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On Finiteness Theorems and Porcupine varieties in Lie algebras

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This is an abridged diary of some of the 'mathematical adventures' we encountered on our way towards a proof of the Divisibility Conjecture, formulated below.

1. Divisibility, Dispersion, and Porcupine Varieties

In this section we give a brief account of how Porcupine Varieties enter the stage naturally in connection with the Divisibility Problem. Let us start with some basic concepts and the appropriate notation from [5].

Let G be a connected Lie group, and let S be a closed subsemigroup containing the identity. We say that S is *divisible* if each of its elements has roots in S of arbitrary order, i.e.,

$$(\forall s \in S,\, n \in \mathbb{N})(\exists s_n \in S) \quad (s_n)^n = s.$$

We say that S is *exponential* if every point of S lies on a one-parameter semigroup lying entirely in S. One sees immediately that an exponential semigroup is divisible, and it is a principal result of [10] that the converse holds.

The property that a semigroup be divisible is quite restrictive, and it reasonable to expect that one could achieve a classification of such semigroups. As a beginning point one would like to pull the problem back to the Lie algebra level. The Lie theory of semigroups provides the machinery for such a transfer.

Indeed, let $\mathfrak{g}$ be the Lie algebra of a connected Lie group G containing a closed subsemigroup S and let $\exp\colon \mathfrak{g} \to G$ denote the exponential function. Recall first that we denote the set of all subtangent vectors at 0 of a subset A of $\mathfrak{g}$ with $L(A)$; for $B \subseteq G$ we let $\mathbf{L}(B) = L(\exp^{-1} B)$. Then the *tangent wedge* $W = \mathbf{L}(S)$ of S satisfies $W = \{x \in \mathfrak{g} : \exp \mathbb{R}^+{\cdot}x \subseteq S\}$; it is a closed wedge in $\mathfrak{g}$, invariant under the inner automorphisms induced by the elements of its *edge* $H(W) = W \cap -W$; that is, $W = e^{\operatorname{ad} H(W)}W$. Conversely, a wedge W satisfying $W = e^{\operatorname{ad} H(W)}W$ is called a *Lie wedge*. The closed subsemigroup S is said to be *infinitesimally generated* if it coincides with the smallest closed subsemigroup containing the exponential image of its Lie wedge W, i.e., if $S = \overline{\langle \exp W \rangle}$; it is said to be *weakly exponential* if $S = \overline{\exp W}$, and one sees at once that S is exponential in the sense of the definition above if and only if $S = \exp W$.

Thus we want to study and characterize those Lie wedges W for which $\exp W$ is a (divisible!) semigroup. Unfortunately, this notion involves G and the exponential mapping; it does not translate to an inherent proberty of W and $\mathfrak{g}$ alone. But a modified notion, that of local divisibility does carry over. The semigroup S is said to be *locally divisible* if there exists an identity neighborhood U such that every element of U has roots of all orders *which lie in* $U \cap S$ (not merely in S). We say that a Lie wedge W is a Lie semialgebra if there is a Campbell-Hausdorff 0-neighborhood B in $\mathfrak{g}$ such that $B \cap W$ is a local semigroup. Using the local bijectivity of exp one sees readily that the Lie wedge W of a closed subsemigroup S is a semialgebra if and only if S is locally divisible. (For more details we refer to [5] and the papers [10], [12], [13].)

Thanks to the work of EGGERT [4], a complete classification of Lie semialgebras is now available. In principle, therefore, we know what a locally divisible subsemigroup of a Lie group looks like, and we have a fairly good understanding of how to construct them (at least locally) from scratch.

While it is easy to see that locally divisible closed subsemigroups need not be divisible (there are, e.g., counterexamples in the universal covering group of $\mathrm{Sl}(2, \mathbb{R})$), there was much evidence to support the following conjecture:

The Divisibility Conjecture. (First version) *Every closed divisible subsemigroup of a Lie group is locally divisible.*

In the light of our earlier remarks this may be rephrased:

The Divisibility Conjecture. (Second version) *The Lie wedge of any divisible closed subsemigroup of Lie group is a Lie semialgebra.*

A verification of this conjecture is crucial to our program since it allows us to invoke EGGERT's classification of Lie semialgebras.

There is indeed now a proof of this conjecture. However, it has turned out to be amazingly hard and long. The efforts to establish such a proof were very fruitful and rewarding, since they led to a variety of problems which are rather deep and of independent interest. We report here on some of these.

The main difficulty in the proof of the Divisibility Conjecture lies in controlling points near the origin. We know [10] that each point s in a divisible closed semigroup S is the exponential image $s = \exp w$ of *some* vector w in its tangent wedge W. But in order to show that S is *locally* divisible one has to find such a vector $w \in W$ *in the vicinity of* 0 whenever s lies in the vicinity of $\mathbf{1}$. We have to exclude the possibility that in every sufficiently small $\mathbf{1}$-neighborhood U there might lurk pockets of points s which can be reached only via 'long detours', i.e., by one-parameter subsemigroups of S which leave and re-enter U at least once before arriving at s. The following definition is taken from [5], p.461.

Definition 1.1. Let W be a wedge in the Lie algebra $\mathfrak{g}$ of a Lie group G (not necessarily a Lie wedge). Then W is said to *disperse in* G if there exists a Campbell Hausdorff neighborhood B in $\mathfrak{g}$ such that a point $p \in \exp B$ lies in $\exp W$ if and only if it lies in $\exp(B \cap W)$. In other words, W disperses in G if the closure of the set $\exp^{-1}(\exp W) \setminus W$ does not contain 0, or equivalently, if $\mathbf{L}(\exp^{-1}(\exp W) \setminus W) = \emptyset$. ∎

Thus if W disperses in G then every long detour one-parameter subsemigroup steering to a point in some vicinity of $\mathbf{1}$ has a shortcut not leaving this vicinity.

It can be shown that the Lie wedge W of a weakly exponential subsemigroup S is a semialgebra if and only if it disperses in G.

The above discussion has given some indication why we are interested in the set $\mathbf{L}(\exp^{-1}(\exp W)\setminus W)$. Our next proposition relates the non-zero vectors in $\mathbf{L}(\exp^{-1}(\exp W)\setminus W)$ with non-zero vectors in the closure of the set $\mathrm{comp}_G(\mathfrak{g}) = \{k \in \mathfrak{g} \mid \exp k$ lies in a compact subgroup of $G\}$.

Proposition 1.2. ([13]) *Let W be the tangent wedge of a closed subsemigroup S of G. Then the following assertions hold:*

(i) $\mathbf{L}(\exp^{-1}(\exp W)\setminus W)$ *lies in the boundary ∂W of W in $\mathfrak{g}$.*
(ii) *W disperses in G if and only if* $\mathbf{L}(\exp^{-1}(\exp W)\setminus W) = \emptyset$.
(iii) *If W has interior points and* $\mathbf{L}(\exp^{-1}(\exp W)\setminus W)$ *is nowhere dense in the boundary ∂W then W disperses in G, and thus W is a semialgebra whenever S is divisible.*
(iv) *If x is a non-zero subtangent vector of* $\exp^{-1}(\exp W)\setminus W$ *then there exists a non-zero vector $y \in H(W)\cap \overline{\mathrm{comp}_G(\mathfrak{g})}$ with $[x,y]=0$.* ∎

(Note that $0 \notin \exp^{-1}(\exp W)\setminus W$, so $\mathbf{L}(\exp^{-1}(\exp W)\setminus W) \neq \emptyset$ always implies that $\mathbf{L}(\exp^{-1}(\exp W)\setminus W)$ contains a non-zero vector.)

