsubsection is [JT]. Let A, B, E be C*-algebras, and $\mathcal{K}$ the ideal of compact operators in some fixed separable Hilbert space. Let us consider the extensions of type

$$0 \longrightarrow B \otimes \mathcal{K} \longrightarrow E \longrightarrow A \longrightarrow 0.$$

Two extensions are said to be *equivalent* iff there is some isomorphism $\psi : E \longrightarrow E'$ such that it induces the identity isomorphisms on the ideal $B \otimes \mathcal{K}$ and on the quotient A, i.e. the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow \psi & & \| & & \\ 0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

The extension is called *trivial* if the the exact sequence splits. Also due to well-known results of R. C. BUSBY, we can identify each extension with some *-homomorphism from A to the algebra of exterior multipliers of $B \otimes \mathcal{K}$, $\tau : A \longrightarrow \mathcal{O}(B \otimes \mathcal{K})$. *The sum* of two extensions

$$\tau_i : A \longrightarrow \mathcal{O}(B \otimes \mathcal{K})$$

can be therefore defined as the extension

$$\tau_1 \oplus \tau_2 : A \longrightarrow \mathcal{O}(B \otimes \mathcal{K}) \oplus \mathcal{O}(B \otimes \mathcal{K}) \hookrightarrow \mathcal{O}(B \otimes \mathcal{K}) \otimes M_2 \cong \mathcal{O}(B \otimes \mathcal{K}),$$

where M_2 is the full algebra of 2×2-matrices over the complexes numbers. Two extensions $\tau_i, i = 1, 2$ are *stably equivalent* if there exist two trivial extensions σ_1 and σ_2 such that the sums $\tau_i + \sigma_i$, $i = 1, 2$ are equivalent.

G. G. KASPAROV [K1] proved that:

i) when A is nuclear separable and B has at least an approximate identity, the set $\mathrm{Ext}(A, B)$ of the stably equivalent classses of extensions is an ablelian group.

ii) $K^*(.) = \mathrm{Ext}_*(A, .)$ is a K-homology theory and $K_* = \mathrm{Ext}_*(., B)$ is the algebraic K-theory of C*-algebras.

iii) There is a natural realization of $KK^{*,*}$ as some K-bifunctor and for its direct relation with the Ext_*-groups see [K2].

J. ROSENBERG and C. SCHOCHET [RS] proved the Künneth formula for these groups, i.e. there is some homomorphism

$$Y : \mathrm{Ext}_i(A, B) \longrightarrow \bigoplus_{j (mod\, 2)} \mathrm{Hom}_{\mathbb{Z}}(K_{i+j}(A), K_{i+j+1}(B)).$$

Let us see this in more detail in the next subsubsection.

4.1.2. Relation with K-groups of C-algebras*

The most important for us is the relation of the theory with K-groups of C*-algebras. Let A be an algebra with unit. By definition, $K_0(A)$ is the Grothendieck group of the semi-group of the stably equivalent classes of projective A-modules of finite type. When A has no unity element, one considers the algebra $A^{\sim}$ with the formally adjoint unit and defines the K-group as

$$K_*(A) := \ker\{K_*(A^{\sim}) \longrightarrow K_*(\mathbb{C}) = \mathbb{Z}\}.$$

This definition is compatible with the above defined K-groups also for algebras with unit element. For $A = C(X)$, there is a natural isomorphism between these K-groups with the corresponding topological groups $K^*(X)$, see for example [K]. One defines the higher groups $K_n(A)$ as

$$K_n(A) := K_0(A \otimes C_0(\mathbb{R}^n)), \forall n \geq 0.$$

The Bott theorem says that $K_0(A) \cong K_2(A)$. The Connes-Kasparov theorem says that for any connected and simply-connected solvable Lie group G,

$$K_0(C^*(G)) = \begin{cases} \mathbb{Z} & \text{if } \dim G \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

$$K_1(C^*(G)) = \begin{cases} 0 & \text{if } \dim G \text{ is even,} \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

For each extension

$$0 \longrightarrow J \longrightarrow E \longrightarrow A \longrightarrow 0,$$

there is a six-term exact sequence of K-groups

$$\begin{tikzcd} & K_0(E) \arrow[r] & K_0(A) \arrow[dr] & \\ K_0(J) \arrow[ur] & & & K_1(J) \arrow[dl] \\ & K_1(A) \arrow[ul] & K_0(E) \arrow[l] & \end{tikzcd}$$

Let us consider the case $J = B \otimes \mathcal{K}$, There is an isomorphism between $K_*(J)$ and $K_*(B)$. The group $K_*(A)$ consists of the formal differences of equivalence classes of projectors in $A \otimes \mathcal{K}$. One obtain therefore the well-known exact sequence

$$\begin{tikzcd} & K_0(E) \arrow[r] & K_0(A) \arrow[dr, "\partial_0"] & \\ K_0(B) \arrow[ur] & & & K_1(B) \arrow[dl] \\ & K_1(A) \arrow[ul, "\partial_1"] & K_1(E) \arrow[l] & \end{tikzcd}$$

It is therefore clear that each element of $\mathrm{Ext}(A, B)$ induces a pair of homomorphisms (∂_0, ∂_1) of K-groups, and one has a homomorphism

$$\gamma : \mathrm{Ext}_i(A, B) \longrightarrow \oplus_{j \in \mathbb{Z}/(2)} \mathrm{Hom}_{\mathbb{Z}}(K_{i+j}(A), K_{i+j+1}(B)),$$

associating to each extension a pair of connecting homomorphisms (∂_0, ∂_1).

J. Rosenberg and S. Schochet [RS] have proved the following exact sequence

$$0 \longrightarrow \sum_{i \in \mathbb{Z}/(2)} \mathrm{Ext}^1_{\mathbb{Z}}(K_{i+j}(A), K_{i+j+1}(B)) \longrightarrow \mathrm{Ext}_i(A, B) \longrightarrow$$

$$\sum_{j \in \mathbb{Z}/(2)} Hom_{\mathbb{Z}}(K_{i+j}(A), K_{i+j+1}(B)) \longrightarrow 0.$$

4.2. Construction and reduction of the K-Theory Invariant Index $C^*(G)$

We review in this section a construction for obtaining the short exact sequence of C*-algebras.

4.2.1. Measurable foliations

In this section we propose a canonical method for constructing the measurable foliations, consisting of the adjoint orbits of fixed dimension, and therefore their C*-algebras [C1]. The last ones are included in group C*-algebras or their quotients.

Let us denote by G a connected and simply connected Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ the dual vector space, $\mathcal{O} = \mathcal{O}(G)$ the space of all the coadjoint orbits of G in $\mathfrak{g}^*$. This space is a disjoint union of subspaces of coadjoint orbits of fixed dimension, i.e.

$$\mathcal{O} = \amalg_{0 \leq 2n \leq \dim G} \mathcal{O}_{2n},$$

$$\mathcal{O}_{2n} := \{\Omega \in \mathcal{O}; \dim \Omega = 2n\}.$$

We define

$$V_{2n} := \bigcup_{\dim \Omega = 2n} \Omega.$$

Then it is easy to see that V_{2n} is the set of points of a fixed rank of the Poisson structure bilinear function

$$\{X, Y\}(F) = \langle F, [X, Y] \rangle.$$

suppose it is a foliation, at least for V_{2n}, with $2n = max$.

First, we shall show that the foliation V_{2n} can be obtained by the associated action of $\mathbb{R}^{2n}$ on V_{2n} via $2n$ times repeated action of $\mathbb{R}$.