For any $y \in \mathfrak{g}$ let us write $\mathfrak{z}(y,\mathfrak{g})$ for the centralizer of y in $\mathfrak{g}$ and abbreviate $H(W)\cap \overline{\mathrm{comp}_G(\mathfrak{g})}$ by $\mathfrak{e}$. Using this notation we can write the above assertion (iv) in the form

$$\mathbf{L}(\exp^{-1}(\exp W)\setminus W)\setminus\{0\} \subseteq \{x \in \mathfrak{g} \mid (\exists y \in \mathfrak{e})\, y \neq 0,\, [x,y]=0\}.$$

If we let $\mathcal{A}$ denote the set of all abelian subalgebras of $\mathfrak{g}$ then the union at the right of this expression is

$$\mathfrak{Z}(\mathfrak{e}) := \bigcup\{\mathfrak{z}(y,\mathfrak{g}) \mid 0 \neq y \in \mathfrak{e}\} = \bigcup\{\mathfrak{a} \mid \mathfrak{a} \in \mathcal{A} \text{ and } \mathfrak{a}\cap\mathfrak{e} \neq \{0\}\}.$$

This is an example of a general construction:

The Porcupine Construction: Let $\mathfrak{e}$ be a fixed subset of a Lie algebra $\mathfrak{g}$, and $\mathcal{A}$ a class of subalgebras of $\mathfrak{g}$. Then form the union

$$\mathfrak{U} := \bigcup\{\mathfrak{a} \in \mathcal{A} \mid \mathfrak{a}\cap\mathfrak{e} \not\subseteq \{0\}\}$$

of all those elements in the class $\mathcal{A}$ which hit the set $\mathfrak{e}$ non-trivially. ∎

(The mental picture behind this concept is that of a porcupine shooting its quills at the target $\mathfrak{e}$; for a more detailed description of the situation we refer to the section on the hystrix in [15], p.78.)

No reasonable result can be expected unless $\mathfrak{e}$ is at least a vector space, better still a subalgebra. Hence for the purposes of attacking the Divisibility

Conjecture we have to have the information that $H(W) \cap \overline{\mathrm{comp}_G(\mathfrak{g})}$ is a subalgebra. This is the case if G has a unique maximal compact subgroup. We therefore formulate

The Restricted Divisibility Conjecture. *If the Lie wedge of a divisible closed subsemigroup of a Lie group with a unique maximal compact subgroup has inner points in $\mathfrak{g}$ then it is a Lie semialgebra.*

A confirmation of the Restricted Divisibility Conjecture will by no means finish a proof of the Divisibility Conjecture, but it provides a very important step in the latter and is an instructive special case. We shall deal here with a proof of the Restricted Divisibility Conjecture and the independent information on Lie algebras which had to be accumulated to settle it. The assertions (iii) and (iv) of Proposition 1.2 show that the Lie wedge W of a weakly exponential subsemigroup S is a semialgebra whenever W has interior points and $\mathfrak{Z}(H(W) \cap \mathrm{comp}_G(\mathfrak{g}))$ is nowhere dense in the boundary ∂W of W. This will be the case, as we shall see, if $\mathfrak{Z}(\mathfrak{e})$ is closed and sufficiently 'smooth', say closed in the Zariski topology, and small enough, say, has a codimension in $\mathfrak{g}$ exceeding 1. Thus we are faced with

Problem 1: Find conditions under which the following conditions are satisfied:
(i) $\mathfrak{Z}(\mathfrak{e})$ is Zariski closed in $\mathfrak{g}$,
(ii) $\dim \mathfrak{Z}(\mathfrak{e}) \leq \dim \mathfrak{g} - 2$. ■

With this goal in mind let us record first some preliminary observations. In connection with divisibility we are primarily interested in the porcupine construction when applied to $\mathfrak{e} = H(W) \cap \overline{\mathrm{comp}_G(\mathfrak{g})}$ and the class $\mathcal{A}$ of abelian subalgebras. It is helpful and instructive to consider the class $\mathcal{C}$ of all Cartan subalgebras of $\mathfrak{g}$ as well.

More Porcupines. Set

$$\mathfrak{H}(\mathfrak{e}) = \bigcup \{\mathfrak{h} \mid \mathfrak{h} \in \mathcal{C} \text{ and } \mathfrak{h} \cap \mathfrak{e} \neq \{0\}\};$$

$$\begin{aligned}\mathfrak{N}(\mathfrak{e}) &= \{x \in \mathfrak{g} \mid (\exists y \in \mathfrak{e} \setminus \{0\}, n \in \mathbb{N}) : (\mathrm{ad}\, x)^n(y) = 0\} \\ &= \{x \in \mathfrak{g} \mid \mathfrak{g}^0(x) \cap \mathfrak{e} \neq \{0\}\}.\end{aligned}$$

Since Cartan subalgebras are much easier to handle than centralizers of single elements, it is natural to search for conditions under which $\mathfrak{H}(\mathfrak{e})$ and $\mathfrak{Z}(\mathfrak{e})$ coincide, or at least are as intimately related as possible. The set $\mathfrak{N}(\mathfrak{e})$ is not derived directly from a porcupine construction, but contains both $\mathfrak{H}(\mathfrak{e})$ and $\mathfrak{Z}(\mathfrak{e})$; in some cases of interest it coincides with the closure of $\mathfrak{H}(\mathfrak{e})$.

Let us write $\mathrm{reg}\, \mathfrak{g}$ for the set of all regular elements in $\mathfrak{g}$.

Remark 1.3. (i) If $\mathfrak{e}$ contains a non-zero central element of $\mathfrak{g}$ then $\mathfrak{Z}(\mathfrak{e}) = \mathfrak{H}(\mathfrak{e}) = \mathfrak{N}(\mathfrak{e}) = \mathfrak{g}$.
(ii) If $\mathfrak{g}$ is nilpotent and $\mathfrak{e} \not\subseteq \{0\}$ then $\mathfrak{N}(\mathfrak{e}) = \mathfrak{H}(\mathfrak{e}) = \mathfrak{g}$.

(iii) If $\mathfrak{g}$ is a compact algebra then $\mathfrak{Z}(\mathfrak{e}) = \mathfrak{H}(\mathfrak{e}) = \mathfrak{N}(\mathfrak{e})$.
(iv) $\operatorname{reg}\mathfrak{g} \cap \mathfrak{Z}(\mathfrak{e}) \subseteq \operatorname{reg}\mathfrak{g} \cap \mathfrak{N}(\mathfrak{e}) \subseteq \mathfrak{H}(\mathfrak{e})$.

Let us pause to have a look on some illustrative examples.

Examples 1.4. (cf. [14], 3.10.) In the following examples we write $\mathbb{K}$ for the ground field of $\mathfrak{g}$, having in mind mainly the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(i) Let $\mathfrak{g}$ be the 'Heisenberg algebra' over $\mathbb{K}$, $\mathfrak{g} = \mathbb{K}{\cdot}x + \mathbb{K}{\cdot}y + \mathbb{K}{\cdot}z$ with $[x,y] = z$ and $[x,z] = [y,z] = 0$, and $\mathfrak{e} = \mathbb{K}{\cdot}x$. Then $\mathfrak{e}$ is ideal free, i.e., does not contain nonzero ideals of $\mathfrak{g}$, and $\mathfrak{Z}(\mathfrak{e}) = \mathbb{K}{\cdot}x + \mathbb{K}{\cdot}z$, but $\mathfrak{N}(\mathfrak{e}) = \mathfrak{H}(\mathfrak{e}) = \mathfrak{g}$.

(ii) Let $\mathfrak{g}$ be the 'motion algebra,' $\mathfrak{g} = \mathbb{R}{\cdot}u + \mathbb{R}{\cdot}x + \mathbb{R}{\cdot}y$ with $[u,x] = y$, $[u,y] = -x$ and $[x,y] = 0$.
  (a) For $\mathfrak{e} = \mathbb{R}{\cdot}x$ we have $\mathfrak{H}(\mathfrak{e}) = \emptyset$, $\mathfrak{Z}(\mathfrak{e}) = \mathfrak{N}(\mathfrak{e}) = \mathbb{R}{\cdot}x + \mathbb{R}{\cdot}y$.
  (b) For $\mathfrak{e} = \mathbb{R}{\cdot}x + \mathbb{R}{\cdot}u$ (which is not a subalgebra) we have $\overline{\mathfrak{H}(\mathfrak{e})} = \mathfrak{Z}(\mathfrak{e}) = \mathfrak{N}(\mathfrak{e}) = \mathfrak{g}$.

(iii) Let $\mathfrak{g}$ be a solvable Lie algebra of type Γ_1, i.e., $\mathfrak{g} = \mathbb{K}{\cdot}x + \mathbb{K}{\cdot}y + \mathbb{K}{\cdot}z_1 + \mathbb{K}{\cdot}z_2$ with $[x,y] = y + z_1$, $[x,z_j] = j{\cdot}z_j$, $[y,x_j] = x_{j+1}$ with $z_3 = 0$ and all other brackets zero (cf. [5], p.124). Set $\mathfrak{e} = \mathbb{K}{\cdot}z_1$. Then $\mathfrak{e}$ is ideal free, $\mathfrak{H}(\mathfrak{e}) = \emptyset$, $\mathfrak{Z}(\mathfrak{e}) = \mathbb{K}{\cdot}z_1 + \mathbb{K}{\cdot}z_2$ and $\mathfrak{N}(\mathfrak{e}) = \mathbb{K}{\cdot}y + \mathbb{K}{\cdot}z_1 + \mathbb{K}{\cdot}z_2$.