Indeed, fixing any basis $X_1, X_2, \ldots, X_{2n}$ of the tangent space $\mathfrak{g}/\mathfrak{g}_F$ of Ω at the point $F \in \Omega$, we can define an action $\mathbb{R}^{2n} \curvearrowright V_{2n}$ as

$$(\mathbb{R} \curvearrowright (\mathbb{R} \curvearrowright (\ldots \mathbb{R} \curvearrowright V_{2n})))$$

by

$$(t_1, t_2, \ldots, t_{2n}) \longmapsto \exp(t_1 X_1) \ldots \exp(t_{2n} X_{2n}) F$$

Thus we have the Hamiltonian vector fields

$$\xi_k := \frac{d}{dt_k}|_{t_k=0} \exp(t_k X_k) F, k = 1, 2, \ldots, 2n$$

and the linear span

$$F_{2n} = \{\xi_1, \xi_2, \ldots, \xi_{2n}\}$$

provides a tangent distribution.

Theorem 4.1. *(V_{2n}, F_{2n}) is a measurable foliation.*

Proof. See [D8].

Corollary 4.2. *The Connes C*-algebra $C^*(V_{2n}, F_{2n}), 0 \leq 2n \leq \dim G$ are well defined.*

4.2.2. Reduction of Index $C^*(G)$ *to* Index $C^*(V_{2n}, F_{2n})$

Now we assume that the orbit method (see [Ki],[D4]-[D6]) gives us a complete list of irreducible representations of G ,

$$\pi_{\Omega_F,\sigma} = \mathrm{Ind}(G, \Omega_F, \sigma, \mathfrak{p}), \sigma \in \mathcal{X}_G(F),$$

the finite set of Duflo's data.

Suppose that

$$\mathcal{O} = \bigcup_{i=1}^{k} \mathcal{O}_{2n_i}$$

is the decomposition of the orbit space as a stratification of orbits of dimensions $2n_i$, where $n_1 > n_2 > \ldots > n_k > 0$.

We include $C^*(V_{2n_1}, F_{2n_1})$ into $C^*(G)$. It is well known that the Connes C*-algebra of foliation can be included in the algebra of pseudodifferential operators of degree 0 as an ideal. This algebra of pseudodifferential operators of degree 0 is included in $C^*(G)$.

We define

$$J_1 = \bigcap_{\Omega_F \in \mathcal{O}(G) \setminus \mathcal{O}_{2n_1}} \ker \pi_{\Omega_F,\sigma},$$

and

$$A_1 = C^*(G)/J_1.$$

Then

$$C^*(G)/C^*(V_{2n_1}, F_{2n_1}) \cong A_1$$

and we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 & \longrightarrow & C^*(G) & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mathrm{Id} & & \downarrow & & \\
0 & \longrightarrow & C^*(V_{2n_1}, F_{2n_1}) & \longrightarrow & C^*(G) & \longrightarrow & C^*(G)/C^*(V_{2n_1}, F_{2n_1}) & \longrightarrow & 0
\end{array}$$

Hence $J_1 \simeq C^*(V_{2n_1}, F_{2n_1})$ and we have

$$O \to C^*(V_{2n_1}, F_{2n_1}) \to C^*(G) \to A_1 \to 0$$

Repeating the procedure in replacing

$$C^*(G),\ C^*(V_{2n_1}, F_{2n_1}),\ A_1,\ J_1$$

by

$$A_1,\ C^*(V_{2n_2}, F_{2n_2}),\ A_2,\ J_2,$$

we have

$$0 \to C^*(V_{2n_2}, F_{2n_2}) \to A_1 \to A_2 \to 0$$

etc...

So we obtain the following result.

Theorem 4.3. *The group C^*-algebra $C^*(G)$ can be included in a finite sequence of extensions*

$$(\gamma_1): \qquad 0 \to C^*(V_{2n_1}, F_{2n_1}) \to C^*(G) \to A_1 \to 0$$

$$(\gamma_2): \qquad 0 \to C^*(V_{2n_2}, F_{2n_2}) \to A_1 \to A_2 \to 0,$$

$$\dots\dots\dots\dots\dots$$

$$(\gamma_k): \qquad 0 \to C^*(V_{2n_k}, F_{2n_k}) \to A_{k-1} \to A_k \to 0,$$

where $\widehat{A_k} \simeq Char(G)$

Corollary 4.4. Index $C^*(G)$ *is reduced to the system* Index $C^*(V_{2n_i}, F_{2n_i}), i = 1, 2, \dots, k$ *by the invariants*

$$[\gamma_i] \in KK(A_i, C^*(V_{2n_i}, F_{2n_i})), \quad i = 1, 2, \dots, k.$$

Remark 4.5. Ideally, all these invariants $[\gamma_i]$ could be computed step-by-step from $[\gamma_k]$ to $[\gamma_1]$.

4.2.3. Reduction of Index $C^*(V_{2n_i}, F_{2n_i})$ *to the computable extension indices with values in topological KK-groups of pairs of spaces*

Let us consider $C^*(V_{2n_i}, F_{2n_i})$ for a fixed i. We introduce the following assumptions which were considered by G. G. KASPAROV in nilpotent cases [K2]:

(A_1) There exists $k \in \mathbb{Z}, 0 < k \leq 2n_i$ such that the foliation

$$V_{gen} := V_{2n_i} \setminus (\text{Lie } \Gamma)^\perp$$

has its C*- algebra

$$C^*(V_{gen}, F|_{V_{gen}}) \cong C(\mathcal{O}^\sim_{gen}) \otimes \mathcal{K}(H),$$

where

$$\Gamma := \mathbb{R}^k \hookrightarrow \mathbb{R}^{2n_i} \hookrightarrow G,$$
$$\text{Lie}\,\Gamma = \mathbb{R}^k \hookrightarrow \mathfrak{g}/\mathfrak{g}_{F_i}, (\text{Lie}\,\Gamma)^\perp \subset \mathfrak{g}^* \cap V_{2n_i}.$$

Example 4.6. If V_{gen} is a principal bundle, or the space $\mathcal{O}_{gen} = V_{gen}/G$ is a Hausdorff space, then $C^*(V_{gen}, F|_{V_{gen}}) \simeq C(\mathcal{O}^\sim_{gen}) \otimes \mathcal{K}(H)$

It is easy to see that if the condition (A_1) holds, $C^*(V_{2n_i}, F_{2n_i})$ is an extension of $C^*(V_{2n_i} \setminus V_{gen}, F_{2n_i}|.)$ by $C(\mathcal{O}^\sim_{gen}) \otimes \mathcal{K}(H)$, where $\mathcal{O}^\sim_{gen} = \{\pi_{\Omega_F,\sigma}; \Omega_F \in \mathcal{O}_{gen}, \sigma \in \mathcal{X}_G(F)\}$, described by the multidimensional orbit method from the previous section. If $k = 2n_i, (\mathbb{R}^{2n_i})^\perp = \{0\}$, $V_{2n_i} = V_{gen}$, we have

$$C^*(V_{2n_i}, F_{2n_i}) \simeq C(\mathcal{O}^\sim_{2n_i}) \otimes \mathcal{K}(H)$$

If $k = k_1 < 2n_i$, then $\mathbb{R}^{2n_i - k_1}$ acts on $V_{2n_i} \setminus V_{gen}$ and we suppose that a similar assumption (A_2) holds

(A_2) There exists $k_2, 0 < k_2 \leq 2n_i - k_1$ such that

$$(V_{2n_i} \setminus V_{gen})_{gen} := (V_{2n_i} \setminus V_{gen}) \setminus (\mathbb{R}^{k_2})^\perp$$

has its C*-algebra

$$C^*((V_{2n_i} \setminus V_{gen})_{gen}, F_{2n_i}|.) \simeq C((\mathcal{O}_{2n_i} \setminus \mathcal{O}_{gen})_{gen})^\sim \otimes \mathcal{K}(H)$$

As above, if $k_2 = 2n_i - k_1$, $C^*(V_{2n_i} \setminus V_{gen}, F_{2n_i}|.) \simeq C((\mathcal{O}_{2n_i} \setminus \mathcal{O}_{gen})^\sim_{gen}) \otimes \mathcal{K}(H)$. In the other case we repeat the procedure and go to assumption (A_3), etc. ...