(iv) Let $\mathfrak{g} = \mathrm{sl}(2,\mathbb{K})$. Then for every vector subspace $\mathfrak{e}$ (over $\mathbb{K}$) we have $\mathfrak{Z}(\mathfrak{e}) = \mathfrak{e}$ and $\mathfrak{N}(\mathfrak{e}) = C \cup \mathfrak{e}$, where C is the set of all nilpotent elements in $\mathfrak{g}$. (If $\mathbb{K} = \mathbb{R}$ then C is the two-dimensional cone-surface $\{x \in \mathfrak{g} \mid k(x,x) = 0\}$ which bounds the compact elements of $\mathfrak{g}$.) Furthermore, $\mathfrak{H}(\mathfrak{e}) \neq \emptyset$ if and only if $\mathfrak{e}$ contains regular elements, and $\mathfrak{H}(\mathfrak{e})$, if non-empty, is made up of 0 and the regular elements contained in $\mathfrak{e}$. Pick elements h, p, q in $\mathfrak{g}$ with $[h,p] = 2p$, $[h,q] = -2q$ and $[p,q] = h$. Then
  (a) for $\mathfrak{e} = \mathbb{K}{\cdot}p$ we have $\mathfrak{H}(\mathfrak{e}) = \emptyset \neq \mathfrak{Z}(\mathfrak{e}) = \mathfrak{e}$ and $\mathfrak{N}(\mathfrak{e}) = C$;
  (b) for $\mathfrak{e} = \mathbb{K}{\cdot}(p-q)$ we have $\mathfrak{H}(\mathfrak{e}) = \mathfrak{Z}(\mathfrak{e}) = \mathfrak{e}$ and $\mathfrak{N}(\mathfrak{e}) = C \cup \mathbb{K}{\cdot}(p-q)$;
  (c) for $\mathfrak{e} = \mathbb{K}{\cdot}h + \mathbb{K}{\cdot}p$ we have $\mathfrak{H}(\mathfrak{e}) = \mathfrak{e} \setminus (\mathbb{K} \setminus \{0\}){\cdot}p \neq \mathfrak{Z}(\mathfrak{e}) = \mathfrak{e}$ and $\mathfrak{N}(\mathfrak{e}) = C \cup (\mathbb{K}{\cdot}h + \mathbb{K}{\cdot}p)$.

Note that in case (c) the porcupine set $\mathfrak{H}(\mathfrak{e})$ is a dense subset of $\mathfrak{Z}(\mathfrak{e})$, it is neither open nor closed in $\mathfrak{Z}(\mathfrak{e})$. ■

We next record that if $\mathfrak{e}$ is a vector subspace of $\mathfrak{g}$ then the sets $\mathfrak{Z}(\mathfrak{e})$ and $\mathfrak{N}(\mathfrak{e})$ are Zariski closed subsets — varieties. (This is a consequence of the fact that an element $x \in \mathfrak{g}$ belongs to $\mathfrak{Z}(\mathfrak{e})$, respectively, $\mathfrak{N}(\mathfrak{e})$ if and only if the restriction of $\operatorname{ad} x$, respectively, $(\operatorname{ad} x)^{\dim \mathfrak{g}}$ to $\mathfrak{e}$ is singular). Note that $\operatorname{comp}_G(\mathfrak{g})$ is a subalgebra of $\mathfrak{g}$ iff G has a unique maximal compact subgroup. In this case $H(W) \cap \operatorname{comp}_G(\mathfrak{g})$ is a subalgebra of $\mathfrak{g}$.

Proposition 1.5. ([14], Proposition 3.6) *Let $\mathfrak{e}$ be a vector subspace of the Lie algebra $\mathfrak{g}$. Then the following assertions hold:*
  (i) *The sets $\mathfrak{Z}(\mathfrak{e})$ and $\mathfrak{N}(\mathfrak{e})$ are Zariski closed subsets of $\mathfrak{g}$. Thus the Zariski closure $\operatorname{Zcl}(\mathfrak{H}(\mathfrak{e}))$ is contained in $\mathfrak{N}(\mathfrak{e})$.*
  (ii) *If $[\mathfrak{h} \cap \mathfrak{e}, \mathfrak{e}] = \{0\}$ for every Cartan subalgebra of $\mathfrak{g}$—in particular, if $\mathfrak{g}$ has abelian Cartan subalgebras—then $\mathfrak{H}(\mathfrak{e})$ is a subset of $\mathfrak{Z}(\mathfrak{e})$, and we have $\operatorname{Zcl}(\mathfrak{Z}(\mathfrak{e}) \cap \operatorname{reg}\mathfrak{g}) = \operatorname{Zcl}(\mathfrak{H}(\mathfrak{e})) \subseteq \mathfrak{Z}(\mathfrak{e})$.* ■

The above Examples 1.4(iii)(b) and 1.4(iii)(c) show that $\mathfrak{N}(\mathfrak{e})$ need not be an irreducible variety. (But if $\mathfrak{N}(\mathfrak{e})$ is irreducible and $\mathfrak{H}(\mathfrak{e}) \neq \emptyset$ then $\mathrm{Zcl}\big(\mathfrak{H}(\mathfrak{e})\big) = \mathfrak{N}(\mathfrak{e})$.) But Proposition 1.5(i) settles Part (a) of Problem 1, and 1.5(ii) gets us started on Part (b) of Problem 2, because $\mathfrak{H}(\mathfrak{e})$ is more accesible to explicit calculations than $\mathfrak{Z}(\mathfrak{e})$ itself.

More subtle information becomes available under special circumstances. Recall that a subset of a Lie algebra is said to be *reductively embedded* if $\mathrm{ad}\, x$ is semisimple on $\mathfrak{g}$ for each $x \in \mathfrak{e}$. We shall say that a subalgebra $\mathfrak{e}$ of $\mathfrak{g}$ is *Cartan reductive* if every Cartan subalgebra of $\mathfrak{g}$ meets $\mathfrak{e}$ in a reductively embedded subalgebra.

Theorem 1.6. ([14], Theorem 3.8) *Suppose that $\mathfrak{e}$ is a nonzero vector subspace of a Lie algebra $\mathfrak{g}$.*

(i) *If $\mathfrak{e}$ is a Cartan reductive subalgebra of $\mathfrak{g}$ containing an $x \neq 0$ such that $\mathrm{ad}\, x$ is semisimple then $\emptyset \neq \mathrm{Zcl}\big(\mathfrak{H}(\mathfrak{e})\big) \subseteq \mathfrak{Z}(\mathfrak{e}) \subseteq \mathfrak{N}(\mathfrak{e})$.*

(ii) *If $\mathfrak{e}$ is reductively embedded then $\mathfrak{Z}(\mathfrak{e}) = \mathrm{Zcl}\big(\mathfrak{H}(\mathfrak{e})\big) \subseteq \mathfrak{N}(\mathfrak{e})$.* ■

For the purposes of dealing with the Restricted Divisibility Problem it will suffice to restrict our attention to the case where $\mathfrak{e}$ is a subalgebra of $\mathfrak{g}$. Furthermore, after factoring out the maximal G-normal subgroup of S it is not hard to see that we may suppose as well that $\mathfrak{e}$ is *ideal-free*, i.e., does not contain any non-zero ideal of $\mathfrak{g}$ (cf. [12]).

2. Porcupines which are proper subsets

In the last section we have set up the task of finding conditions under which $\mathfrak{Z}(\mathfrak{e})$ is ‘very thin.’ But it is not even clear whether there are workable conditions preventing $\mathfrak{Z}(\mathfrak{e})$ from being $\mathfrak{g}$—we know that $\mathfrak{Z}(\mathfrak{e}) = \mathfrak{g}$ if the center of $\mathfrak{g}$ meets $\mathfrak{e}$ non-trivially (cf. 1.3). Our next proposition essentially says that (under a mild additional condition) $\mathfrak{Z}(\mathfrak{e}) = \mathfrak{g}$ if and only if every Cartan subalgebra of $\mathfrak{g}$ meets $\mathfrak{e}$ non-trivially.