The procedure must be finished after a finite number of steps, say in the m-th step,

$$C^*((\dots(V_{2n_i} \setminus V_{gen}) \setminus (V_{2n_i} \setminus V_{gen})_{gen} \setminus \dots, F_{2n_i}|.) \simeq C((\dots(\mathcal{O}_{2n_i} \setminus \mathcal{O}_{gen}) \setminus \dots)) \otimes \mathcal{K}(H)$$

Thus we have the following result.

Theorem 4.7. *If all the arizing assumptions $(A_1), (A_2), \dots$ hold, the C*-algebra $C^*(V_{2n_i}, F_{2n_i})$ can be included in a finite sequence of extensions*

$$0 \to C(\mathcal{O}^\sim_{gen}) \otimes \mathcal{K}(H) \to C^*(V_{2n_i}, F_{2n_i}) \to C^*(V_{2n_i} \setminus V_{gen}, F_{2n_i}|.) \to 0$$

$$0 \to C((\mathcal{O}_{2n_i} \setminus \mathcal{O}_{gen})^\sim_{gen}) \otimes \mathcal{K}(H) \to C^*(V_{2n_i} \setminus V_{gen}, F_{2n_i}|.) \to C^*(\dots) \to 0$$

$$\dots\dots\dots\dots\dots$$

$$0 \to C((\dots(\mathcal{O}_{2n_i} \setminus \mathcal{O}_{gen}) \setminus (\mathcal{O}_{2n_i} \setminus \mathcal{O}_{gen}))_{gen} \dots{}^\sim) \otimes \mathcal{K}(H) \to C^*(\dots) \to C^*(\dots) \otimes \mathcal{K}(H) \to 0.$$

4.2.4. General remarks concerning computation of Index C(G) via $KK(X_i, Y_i)$*

We see that the general computation procedure of Index C*(G) is reduced to the case of short exact sequences of type

$$(\gamma) \qquad 0 \to C(Y) \otimes \mathcal{K}(H) \to \mathcal{E} \to C(X) \otimes \mathcal{K}(H) \to 0,$$

and the index is

$$[\gamma] = \operatorname{Index} \mathcal{E} \in KK(X, Y)$$

The group $KK_i(X, Y)$ can be mapped onto

$$\bigoplus_{j \in \mathbb{Z}/(2)} \operatorname{Hom}_{\mathbb{Z}}(K^{i+j}(X), K^{i+j+1}(Y))$$

with kernel

$$\bigoplus_{j \in \mathbb{Z}/(2)} \operatorname{Ext}^1_{\mathbb{Z}}(K^{i+j}(X), K^{i+j+1}(Y))$$

by the well known cap-product, see [K2]. So $[\gamma] = (\delta_0, \delta_1)$

$$\delta_0 \in \operatorname{Hom}_{\mathbb{Z}}(K^0(X), K^1(Y)) = \operatorname{Ext}_0(X) \wedge K^1(Y)$$

$$\delta_1 \in \operatorname{Hom}_{\mathbb{Z}}(K^1(X), K^0(Y)) = \operatorname{Ext}_1(X) \wedge K^0(Y).$$

Suppose $e_1, e_2, \ldots, e_n \in \pi^1(X)$ to be generators and $\phi_1, \phi_2, \ldots, \phi_n \in \mathcal{E}$ the corresponding Fredholm operators, $T_1, T_2, \ldots, T_n$ the Fredholm operators, representing the generators of $K^1(Y) = \mathrm{Index}[Y, Fred]$ We have therefore

$$[\delta_0] = \sum_j c_{ij}\ index\, T_j,$$

where

$$\delta_0 = (c_{ij}) \in \mathrm{Mat}_{\mathrm{rank}\, K^0(X) \times \mathrm{rank}\, K^1(Y)}(\mathbb{Z})$$

In the same way δ_1 can be computed.

4.3. Case-by-Case Examples

We finish this section by an overview of well-treated examples, demonstrating effectiveness of the above reduction and conputation.

Now we introduce the classes $\overline{MD}$ and $MD4$ of Lie groups, the Lie algebras of which are well-defined by the method exposed above.

Definition 4.8. Lie algebra $\mathfrak{g}$ is said to be in the class $\overline{MD}$ (resp., MD), iff for every $F \in \mathfrak{g}^*$, $\dim \Omega_F$ is either 0 or equal to $\dim \mathfrak{g}$ (resp., or maximal).

We finish this section by stating some current new research results, see [Vu2],[R1],[Vu1],[SV].

Theorem 4.9. (Classification for $\overline{MD}$ and $MD4$). *(1) The Lie algebra $\mathfrak{g}$ is of class $\overline{MD}$ if and only if either $\mathfrak{g}$ is commutative or it is the Lie algebra of the Lie group of affine transformations of the real straight line* $\mathrm{Aff}\,\mathbb{R}$ *, and its universal covering* $\widetilde{\mathrm{Aff}\,\mathbb{R}}$ *, generated by two generators X, Y with the only non trivial commutation relation*

$$[X, Y] = Y,$$

or the Lie algebra of the group of affine transformations of the complex straight line $\mathrm{Aff}\,\mathbb{C}$ *and its universal covering* $\widetilde{\mathrm{Aff}\,\mathbb{C}}$*, generated by the generators T, X, Y, Z with the only non trivial commutation relations*

$$[T, X] = X, \quad [T, Y] = Y, \quad [X, Y] = -Z, \quad [X, Z] = Y$$

(2) Let us denote $MD4$ the subclass of MD , consisting only of the 4-dimensional ones. Suppose that $\mathfrak{g}$ is indecomposable *into a direct product of two proper ideals. Then $\mathfrak{g}$ is of class $MD4$ if and only if it is generated by the generators T, X, Y, Z with* the only non-trivial *commutation relations in one of the following cases :*

$\mathfrak{g}_{4,1,1}$: $$[T, X] = Z,$$

$\mathfrak{g}_{4,1,2}$: $$[T, Z] = Z,$$

$\mathfrak{g}_{4,2,1(\lambda)}$: $$[T, X] = \lambda X, \quad [T, Y] = Y; \quad \lambda \in \mathbb{R}^* = \mathbb{R} \setminus (0),$$

$\mathfrak{g}_{4,2,2}:$ $$[T,X] = X, \quad [T,Y] = X+Y,$$

$\mathfrak{g}_{4,2,3(\varphi)}:$ $$\operatorname{ad} T = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\mathfrak{g}_{4,2,4} = \operatorname{Lie}(\operatorname{Aff}\mathbb{C}):$ $$\operatorname{ad} T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad} X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\mathfrak{g}_{4,3,1(\lambda_1,\lambda_2)}:$ $$\operatorname{ad} T = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}^*,$$

$\mathfrak{g}_{4,3,2(\lambda)}:$ $$\operatorname{ad} T = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}^*,$$

$\mathfrak{g}_{4,3,3}:$ $$\operatorname{ad} T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathfrak{g}_{4,3,4(\lambda)}:$ $$\operatorname{ad} T = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}^*, \quad \varphi \in (0,\pi),$$

$\mathfrak{g}_{4,4,1} = \operatorname{Lie}(\mathbb{R} \ltimes_J \mathbb{H}_3):$ $$\operatorname{ad} T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [X,Y] = Z,$$

$\mathfrak{g}_{4,4,2} = \operatorname{Lie}(\mathbb{R} \ltimes \mathbb{H}_3):$ $$\operatorname{ad} T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [X,Y] = Z,$$

(in this case the group is the so called real diamond group*).*

Proof. See [Vu2].

It is not hard to classify all the K-orbits of these groups and then construct the corresponding representations by the orbit method. Now we consider the C*-algebras of the corresponding measurable foliations of generic orbits. Most of them can be described by the analytic method via the corresponding Fourier-Gel'fand transforms, but we consider only the cases $\mathfrak{g}_{4,3,4(\lambda,\varphi)} = \operatorname{Lie}(G_{(\varphi,\lambda)})$, $\mathfrak{g}_{4,4,1} = \operatorname{Lie}(\mathbb{R} \ltimes_J \mathbb{H}_3)$ and the real diamond group $\mathbb{R} \ltimes \mathbb{H}_3$.