Proposition 2.1. *Let $\mathfrak{e}$ be a vector subspace of $\mathfrak{g}$ such that $[\mathfrak{h} \cap \mathfrak{e}, \mathfrak{h}] = \{0\}$ for every Cartan subalgebra of $\mathfrak{g}$. Then the following statements are equivalent:*

(i) $\mathfrak{g} \neq \mathfrak{Z}(\mathfrak{e})$;

(ii) *there exists a regular element x with $x \notin \mathfrak{N}(\mathfrak{e})$;*

(iii) *there exists a Cartan subalgebra $\mathfrak{h}$ with $\mathfrak{e} \cap \mathfrak{h} = \{0\}$.*

Proof. From 1.3(iv) we know that $\mathrm{reg}\, \mathfrak{g} \cap \mathfrak{N}(\mathfrak{e}) \subseteq \mathfrak{H}(\mathfrak{e})$, and from 1.5(ii) that $\mathfrak{H}(\mathfrak{e}) \subseteq \mathfrak{Z}(\mathfrak{e})$. Since $\mathfrak{Z}(\mathfrak{e})$ is closed we conclude that

$$\overline{\mathrm{reg}\, \mathfrak{g} \cap \mathfrak{N}(\mathfrak{e})} \subseteq \overline{\mathfrak{H}(\mathfrak{e})} \subseteq \mathfrak{Z}(\mathfrak{e}) \subseteq \mathfrak{N}(\mathfrak{e}).$$

Moreover, since $\mathfrak{N}(\mathfrak{e})$ is closed and the regular elements are dense in $\mathfrak{g}$ we have $\mathfrak{N}(\mathfrak{e}) = \mathfrak{g}$ if and only if $\mathrm{reg}\, \mathfrak{g} \subseteq \mathfrak{N}(\mathfrak{e})$. Thus $\mathfrak{Z}(\mathfrak{e}) = \mathfrak{g}$ if and only if $\mathfrak{N}(\mathfrak{e}) = \mathfrak{g}$. By the defintion of $\mathfrak{N}(\mathfrak{e})$ this establishes the assertion. ■

Proposition 2.1 shows that our task involves as an intermediate step the solution of

Problem 2: Suppose that $\mathfrak{e}$ is an ideal-free subalgebra of $\mathfrak{g}$ such that $[\mathfrak{h}\cap\mathfrak{e},\mathfrak{h}] = \{0\}$ for all Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$. Find additional conditions on $\mathfrak{e}$ under which there exists a Cartan subalgebra $\mathfrak{h}$ with $\mathfrak{h}\cap\mathfrak{e} = \{0\}$. ■

Note that such conditions would imply that the codimension of $\mathfrak{e}$ in $\mathfrak{g}$ is not less than the rank of $\mathfrak{g}$. Thus it is hopeless to try a proof of $\mathfrak{Z}(\mathfrak{e}) \neq \mathfrak{g}$ without, say, the assumption that $\mathfrak{e}$ is ideal-free.

Since there are only finitely many conjugacy classes of Cartan subalgebras of $\mathfrak{g}$ it is natural to fix some Cartan subalgebra $\mathfrak{h}$ and to consider the intersections $\mathfrak{h}\cap\varphi(\mathfrak{e})$, where φ is an inner automorphism of $\mathfrak{g}$. Let m be the minimal dimension of the vector spaces $\mathfrak{h}\cap\varphi(\mathfrak{e})$. We want to show that $m = 0$.

Let $\mathcal{G}$ be the group of all inner automorphisms of $\mathfrak{g}$, endowed with its intrinsic topology, and $\mathcal{O}$ the subset of those $\varphi\in\mathcal{G}$ which satisfy $\dim\big(\mathfrak{h}\cap\varphi(\mathfrak{e})\big) = m$. We write $\Sigma(\mathfrak{h})$ for the space of m-dimensional vector subspaces of $\mathfrak{h}$, with the usual topology. Then $\mathcal{O}$ is Zariski open in $\mathcal{G}$ and it can be shown that the map

$$\Delta\colon\mathcal{O}\to\Sigma(\mathfrak{h}),\quad \varphi\mapsto\mathfrak{h}\cap\varphi(\mathfrak{e})$$

is continuous. (We omit all details, cf. [14], Lemma 4.2).

We fix an element $\psi\in\mathcal{O}$ and define for every subset $\mathcal{X}\subseteq\mathcal{G}$ the vector space $\mathfrak{e}(\mathcal{X}) = \bigcap_{\varphi\in\mathcal{X}}\varphi\psi(\mathfrak{e})$. If $\mathcal{U}$ ranges through the neighborhoods of $\mathbf{1}$ in $\mathcal{G}$, then the spaces $\mathfrak{e}(\mathcal{U})$ form an updirected family of vector subspaces of $\mathfrak{h}$, so for dimensional reasons there must be a $\mathcal{U}_1$ such that $\mathcal{U}\subseteq\mathcal{U}_1$ implies $\mathfrak{e}(\mathcal{U}) = \mathfrak{e}(\mathcal{U}_1)$.

Suppose now that Δ *is locally constant.*
Then there exists an open symmetric identity neighborhood $\mathcal{V}$ in $\mathcal{G}$, so small that $\mathcal{V}\mathcal{V}\subseteq\mathcal{U}_1$, and such that Δ is constant on $\mathcal{V}\psi$, that is, $\mathfrak{h}\cap\varphi\psi(\mathfrak{e}) = \mathfrak{h}\cap\psi(\mathfrak{e})$ for all $\varphi\in\mathcal{V}$. Then

$$\mathfrak{h}\cap\mathfrak{e}(\mathcal{V}) = \bigcap_{\varphi\in\mathcal{V}}\big(\mathfrak{h}\cap\varphi\psi(\mathfrak{e})\big) = \mathfrak{h}\cap\psi(\mathfrak{e}).$$

On the other hand, $\varphi\in\mathcal{V}$ implies $\varphi\big(\mathfrak{e}(\mathcal{V})\big)\supseteq\mathfrak{e}(\mathcal{V}\mathcal{V})\supseteq\mathfrak{e}(\mathcal{U}_1) = \mathfrak{e}(\mathcal{V})$. Since $\mathcal{V} = \mathcal{V}^{-1}$ we therefore have $\varphi\big(\mathfrak{e}(\mathcal{V})\big) = \mathfrak{e}(\mathcal{V})$ for all $\varphi\in\mathcal{V}$. But $\mathcal{V}$ generates $\mathcal{G}$, so the set $\mathfrak{e}(\mathcal{V})$ is invariant under $\mathcal{G}$ and is therefore an ideal contained in $\psi(\mathfrak{e})$. Since ψ is an automorphism and $\mathfrak{e}$ is ideal-free it follows that $\mathfrak{e}(\mathcal{V}) = \{0\}$ and we get $\mathfrak{h}\cap\psi(\mathfrak{e}) = \mathfrak{h}\cap\mathfrak{e}(\mathcal{V}) = \{0\}$. We summarize this discussion in the following Lemma.

Lemma 2.2.(cf. [14], Lemmas 4.2 and 4.3) *Let* $\mathfrak{e}$ *be an ideal-free subalgebra, and* $\mathfrak{h}$ *a Cartan subalgebra of our given Lie algebra* $\mathfrak{g}$. *We assume that the above map* Δ *is locally constant. Then there exists an inner automorphism* $\psi\in\mathcal{G}$ *such that* $\mathfrak{e}\cap\psi(\mathfrak{h}) = \{0\}$. ■

This encouraging, though somewhat incomplete result motivates the hunt for a practicable condition under which Δ is locally constant. Since Δ must be locally constant if $\{\mathfrak{h} \cap \varphi(\mathfrak{e}) \mid \varphi \in \mathcal{G}\}$ is finite, an obvious strategy is to look for finiteness.

Problem 3: Suppose that $\mathfrak{e}$ is an ideal-free subalgebra of $\mathfrak{g}$ such that $[\mathfrak{k} \cap \mathfrak{e}, \mathfrak{k}] = \{0\}$ for all Cartan subalgebras $\mathfrak{k}$ of $\mathfrak{g}$. Find additional conditions on $\mathfrak{e}$ under which for a fixed Cartan subalgebra $\mathfrak{h}$ the set $\{\mathfrak{h} \cap \varphi(\mathfrak{e}) \mid \varphi \in \mathcal{G}\}$ is finite. ∎

We note that an adequate answer to Problem 3 provides one for Problem 2.