The structure of the C^*-algebras of these examples is described in the following results which we state, again without detailed proof, see [D1], [D2], [R1], [R2] [Vu1], [Vu2].

Theorem 4.10. *(1) The C^*-algebra $C^*(\mathrm{Aff}\,\mathbb{R})$ of the group of affine transformations of the real straight line $\mathbb{R}$ can be included in an C^*-algebra extension*

$$0 \to \mathcal{K}(H) \to C^*(\mathrm{Aff}\,\mathbb{R}) \to C(\mathbb{S}^1 \vee \mathbb{S}^1) \to 0$$

and its structure is uniquely defined by the K-theory invariant

$$\mathrm{Index}\ \ C^*(\mathrm{Aff}\,\mathbb{R}) = (1,1) \in \mathcal{E}xt(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

(2) The C^*-algebra $C^*((\mathrm{Aff}\,\mathbb{R})_0)$ of the group of proper affine transformations of the real straight line can be included in an extension

$$0 \to \mathcal{K}(H) \oplus \mathcal{K}(H) \to C^*((\mathrm{Aff}\,\mathbb{R})_0) \to C(\mathbb{S}^1) \to 0$$

and its structure is uniquely defined by the K-theory invariant

$$\mathrm{Index}\, C^*((\mathrm{Aff}\,\mathbb{R})_0) = (1,1) \in \mathcal{E}xt(\mathbb{S}^1) \oplus \mathcal{E}xt(\mathbb{S}^1) = \mathbb{Z} \oplus \mathbb{Z}$$

(3) The C^*-algebra $C^*(\mathrm{Aff}\,\mathbb{C})$ of the group of affine transformations of the complex straight line $\mathbb{C}$ can be included in an extension

$$0 \to \mathcal{K}(H) \to C^*(\mathrm{Aff}\,\mathbb{C}) \to C(X) \to 0,$$

where

$$X := \{z \in \mathbb{C}; |z - 2^{-n}| = 2^{-n}, n = 1, 2, ...\}$$

is the so called "Hawaiian necklace", and its structure is uniquely defined by the K-theory invariant

$$\mathrm{Index}\, C^*(\mathrm{Aff}\,\mathbb{C}) = (-1, -1, , \dots) \in KK(X, pt) = \mathcal{E}xt(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \dots$$

(4) The C^*-algebra $C^*(\widetilde{\mathrm{Aff}\,\mathbb{C}})$ of the universal covering $\widetilde{\mathrm{Aff}\,\mathbb{C}}$ of the group $\mathrm{Aff}\,\mathbb{C}$ of the affine transformations of the complex straight line can be included in an extension

$$0 \to C(\mathbb{S}^1) \otimes \mathcal{K}(H) \to C^*(\widetilde{\mathrm{Aff}\,\mathbb{C}}) \to C(\mathbb{S}^2) \to 0$$

and its structure is uniquely defined by the invariant

$$\mathrm{Index}\, C^*(\widetilde{\mathrm{Aff}\,\mathbb{C}}) = 1 \in KK(\mathbb{S}^2, \mathbb{S}^1) = \mathbb{Z}.$$

(5) The C^*-algebra $C^*(G_{\varphi,\lambda})$ can be included into two subsequent extensions

$$(\gamma_1) \qquad 0 \to C^*(V_{\varphi,\lambda}, F) \to C^*(G_{\varphi,\lambda}) \to C(\mathbb{S}^1) \to 0,$$

$$(\gamma_2) \qquad 0 \to C(\mathbb{S}^2 \vee \mathbb{S}^2) \otimes \mathcal{K}(H) \to C^*(V_{\varphi,\lambda}, F) \to C(\mathbb{S}^1) \otimes \mathcal{K}(H) \to 0,$$

and its structure is uniquely defined by the KK-theory invariant

$$\mathrm{Index}\, C^*(G_{\varphi,\lambda}) = ([\gamma_1], [\gamma_2]),$$

where

$$[\gamma_1] \in KK(C(\mathbb{S}^1), C^*(V_{\varphi,\lambda}, F)),$$

$$[\gamma_2] \in KK(\mathbb{S}^1, \mathbb{S}^2 \vee \mathbb{S}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