Warning: The condition '$\{\mathfrak{h} \cap \varphi(\mathfrak{e}) \mid \varphi \in \mathcal{G}\}$ is finite' is *not* equivalent to '$\{\mathfrak{e} \cap \varphi(\mathfrak{h}) \mid \varphi \in \mathcal{G}\}$ is finite.'

3. Intersections with Cartan subalgebras

Problem 3 of the last section triggers associations with well-known finiteness properties of Cartan subalgebras (notably in semisimple Lie algebras). Thus a natural start for our attempts is the investigation of the case where $\mathfrak{e}$ is a Cartan subalgebra itself, producing

Problem 4: Find conditions under which the Lie algebra $\mathfrak{g}$ is *Cartan finite*, that is, the set

$$\{\mathfrak{h} \cap \mathfrak{k} \mid \mathfrak{k} \text{ is a Cartan subalgebra of } \mathfrak{g}\}$$

is finite for every fixed Cartan subalgebra $\mathfrak{h}$.

We first remark that it suffices to consider this question for complex Lie algebras $\mathfrak{g}$ (this follows easily from the fact that the Cartan subalgebras of $\mathfrak{g}_{\mathbb{C}}$ are the complexifications of the Cartan subalgebras of $\mathfrak{g}$). Next we notice that if the intersection of two Cartan algebras contains a regular element x, then the two Cartan algebras are both equal to $\mathfrak{g}^0(x)$, the nilspace of $\operatorname{ad} x$. Thus a non-trivial intersection of a fixed Cartan subalgebra $\mathfrak{h}$ with another one, $\mathfrak{k}$, must miss the regular elements, and hence be contained in the kernel of some root. We therefore wonder whether $\mathfrak{h} \cap \mathfrak{k}$ can be characterized uniquely as the intersection of the kernels of certain roots. Since there are only finitely many roots, a positive answer would confirm the *conjecture* that every Lie algebra is Cartan finite and thus answer Problem 4. In addition, this information would yield an estimate for the number of such intersections $\mathfrak{h} \cap \mathfrak{k}$.

Unfortunately, the following example shows that we cannot hope to confirm this tentative conjecture in general, even if $\mathfrak{h}$ is abelian.

Example 3.1. Let $\mathfrak{g}$ denote the Lie algebra of all matrices of the form

$$[c; x, z; u, v, w] := \begin{pmatrix} c & x & z & u \\ 0 & c & 0 & v \\ 0 & 0 & c & w \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c, x, z, u, v, w \in \mathbb{C}.$$

(Note that $\mathfrak{g}$ is the Lie algebra of an algebraic subgroup of $\mathrm{Gl}(\mathbb{C},4)$.) Further, we define $\mathfrak{h}:=\{[c;x,z;0,0,0;0,0]: c,x,z\in\mathbb{C}\}$; this is an abelian Cartan subalgebra. The set Λ of roots contains only one element λ, with $\lambda([c;x,z;0,0,0])=c$, and the corresponding root space is

$$\mathfrak{g}^{\lambda_1}=\{[0;0,0;u,v,w]: u,v,w\in\mathbb{C}\}.$$

But for every $v\in\mathbb{C}$ the subspace $\mathfrak{k}_v:=\{[c;x,z;xv+z;cv;c,0,0]\mid c,x,z\in\mathbb{C}\}$ is a Cartan subalgebra and the intersections $\mathfrak{h}\cap\mathfrak{k}_v=\{[0;x,-xv;0;0,0,0]\mid x\in\mathbb{C}\}$ form a one-parameter family over $\mathbb{C}$. ∎

However, a closer study of the situation shows that our conjecture is not so far off the mark. By Theorem 5.9 of [11] the conjecture is valid under fairly mild restrictions. For lack of space we only cite an important special case where the above proof strategy can be carried out.

Theorem 3.2. ([11], Proposition 5.18) *If $\mathfrak{g}$ is a Lie algebra containing a reductively embedded Cartan subalgebra then $\mathfrak{g}$ is Cartan finite.* ∎

Note that if one Cartan subalgebra of $\mathfrak{g}$ is reductively embedded then all of them are reductively embedded and abelian. If $\mathfrak{g}$ is a reductive algebra then the Cartan subalgebras are reductively embedded, so every reductive Lie algebra is Cartan finite, as is well known. Theorem 3.2 answers Problem 4.

If $\mathfrak{g}$ is Cartan finite and $\mathfrak{v}$ is any subset of a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ then the set $\{\mathfrak{v}\cap\varphi(\mathfrak{h})\mid\varphi\in\mathcal{G}\}$ is finite. Also, every reductively embedded abelian subalgebra is contained in the center of a Cartan subalgebra. Thus the following Theorem is a generalization of Theorem 3.2. Its proof follows the same strategy but requires some additional care in the details.

Theorem 3.3. ([14], Proposition 2.6) *For any reductively embedded abelian subalgebra $\mathfrak{v}$ of a Lie algebra $\mathfrak{g}$ the intersections $\mathfrak{v}\cap\mathfrak{k}$, $\mathfrak{k}$ a Cartan subalgebra, form a finite set. In particular, the set $\{\mathfrak{v}\cap\varphi(\mathfrak{h})\mid\varphi\in\mathcal{G}\}$ is finite.* ∎

Problem 3 asks for conditions under which the set $\{\mathfrak{h}\cap\varphi(\mathfrak{e})\mid\varphi\in\mathcal{G}\}$ is finite. Thus the above theorems do not directly provide answers for $\mathfrak{e}=\mathfrak{v}$, since it is not obvious that the finiteness of $\{\mathfrak{v}\cap\varphi(\mathfrak{h})\mid\varphi\in\mathcal{G}\}$ implies the finiteness of $\{\mathfrak{h}\cap\varphi(\mathfrak{v})\mid\varphi\in\mathcal{G}\}$: If $\varphi(\mathfrak{h})=\psi(\mathfrak{h})$ for some $\varphi\neq\psi\in\mathcal{G}$ then in general $\varphi^{-1}\big(\mathfrak{v}\cap\varphi(\mathfrak{h})\big)=\mathfrak{h}\cap\varphi^{-1}(\mathfrak{v})$ will not coincide with $\psi^{-1}\big(\mathfrak{v}\cap\varphi(\mathfrak{h})\big)=\mathfrak{h}\cap\psi^{-1}(\mathfrak{v})$, so it is well conceivable that there are infinitely many φ's producing different sets $\mathfrak{h}\cap\varphi(\mathfrak{v})$ but only one set $\varphi^{-1}(\mathfrak{h})\cap\mathfrak{v}$. We need a condition which guarantees that for a fixed φ and $\mathfrak{a}:=\mathfrak{v}\cap\varphi^{-1}(\mathfrak{h})$ the restrictions to $\mathfrak{a}$ of the maps $\psi\in\mathcal{G}$ with $\mathfrak{v}\cap\psi^{-1}(\mathfrak{h})=\mathfrak{a}$ form a finite family. (Note that $\mathfrak{v}\cap\psi^{-1}(\mathfrak{h})\subseteq\mathfrak{a}=\mathfrak{v}\cap\varphi^{-1}(\mathfrak{h})$ if and only if $\psi(\mathfrak{a})\subseteq\mathfrak{h}$.)

This raises

Problem 5: Suppose that $\mathfrak{a}$ is a subalgebra of a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. Find conditions under which the set

$$\Phi(\mathfrak{a},\mathfrak{b}):=\{f\colon\mathfrak{a}\to\mathfrak{b}\mid \text{there exists a }\varphi\in\mathcal{G}\text{ with }f(x)=\varphi(x),\text{ for every }x\in\mathfrak{a}\}$$

is finite for every conjugate $\mathfrak{b}$ of $\mathfrak{a}$.

If $f \in \Phi(\mathfrak{a},\mathfrak{b}) \neq \emptyset$ then $f^{-1}\Phi(\mathfrak{a},\mathfrak{b}) = \Phi(\mathfrak{a},\mathfrak{a})$, so $\Phi(\mathfrak{a},\mathfrak{b})$ is finite if and only if $\Phi(\mathfrak{a},\mathfrak{a})$ is finite. If $\mathfrak{a} = \mathfrak{h}$ and $\mathfrak{g}$ is semisimple then $\Phi(\mathfrak{a},\mathfrak{a}) = \Phi(\mathfrak{h},\mathfrak{h})$ can be identified with the traditional *Weyl group* with respect to $\mathfrak{h}$.

For easier reference in the following let us call $\mathfrak{g}$ *Weyl finite* if for every subalgebra $\mathfrak{a}$ of a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ the sets $\Phi(\mathfrak{a},\mathfrak{b})$ of Problem 5 are finite.

Lemma 3.4. *Suppose that $\mathfrak{g}$ is Weyl finite as well as Cartan finite. Then the set $\{\mathfrak{h} \cap \varphi(\mathfrak{v}) \mid \varphi \in \mathcal{G}\}$ is finite for every subset $\mathfrak{v}$ of $\mathfrak{h}$.*

Proof. ([11], 6.4) Since $\mathfrak{g}$ is supposed to be Cartan finite, the family of all intersections $\mathfrak{h} \cap \mathfrak{k}$, where $\mathfrak{k}$ is any Cartan subalgebra of $\mathfrak{g}$, is finite. A fortiori, the family $\mathfrak{I} := \{\mathfrak{a} = \mathfrak{h} \cap \varphi^{-1}(\mathfrak{h}) \mid \varphi \in \mathcal{G}\}$ is finite. Thus the intersections $\mathfrak{v} \cap \varphi(\mathfrak{a}) = \mathfrak{v} \cap \mathfrak{h} \cap \varphi(\mathfrak{h})$ form a finite set.

By assumption, the set $\Phi(\mathfrak{a},\mathfrak{b})$ is finite for any pair $\mathfrak{a},\mathfrak{b} \in \mathfrak{I}$. Thus the set

$$\mathcal{S} = \{f(\mathfrak{v} \cap \mathfrak{a}) \mid \mathfrak{a} \in \mathfrak{I},\ f \in \Phi(\mathfrak{a},\mathfrak{b}) \ \text{ for some } \mathfrak{b} \in \mathfrak{I}\}$$

is finite. But for every $\varphi \in \mathcal{G}$ we have $\mathfrak{h} \cap \varphi(\mathfrak{v}) = f(\mathfrak{v} \cap \mathfrak{a})$ where $\mathfrak{a} = \mathfrak{h} \cap \varphi^{-1}(\mathfrak{h}) \in \mathfrak{I}$, $\mathfrak{b} = \varphi(\mathfrak{a}) \in \mathfrak{I}$ and $f = (\varphi|\mathfrak{a}) \in \Phi(\mathfrak{a},\mathfrak{b})$. So $\mathfrak{h} \cap \varphi(\mathfrak{v}) \in \mathcal{S}$ and the assertion follows. ∎

Similar to the situation of Theorem 3.2, a careful study of this discussion shows that it can be adapted to the case where $\mathfrak{g}$ is not supposed to be Cartan finite, but $\mathfrak{v}$ is supposed to be reductively embedded in $\mathfrak{g}$. This yields

Lemma 3.5.([14], Proposition 2.7) *Suppose that $\mathfrak{g}$ is Weyl finite. Then the set $\{\mathfrak{h} \cap \varphi(\mathfrak{v}) \mid \varphi \in \mathcal{G}\}$ is finite for every subset $\mathfrak{v}$ of $\mathfrak{h}$ which is reductive in $\mathfrak{g}$.* ∎

Starting with the above result and exploiting the properties of reductively embedded subalgebras it is now not difficult to get a satisfying answer to Problem 3—if we take for granted that Problem 5 can be solved in a practical way.

Remember that we called a subalgebra $\mathfrak{e}$ of $\mathfrak{g}$ Cartan reductive if every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ meets $\mathfrak{e}$ in a reductively embedded subset. Note that $\mathfrak{e}$ always contains maximal abelian $\mathfrak{g}$-reductive subalgebras and each of these is contained in the center of some Cartan subalgebra $\mathfrak{h}$.

Proposition 3.6. ([14], Proposition 2.9) *Let $\mathfrak{e}$ be a Cartan reductive subalgebra of a Weyl finite Lie algebra $\mathfrak{g}$. Furthermore, we let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing a maximal abelian $\mathfrak{g}$-reductive subalgebra $\mathfrak{v}$ of $\mathfrak{e}$. Then the set $\{\mathfrak{h} \cap \varphi(\mathfrak{e}) \mid \varphi \in \mathcal{G}\}$ is finite.* ∎

In particular, this proposition gives us a satisfactory answer to Problem 3 and allows us to conclude that $\mathfrak{Z}(\mathfrak{e}) \neq \mathfrak{g}$ if $\mathfrak{e}$ is a Cartan reductive and ideal-free subalgebra of a Weyl finite Lie algebra $\mathfrak{g}$.

Let us take care of the problem which is now at hand:

Problem 6: Which Lie algebras are Weyl finite?

4. General Weyl groups

For a subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ we want to define a group which deserves the name of the "Weyl group of $\mathfrak{a}$ in $\mathfrak{g}$". This requires a little preparation. We let $\mathbb{G}$ denote the group $\langle e^{\mathrm{ad}\,\mathfrak{g}}\rangle$ of inner automorphisms of $\mathfrak{g}$ in its intrinsic Lie group topology. Now for a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ we set

$$\mathcal{N}(\mathfrak{a},\mathcal{G})=\{\varphi\in\mathcal{G}\mid\varphi(\mathfrak{a})=\mathfrak{a}\},$$
$$\mathcal{Z}(\mathfrak{a},\mathcal{G})=\{\varphi\in\mathcal{N}(\mathfrak{a},\mathcal{G})\mid(\forall a\in\mathfrak{a})\quad\varphi(a)-a\in[\mathfrak{a},\mathfrak{a}]\}.$$

Definition 4.1. We say that $\mathcal{W}(\mathfrak{a},\mathfrak{g})=\mathcal{N}(\mathfrak{a},\mathcal{G})/\mathcal{Z}(\mathfrak{a},\mathcal{G})$ is the *generalized Weyl group of* $\mathfrak{a}$ *in* $\mathfrak{g}$. ∎

We note that the generalized Weyl group acts in a natural way on $\mathfrak{a}/[\mathfrak{a},\mathfrak{a}]$ by the formula $\big(\varphi+\mathcal{Z}(\mathfrak{a},\mathcal{G})\big)(x+[\mathfrak{a},\mathfrak{a}])=\varphi(x)+[\mathfrak{a},\mathfrak{a}]$. This yields a well-defined linear action of $\mathcal{W}(\mathfrak{a},\mathfrak{g})$ on $\mathfrak{a}/[\mathfrak{a},\mathfrak{a}]$ and this linear representation is faithful on $\mathcal{W}(\mathfrak{a},\mathfrak{g})$. In particular, if $\mathfrak{a}$ is abelian, then $\mathcal{W}(\mathfrak{a},\mathfrak{g})$ acts faithfully and linearly on $\mathfrak{a}$.

Proposition 4.2. *Assume that* $\mathfrak{a}$ *is a subalgebra of a Cartan subalgebra* $\mathfrak{h}$ *satisfying* $[\mathfrak{a},\mathfrak{h}]=\{0\}$. *Then if* $\alpha\in\mathbb{G}$, $\alpha(\mathfrak{a})=\mathfrak{b}\subseteq\mathfrak{h}$, *and* $[\mathfrak{b},\mathfrak{h}]=0$, *the assignment*

$$\theta\colon\mathcal{W}(\mathfrak{a},\mathfrak{g})\to\Phi(\mathfrak{a},\mathfrak{b}),\quad\theta(\varphi{\cdot}\mathcal{Z}(\mathfrak{a},\mathcal{G}))=\alpha\varphi|\mathfrak{a}$$

is a well defined bijection. In particular, $|\Phi(\mathfrak{a},\mathfrak{b})|$ *is finite iff* $|\mathcal{W}(\mathfrak{a},\mathfrak{g})|$ *is finite. Furthermore, each member of* $\Phi(\mathfrak{a},\mathfrak{a})$ *can be viewed as a restriction and corestriction of a member of the Weyl group of* $\mathfrak{h}$.