(6) The group C^*-algebra $C^*(\mathbb{R} \ltimes_J \mathbb{H}_3)$ can be included into two subsequent extensions

$$(\gamma_1) \qquad 0 \to C^*(V_J, F) \to C^*(\mathbb{R} \ltimes_J \mathbb{H}_3) \to C(\mathbb{S}^1) \otimes \mathcal{K}(H) \to 0,$$

$$(\gamma_2) \qquad 0 \to C^*((\mathbb{R}^\times \times \mathbb{R})_{cpt}) \otimes \mathcal{K}(H) \to C^*(V_J, F) \to C(\mathbb{S}^1) \otimes \mathcal{K}(H) \to 0,$$

and its structure is uniquely defined by the KK-theory invariant

$$\operatorname{Index} C^*(\mathbb{R} \ltimes_J \mathbb{H}_3) = ([\gamma_1], [\gamma_2]),$$

where

$$[\gamma_1] \in KK(C(\mathbb{S}^1), C^*(V_J, F)),$$
$$[\gamma_2] = (1, 1) \in KK(\mathbb{S}^1, \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$$

(7) Finally, the C^*-algebra $C^*(\mathbb{R} \ltimes \mathbb{H}_3)$ of the real diamond group can be included in three subsequent extensions

$$(\gamma_1) \qquad 0 \to C(\mathbb{S}^2 \vee \mathbb{S}^2) \otimes \mathcal{K}(H) \to C^*(\mathbb{R} \ltimes \mathbb{H}_3) \to A^1 \to 0,$$

$$(\gamma_2) \qquad 0 \to C(\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2) \otimes \mathcal{K}(H) \to A^1 \to A^2 \to 0,$$

$$(\gamma_3) \qquad 0 \to \mathbb{C}^4 \otimes \mathcal{K}(H) \to A^2 \to C(\mathbb{S}^1) \to 0,$$

and the structure of $C^*(\mathbb{R} \ltimes \mathbb{H}_3)$ is uniquely defined by the KK-theory invariant

$$\operatorname{Index} C^*(\mathbb{R} \ltimes \mathbb{H}_3) = ([\gamma_1], [\gamma_2], [\gamma_3]),$$

where

$$[\gamma_1] = (1, 1) \in KK(A^1, \mathbb{C}(\mathbb{S}^2 \vee \mathbb{S}^2)) = \mathbb{Z} \oplus \mathbb{Z},$$

$$[\gamma_2] = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \in KK(A^2, \mathbb{C}(\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2)) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^4, \mathbb{Z}^4),$$

$$[\gamma_3] = (1, 1, -1, -1) \in KK(\mathbb{C}(\mathbb{S}^1), \mathbb{C}^4) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}^4) = \mathbb{Z}^4$$

5. Deformation Quantization and Cyclic Theories

Let us finish this survey with some indication about some relations of the problem with some new developments. Recall that the group algebra of finite or compact groups are in fact some Hopf bialgebras. One deforms this Hopf bialgebra structure to obtain the corresponding quantum groups. Our problem is therefore closely related to the interesting problem to describe these quantum groups. One of the methods is deformation quantization which is closely related to the orbit method. The others which are closely related with KK-theory are the periodic cyclic (co-)homologies. We finish this survey by indicating the subjects and the author who is working in this area.

5.1. Star-Products and Star-Representations

See [Gu] and the references there.

5.2. Periodic Cyclic Homology

See [Cu], [CQ1] - [CQ3].

5.3. Chern Characters

See for example [Cu], [CQ3], [Pu].

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