Proof. The member of the Weyl group $\varphi\cdot\mathcal{Z}(\mathfrak{a},\mathcal{G})$ acts on $\mathfrak{a}$ by restricting φ to $\mathfrak{a}$, since $[\mathfrak{a},\mathfrak{a}]=\{0\}$. Suppose that $\beta\in\mathcal{G}$ is such that $\beta(\mathfrak{a})=\mathfrak{b}=\alpha(\mathfrak{a})$. Then $\varphi{:}=\alpha^{-1}\beta\in\mathcal{G}$ satisfies $\varphi(\mathfrak{a})=\mathfrak{a}$ and is, therefore in $\mathcal{N}(\mathfrak{a},\mathcal{G})$. If $\beta'\in\mathcal{G}$ satisfies $\beta'(y)=\beta(y)$ for all $y\in\mathfrak{a}$, then $\zeta{:}=\beta^{-1}\beta'\in\mathcal{Z}(\mathfrak{a},\mathcal{G})$ and $\beta'=\alpha\varphi\zeta$. These remarks show that θ is well defined and bijective. ∎

As a corollary, if one, hence all, Cartan subgroups of $\mathfrak{g}$ are abelian, then $\mathfrak{g}$ is Weyl finite (terminology introduced preceding Lemma 3.4) if for all subalgebras (vector subspaces) $\mathfrak{a}$ of a Cartan subalgebra $\mathfrak{h}$, the generalized Weyl group $\mathcal{W}(\mathfrak{a},\mathfrak{g})$ is finite. Thus for Lie algebras with abelian Cartan subalgebras we have an affirmative answer to Problem 6 provided we have an affirmative answer to the following Problem:

Problem 7: Are the generalized Weyl groups $\mathcal{W}(\mathfrak{a},\mathfrak{g})$ finite for all subalgebras $\mathfrak{a}$ of a Cartan subalgebra such that $[\mathfrak{a},\mathfrak{h}]\subseteq[\mathfrak{a},\mathfrak{a}]$? ∎

The answer is given by the following result:

Theorem 4.3. ([11], Theorem 3.9) *Let $\mathfrak{a}$ be a subalgebra of a real or complex Lie algebra $\mathfrak{g}$ which is contained in a Cartan subalgebra $\mathfrak{h}$ such that $[\mathfrak{a}, \mathfrak{h}] \subseteq [\mathfrak{a}, \mathfrak{a}]$. Then $\mathcal{W}(\mathfrak{a}, \mathfrak{g})$, and in particular $\mathcal{W}(\mathfrak{h}, \mathfrak{g})$ is finite.* ■

The proof of this fact uses the theory of algebraic groups and is rather lengthy (see [11]). We now have covered all of our problems except for Problem 1, and we have, in particular, the following theorem which answers Problem 2:

Theorem 4.4. ([14], Theorem 4.4) *Let $\mathfrak{g}$ be a Lie algebra with a Cartan reductive subalgebra $\mathfrak{e}$ which does not contain any nonzero ideal of $\mathfrak{g}$. Then the following assertions hold:*

(i) *In each conjugacy class of Cartan subalgebras of $\mathfrak{g}$ there is a member $\mathfrak{h}$ which satisfies $\mathfrak{h} \cap \mathfrak{e} = \{0\}$.*

(ii) $\dim \mathfrak{g} - \dim \mathfrak{e} \geq \operatorname{rank} \mathfrak{g}$.

(iii) $\mathfrak{N}(\mathfrak{e}) \neq \mathfrak{g}$. *A fortiori,* $\mathfrak{Z}(\mathfrak{e}) \neq \mathfrak{g}$ *and* $\operatorname{Zcl}\big(\mathfrak{H}(\mathfrak{e})\big) \neq \mathfrak{g}$. ■

It is interesting to observe that for the case that $\mathfrak{a}$ is a Cartan subalgebra $\mathfrak{h}$ the Weyl group has an alternate geometric interpretation which we shall briefly describe. We let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\exp\colon \mathfrak{g} \to G$, be an exponential function to some corresponding connected Lie group G. (The complex set-up is analogous but simpler.) We let Λ denote the (finite!) set of nonzero roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C}$. We set

$$N(\mathfrak{h}, G) = \{g \in G : \operatorname{Ad}(g)\mathfrak{h} \subseteq \mathfrak{h}\},$$
$$Z(\mathfrak{h}, G) = \{g \in N(\mathfrak{h}, G) \colon (\forall h \in \mathfrak{h})\ \operatorname{Ad}(g)(h) - h \in [\mathfrak{h}, \mathfrak{h}]\}.$$

We may then identify the Weyl group $\mathcal{W}(\mathfrak{h}, \mathfrak{g})$ with $N(\mathfrak{h}, G)/Z(\mathfrak{h}, G)$, irrespective of the particular choice of the Lie group G as long as G is connected and has $\mathfrak{g}$ as its Lie algebra. (see [11], Lemma 3.3). Every $g \in N(\mathfrak{h}, G)$ defines an automorphism $\operatorname{Ad}(g)$ of $\mathfrak{g}$ preserving $\mathfrak{h}$ as a whole. Thus it induces a unique automorphism $\operatorname{Ad}_\mathbb{C}(g)$ on $\mathfrak{g}_\mathbb{C}$ respecting $\mathfrak{h}_\mathbb{C}$. If $\lambda\colon \mathfrak{h}_\mathbb{C} \to \mathbb{C}$ is a root from Λ, then $\lambda \circ (\operatorname{Ad}_\mathbb{C}(g)|\mathfrak{h}_\mathbb{C})^{-1}$ is again a root. This is in fact readily verified to be the case for any automorphism of $\mathfrak{g}_\mathbb{C}$ preserving $\mathfrak{h}_\mathbb{C}$ in place of $\operatorname{Ad}_\mathbb{C}(\mathfrak{g})$. (See e.g. [9].) Thus if $S(\Lambda)$ denotes the permutation group of all bijections of Λ, then $S(\Lambda)$ is a group of order $|\Lambda|!$ and the function

$$\pi\colon N(\mathfrak{h}, G) \to S(\Lambda), \quad \pi(g)(\lambda) = \lambda \circ (\operatorname{Ad}_\mathbb{C}(g)|\mathfrak{h}_\mathbb{C})^{-1}$$

is a homomorphism of groups. Its image is a finite group of permutations of Λ, which we shall call the *geometric Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$*

$$\mathcal{W}_{\text{geo}}(\mathfrak{h}, \mathfrak{g}) = \pi\big(N(\mathfrak{h}, G)\big).$$

Since $\operatorname{Ad}(G) = \mathbb{G}$, this definition indeed depends only on $\mathfrak{g}$ and $\mathfrak{h}$ and not on the choice of G (see [11], Lemma 3.3).

Definition 4.5. The kernel $\ker\pi \subseteq G$ of $\pi\colon N(\mathfrak{h}, G) \to S(\Lambda)$ is called the *Cartan subgroup of* G *with respect to* $\mathfrak{h}$ and is denoted by $C(\mathfrak{h})$. (See [9] Definition 1.1.a.) ■

In other words, $\mathcal{W}_{\rm geo}(\mathfrak{h}, \mathfrak{g}) \cong \frac{N(\mathfrak{h},G)}{C(\mathfrak{h})}$.

Evidently $g \in Z(\mathfrak{h}, G)$ implies $\mathrm{Ad}(g)(x) - x \in [\mathfrak{h}, \mathfrak{h}]$ for all $x \in \mathfrak{h}$. But $\lambda([\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}]) = \{0\}$ for all roots λ. Consequently $\lambda \circ \mathrm{Ad}_{\mathbb{C}}(g)|\mathfrak{h}_{\mathbb{C}} = \lambda$ for all λ and $g \in Z(\mathfrak{h}, G)$. Thus $Z(\mathfrak{h}, G) \subseteq \ker\pi$. Therefore, $\pi\colon N(\mathfrak{h}, \mathbb{G}) \to \mathcal{W}_{\rm geo}(\mathfrak{h}, \mathfrak{g})$ factors uniquely through a surjective morphism $\pi'\colon \mathcal{W}(\mathfrak{h}, \mathfrak{g}) \to \mathcal{W}_{\rm geo}(\mathfrak{h}, \mathfrak{g})$ with kernel $\frac{C(\mathfrak{h})}{Z(\mathfrak{h},G)}$.

Proposition 4.6. ([11], Proposition 4.2) *For any connected Lie group* G *with Lie algebra* $\mathfrak{g}$ *and with a Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$, *set* $H = \exp\mathfrak{h}$. *Then the Cartan subgroup* $C(\mathfrak{h})$ *equals* $Z(\mathfrak{h}, G)$ *and* $\pi'\colon \mathcal{W}(\mathfrak{h}, \mathfrak{g}) \to \mathcal{W}_{\rm geo}(\mathfrak{h}, \mathfrak{g})$ *is an isomorphism of groups.* ■

Hence the Weyl group $\mathcal{W}(\mathfrak{h}, \mathfrak{g})$ has an alternate geometric interpretation.

5. Codimension two

As we have seen in the first section, knowing that $\mathfrak{Z}(\mathfrak{e})$ is a proper subvariety of $\mathfrak{g}$ is not enough. We need more precise information on the dimension of $\mathfrak{Z}(\mathfrak{e})$. If $\mathfrak{Z}(\mathfrak{e})$ contains regular elements at all, then 1.3 shows $\mathrm{reg}\,\mathfrak{g} \cap \mathfrak{Z}(\mathfrak{e}) \subseteq \mathfrak{H}(\mathfrak{e})$, and if $\mathfrak{g}$ has abelian Cartan subalgebras, then 1.5(ii) gives us $\mathrm{Zcl}(\mathrm{reg}\,\mathfrak{g} \cap \mathfrak{Z}(\mathfrak{e})) = \mathrm{Zcl}\big(\mathfrak{H}(\mathfrak{e})\big) \subseteq \mathfrak{Z}(\mathfrak{e})$. As a consequence, in a finer analysis of the dimension of $\mathfrak{Z}(\mathfrak{e})$ the focus shifts to $\mathfrak{H}(\mathfrak{e})$.

The group $\mathcal{E} = \langle e^{\mathrm{ad}\,\mathfrak{e}}\rangle$ of inner automorphisms generated by $\mathfrak{e}$ acts naturally on the set $\mathcal{I} := \big\{\mathfrak{h} \cap \mathfrak{e} : \mathfrak{h} \in \mathcal{C},\quad \mathfrak{h} \cap \mathfrak{e} \neq \{0\}\big\}$. Let $\mathtt{F}$ denote a system of representatives for the orbits of $\mathcal{I}$.

Lemma 5.1.([14], Lemmas 5.1 and 5.2) *Suppose that* $\mathfrak{e}$ *is a Cartan reductive subalgebra of a Lie algebra* $\mathfrak{g}$. *Then*

$$\mathfrak{H}(\mathfrak{e}) = \bigcup_{\mathfrak{a}\in\mathtt{F}} \mathcal{E}\cdot\mathfrak{z}(\mathfrak{a}, \mathfrak{g}) = \bigcup_{\varphi\in\mathcal{E},\,\mathfrak{a}\in\mathtt{F}} \mathfrak{z}(\varphi(\mathfrak{a}), \mathfrak{g}) \subseteq \mathfrak{Z}(\mathfrak{e}).$$

Further, $\mathcal{I}/\mathcal{E}$ *is finite and thus any cross section* $\mathtt{F}$ *is finite.* ■

This lemma allows us to concentrate on $\mathfrak{z}(\mathfrak{a}, \mathfrak{g})$. The result is the following:

Proposition 5.2. ([14], Theorem 5.12) *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{e}$ *a Cartan reductive subalgebra of* $\mathfrak{g}$ *such that* $\mathfrak{H}(\mathfrak{e}) \neq \emptyset$. *Then the dimension of the Zariski closed set* $\mathrm{Zcl}\big(\mathfrak{H}(\mathfrak{e})\big) = \mathrm{Zcl}\big(\mathfrak{Z}(\mathfrak{e}) \cap \mathrm{reg}(\mathfrak{g})\big)$ *is*

$$\dim \mathrm{Zcl}\big(\mathfrak{Z}(\mathfrak{e} \cap \mathrm{reg}(\mathfrak{g}))\big) = \max\{\dim\big(\mathfrak{z}(\mathfrak{a}, \mathfrak{g}) + \mathfrak{e}\big) \mid \mathfrak{a} \in \mathtt{F}\}. \qquad ■$$

In the process of exploiting this proposition once again some shorthand terminology is convenient.

Definition 5.3. A subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ is called *Cartan compact* in $\mathfrak{g}$ if every Cartan subalgebra of $\mathfrak{g}$ meets $\mathfrak{g}_1$ in a compactly embedded subalgebra. ■

One notices at once two principal sufficient conditions under which a subalgebra is Cartan compact in a containing algebra:

Proposition 5.4. *A subalgebra $\mathfrak{g}_1$ of a Lie algebra $\mathfrak{g}$ is Cartan compact in $\mathfrak{g}$ if at least one of the following two conditions is satisfied:*

(i) *All Cartan subalgebras of $\mathfrak{g}$ meeting $\mathfrak{g}_1$ nontrivially are compactly embedded.*

(ii) *$\mathfrak{g}_1$ is compactly embedded.* ■

Proposition 5.5. ([14], Theorem 5.15) *Suppose that $\mathfrak{e}$ is a Cartan compact proper subalgebra of a Lie algebra $\mathfrak{g}$ with $\mathfrak{H}(\mathfrak{e}) \neq \emptyset$ and suppose that $\mathfrak{e}$ does not contain any nonzero ideals of $\mathfrak{g}$. Then* $\dim \mathrm{Zcl}\big(\mathfrak{H}(\mathfrak{e})\big) \leq \dim \mathfrak{g} - 2$. ■

By Theorem 1.6, the difference between $\mathrm{Zcl}\big(\mathfrak{H}(\mathfrak{e})\big)$ and $\mathfrak{Z}(\mathfrak{e})$ vanishes if $\mathfrak{e}$ is reductively embedded. This is certainly the case if $\mathfrak{e}$ is compactly embedded. It is this last condition which allows us via an estimate of the dimension of $\mathfrak{z}(\mathfrak{a}, \mathfrak{g})$ to conclude the following decisive answer to our Problem 1:

Theorem 5.6. ([14], Theorem 5.16) *Suppose that $\mathfrak{e}$ is a compactly embedded proper subalgebra of a Lie algebra $\mathfrak{g}$, and that it does not contain any nonzero ideals of $\mathfrak{g}$. Then* $\dim \mathfrak{Z}(\mathfrak{e}) \leq \dim \mathfrak{g} - 2$. ■

Corollary 5.7. *Suppose that $\mathfrak{e}$ is a proper subalgebra of a compact Lie algebra $\mathfrak{g}$ and that $\mathfrak{e}$ does not contain any nonzero ideals of $\mathfrak{g}$. Then* $\dim \mathfrak{Z}(\mathfrak{e}) \leq \dim \mathfrak{g} - 2$. ■

We point out that passing to quotient algebras is not compatible with porcupine varieties. In order to see this let $\mathfrak{g}$ be a compact Lie algebra with nonzero center $\mathfrak{z}$ contained in a subalgebra $\mathfrak{e}$ of $\mathfrak{g}$ in such a way that any ideal $\mathfrak{j}$ of $\mathfrak{g}$ with $\mathfrak{z} \subseteq \mathfrak{j} \subseteq \mathfrak{e}$ agrees with either $\mathfrak{z}$ or $\mathfrak{e}$. Let $p\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ denote the quotient morphism. Then $\mathfrak{Z}(\mathfrak{e}) = \mathfrak{g}$, but $\dim \mathfrak{Z}\big(p(\mathfrak{e})\big) \leq \dim p(\mathfrak{g}) - 2$ by Corollary 5.7. Thus $p\big(\mathfrak{Z}(\mathfrak{e})\big) = p(\mathfrak{g}) \neq \mathfrak{Z}\big(p(\mathfrak{e})\big)$.

Theorem 5.6 yields a proof of the Restricted Divisibility Conjecture formulated in Section 1: Suppose that G has a unique maximal subgroup. Then $\mathrm{comp}_G(\mathfrak{g})$ is a subalgebra and thus $\mathfrak{e} := H(W) \cap \mathrm{comp}_G(\mathfrak{g})$ is a compactly embedded subalgebra. Hence by Theorem 5.6, $\mathfrak{Z}(\mathfrak{e})$ is a Zariki-closed subvariety in $\mathfrak{g}$ of codimension at least 2. Assume also that the Lie wedge W of the weakly exponential subsemigroup S has inner points. Then by 1.2(iv) and the subsequent section the hypothesis of 1.2(iii) is satisfied and shows that W is a Lie semialgebra. Thus the Restricted Divisibility Conjecture is true. The hypothesis that W has to have interior is, at this time, a technical matter. Eventually, we can get around it, but that is another matter, not exactly trivial, involving a good deal of Lie semigroup theory.

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